Growing Random Squarings

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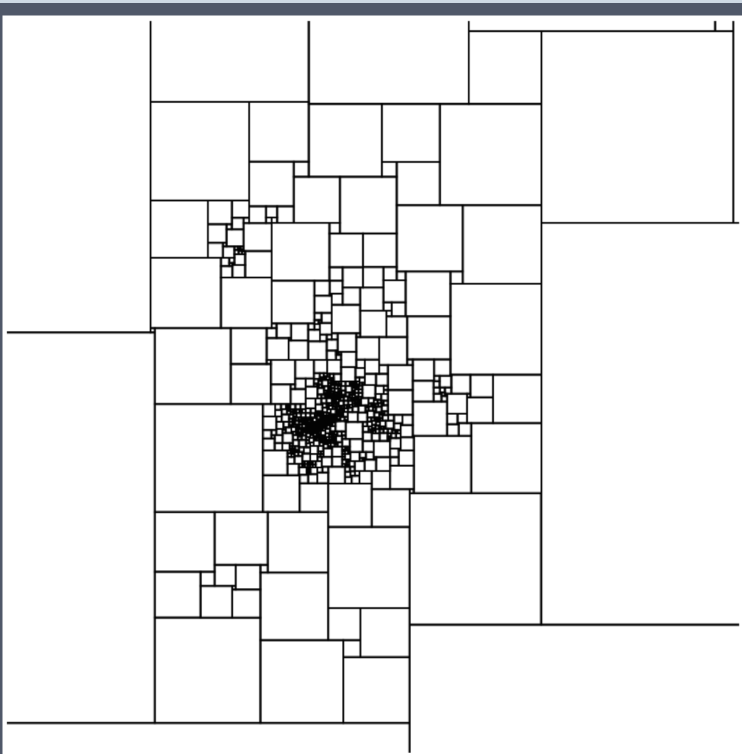


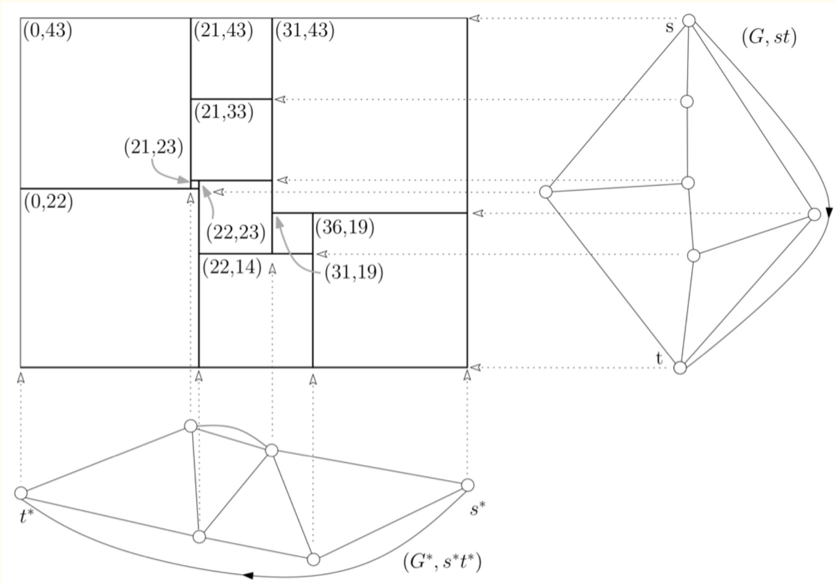
Random Squarings

Brooks, Smith, Stone, Tutte (1940):

Planar map $G = (G, st)$ s.t. $st \in E(G)$

$\longleftrightarrow$

Squaring of a rectangle $S = S(G) \leftarrow$ Collection of lines.



Algorithm: 1) Put potential diff 1 btw s & t ($s \to$ potential 1; $t \to$ ground)

2) Current flows through network; write $p(v)$ for potential at v, $c(e)$ for current throug $e = p(v) - p(w)$ if $e = vw$ & $p(v) > p(w)$

3) For each edge e, make a square of side length $c(e)$.

Horizontal Lines: Height = Potential of vertex

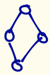
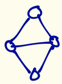
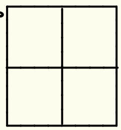
Vertical lines: Height = Potential of dual vertex (face)

Kirchoff's Laws say squares fit together!

(Vertices $\longleftrightarrow$ Horizontal lines; Conservation of charge $\longleftrightarrow$ squares above line & below have same total width)

(Conservation of energy: If $C = v_1, \ldots, v_k = v_1$ cycle in G then $\sum_1^{k-1} (p(v_{i+1}) - p(v_i)) = 0$

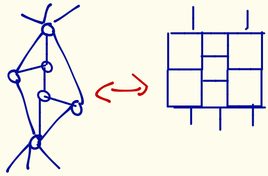
$\longleftrightarrow$ The height of a line is well-defined; calculate by summing pot. diffs on any path from s to v)

NB: Map $G \to S$ not invertible; consider  vs. . Both give 

But is invertible up to 0-current edges.

Note: A 2-cut in $G \longleftrightarrow$ A nontrivial sub-squaring in S

3-connected map $\longleftrightarrow$ Irreducible squaring.



Thm (A-B, Leavitt): Can construct a stoch. proc.

$(S_n, 1 \le n \le \infty) = (S_n(G_n), 1 \le n \le \infty)$, with $G_n = (G_n, st)$

(a) S_n a rand. squaring, n rectangles

(b) $S_n \xrightarrow{a.s.} S_\infty$ for the Hausdorff dist. (i.e. $\forall \varepsilon$, $\forall n \geq n_0(\varepsilon)$, $S_\infty \subset B(S_n, \varepsilon)$)

(c) $G_n \xrightarrow{a.s.} G_\infty$, i.e. $\forall r$, $B_{G_n}(r,s)$ is eventually constant.

(d) G_∞ is "the random infinite 3-connected planar graph":

(i.e. if $\hat{G}_n$ is a u. rand. rooted 3-con. planar map, n edges, then $\hat{G}_n \xrightarrow{d} G_\infty$. In fact $\mathbb{P}(G_n \text{ 3-con}) \to 2^8/3^6$ and $(G_n \mid G_n \text{ 3-con}) \stackrel{d}{=} \hat{G}_n$.)

Proof Idea

Squaring $\xleftrightarrow{\text{BSST}}$ Map $\xleftrightarrow[\text{bijection"}]{\text{"trivial}}$ Quadrangulation $\xleftrightarrow[\text{bijection"}]{\text{"Closure}}$ Binary Tree

Grow Squaring $\longleftarrow$ Grow maps $\longleftarrow$ Grow quads. $\longleftarrow$ Grow trees

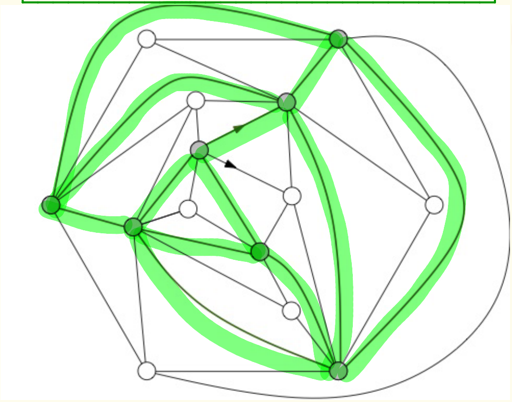
Properties of The Limit

Prop: If $(H_n, st) \to (H_\infty, st)$ a recurrent, locally finite map, then $S(H_n, st) \to S(H_\infty, st)$ for the Hausdorff distance.

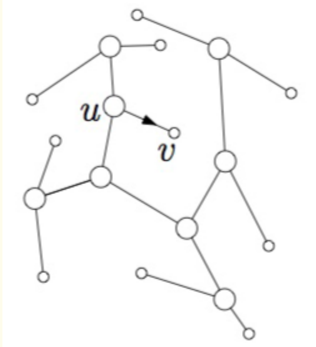
(Part of proposition is that $S(H_\infty)$ is a squaring in this case; related results by Benjamini & Schramm for (certain) transient graphs.)

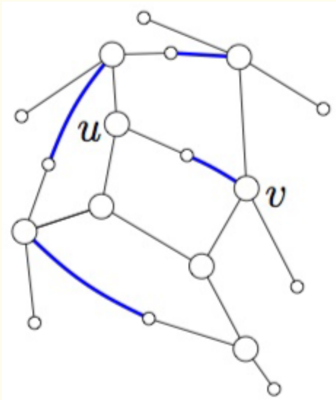
Easy to show G_∞ a.s. recurrent using a recent result of Gurel-Gurevich & Nachmias. $\Rightarrow S_n \xrightarrow{a.s.} S_\infty$

Map $\xleftrightarrow[\text{bijection"}]{\text{"trivial}}$ Quadrangulation



Quadrangulation $\xleftrightarrow[\text{bijection"}]{\text{"Closure}}$ Binary Tree





Thm (Fusy, Poulalhon, Schaeffer 2008)

Closure is a bij. between

T_n = Edge-rooted binary trees, n internal nodes

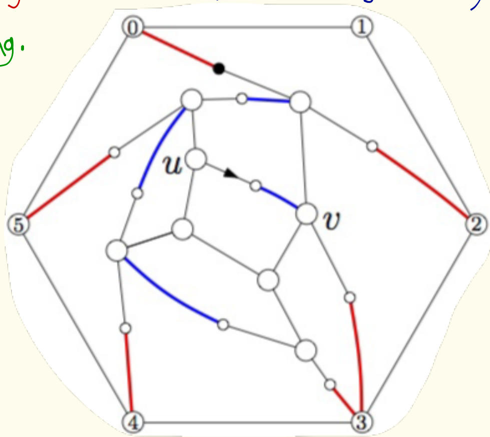
and

Q_n = Irreducible quadrangulations of a hexagon with n+6 nodes.

Let $T_n \in_U T_n$, Q_n its closure $\in_U Q_n$

(To get a true quad, add a random diag to hexagon so not really in Q_n; gloss this)

NB: Q_n may not be irred. after adding diag. e.g. 03 makes irreducibility fail.

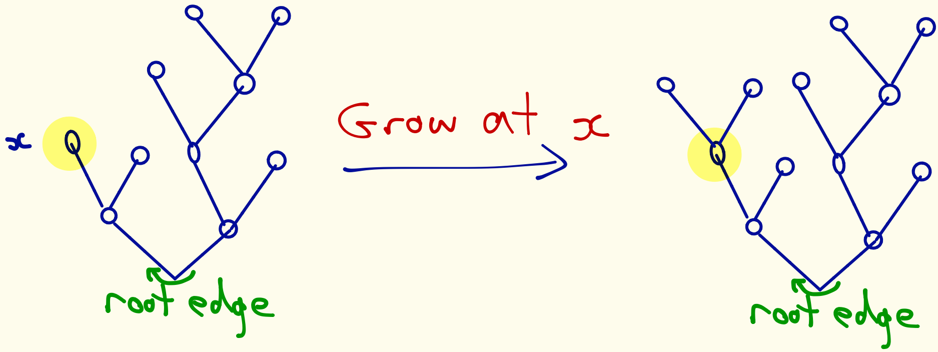


G_n = image of Q_n under trivial bijection.

Fact: G_n 3-conn $\longleftrightarrow$ Q_n irred.

Fact: $(G_n \mid G_n \text{ 3-conn.}) \overset{d}{=}$ U. rand 3-conn. map, n+4 edges.

Growing T_n



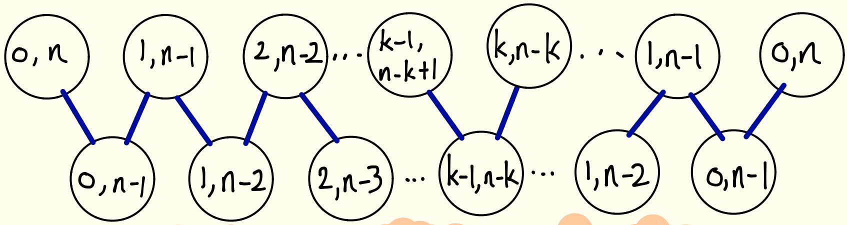
Want: Rule for choosing x s.t. if $T_n \in_u \mathcal{T}_n$ then growing at x yields $T_{n+1} \in_u \mathcal{T}_{n+1}$.

Luczak & Winkler (2004): Can do this!

(Idea: Just need to calculate $p(n,k) = \mathbb{P}(\text{choose } x \text{ from left subtree} \mid \text{left subtree size} = k)$
Then apply recursively. Can calculate these probs. using Catalan #s and they work out.)

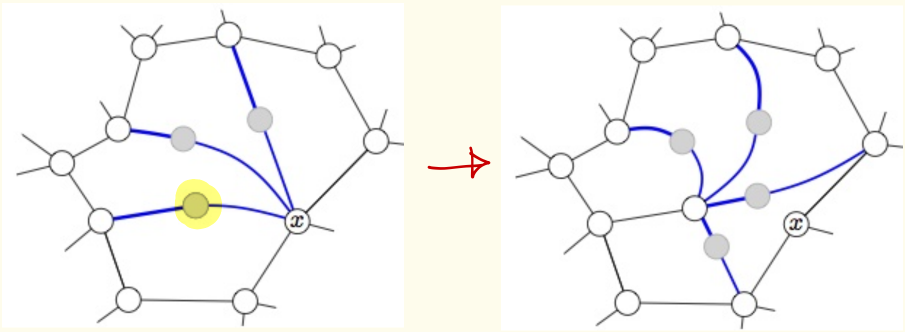
L-R split at root in T_{n+1}

L-R split at root in T_n



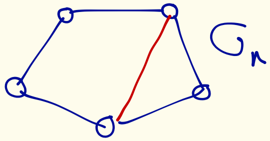
Need: $\displaystyle\sum_{i=0}^{k-1} \frac{C_{i,n-i}}{C_{n+1}} \leqslant \sum_{i=0}^{k-1} \frac{C_{i,n-i-1}}{C_n} \leqslant \sum_{i=0}^{k} \frac{C_{i,n-i}}{C_{n+1}}$

Growing Q_n
(push-forward of $T_n \to T_{n+1}$)

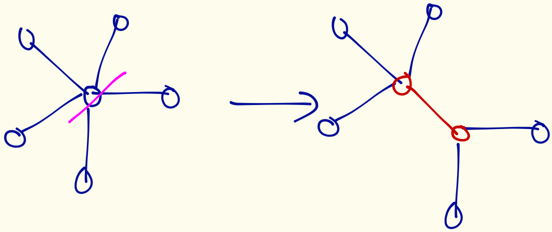


Growing G_n

If leaf becomes a "facial vertex" in Q_n:

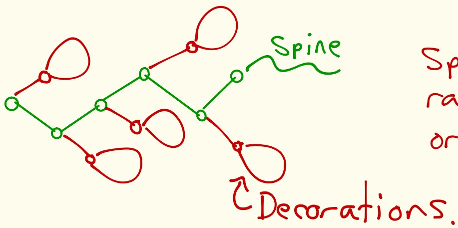


If leaf becomes a "primal vertex":



Convergence

$T_n \to T_\infty =$



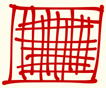
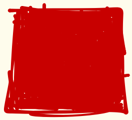
Spine: infinite path, randomly chooses left or right at each step.

Decorations:
iid critical $GW(B)$
trees, $\mathbb{P}(B=0) = \mathbb{P}(B=2) = 1/2$.

Convergence for Q_n : Show that eventually no closures occur near root.

Convergence for G_n: follows from convergence for Q_n easily.

NB: Our result is pretty boring if limit is a black square.

e.g.  $n \times n$ grid, mesh $\frac{1}{n}$ $\xrightarrow{\text{Haus.}}$ 

To rule this out we use a criterion of He and Schramm which ensures the limit packing S_∞ will have ≤ 1 point of accumulation.

Vertex- and edge- parabolicity

$G=(V,E)$ graph

$m: V \to [0,\infty)$.

Def: m-dist $d_m(u,v) = \inf_{\gamma: u \to v} \sum_{w \in \gamma} m(w)$

If G is infinite then $d_m(u,\infty) = \inf_{\gamma: u \to \infty} \sum_{w \in \gamma} m(w)$.

G is vertex-parabolic if $\exists\, m$ s.t. $\|m\|_2 < \infty$ and s.t. $d_m(u,\infty) = \infty$.

G is edge-parabolic if $\exists\, m: E \to \infty$: $\|m\|_2 < \infty$, $\underbrace{d_m(u,\infty)}_{\inf_{\gamma: u \to \infty} \sum_{e \in \gamma} m(e)} = \infty$

Thm: (Duffin, 1962)

G edge-parabolic $\iff$ G recurrent

Ends in graphs

$\#$ ends of $G = \sup_{D \subset V,\, |D| < \infty} \#$ components of $G - D$

ex: $\mathbb{Z}$ 2-ended, $\mathbb{Z}^2$ one-ended, complete binary tree ∞-ended.

He & Schramm

Theorem: Let P be a squaring of a compact set in $\mathbb{R}^2$.
If contacts graph of P is vertex parabolic then P has at most one point of accumulation.

To prove contacts graph of S_∞ vertex parabolic:
first note G_∞ is recurrent (by Gurel-Gurevich - Nachmias) so edge-parabolic.

Then relate edge-metrics in G_∞ to vertex-metrics in contacts graph to show contacts graph vertex parabolic.

(Natural because vertices of contacts graph = squares of S_∞ = edges of G_∞)

Open Questions

(1) We begin with an analogue of Conjecture 7.1 of [10] and of Conjecture 1 (a) of [26], for the random squarings S_n. There is a unique translation and scaling under which the image S'_n of S_n is centred at 0 and such that when S'_n is stereographically projected to the Riemann sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, the image of the unbounded region of $\mathbb{R}^2 \setminus S_n$ has area $1/n$. Apply this transformation, and let μ_n be the measure on $\mathbb{C}^*$ obtained by letting each connected component of $\mathbb{C}^* \setminus S'_n$ have measure $1/n$.[11] Then μ_n should converge weakly to a measure μ on $\mathbb{C}^*$ which is some version of the Liouville quantum gravity measure (possibly the "γ-unit area quantum sphere measure with $\gamma = \sqrt{8/3}$", introduced in [26]). In particular, μ should satisfy a version of the KPZ dimensional scaling relation.

(2) We expect that the box-counting dimension of S_∞ is a.s. well-defined and constant. More precisely, write n_ϵ for the number of balls of radius ϵ required to cover S_∞. We expect that $\log n_\epsilon / \log(1/\epsilon) \to c$ almost surely, where c is non-random. Is this true? If so, what is c? Is $c > 1$? (Note that for the Hausdorff dimension, if $(C_n, n \in \mathbb{N})$ are measurable sets in $\mathbb{R}^2$ then $\dim_{\mathrm{Haus}}(\bigcup_{n\in\mathbb{N}} C_n) = \sup_{n\in\mathbb{N}} \dim_{\mathrm{Haus}}(C_n)$. Since S_∞ is a countable union of line segments, it follows that $\dim_{\mathrm{Haus}}(S_\infty) = 1$ almost surely.)

(3) Let Z be the a.s. unique accumulation point of S_∞. Can the law of Z be explicitly described?

(4) Write $G_\infty(\epsilon)$ for the graph induced by those vertices for which all incident squares are disjoint from $B(Z, \epsilon)$. How quickly does $G_\infty(\epsilon)$ grow as ϵ decreases? Relatedly, how does the diameter of $G_\infty(\epsilon)$ grow? Existing results about random maps suggest that if the diameter grows as $\epsilon^{-\alpha}$ then the volume should grow as $\epsilon^{-4\alpha}$.

(5) The structure of S_∞ near Z should be independent of its structure near the root; here is one question along these lines. Reroot G_∞ by taking one step along a random walk path from the root, write $\hat{S}_\infty$ for the resulting squaring and $\hat{Z}$ for its point of accumulation. Then recenter S_∞ and $\hat{S}_\infty$ so that Z and $\hat{Z}$ sit at the origin. Does $\epsilon^{-1} d_H(S_\infty \cap B(0,\epsilon), \hat{S}_\infty \cap B(0,\epsilon)) \to 0$ almost surely, as $\epsilon \to 0$? Here d_H denotes Hausdorff distance.

(6) Let $e_n(1), \ldots, e_n(k)$ be independent, uniformly random oriented edges of the contacts graph $R(S_n)$, and for $1 \le i \le k$ let $r_n(i)$ be the ratio of the side length of the "tail square" of $e_n(i)$ to that of its "head square". The vector $(r_n(i), 1 \le i \le k)$ should converge in distribution to a limit $(r(i), 1 \le i \le k)$, whose entries are iid. This would be a very small first step towards establishing that the random squaring in some sense "looks like the exponential of a Gaussian free field".

(7) Let A_n be the adjacency matrix of G_n. The areas of squares may be calculated as determinants of minors of A_n. However, these determinants grow very quickly, and even finding logarithmic asymptotics seems challenging. A simpler, still challenging project is to study the determinant of any principal minor of A_n or, equivalently, to study the number of spanning trees of G_n.

(8) The height of S_∞ is 1 but its width W_∞ is random, and by considering the graph structure near the root of G_∞ it is not hard to see that W_∞ is an honest random variable (rather than a.s. constant) On the other hand, duality implies that W_∞ and $1/W_\infty$ have the same law. Can anything explicit be said about this law? In particular, is $\mathbb{P}(W_\infty = 1) > 0$?

(9) Simulations suggest that for n large, S_n is unlikely to contain four squares with common intersection. Does this probability indeed tend to zero as n becomes large? This question looks innocent. However, recall that such intersections are the reason the function sending a rooted planar graph to its squaring is non-invertible. A positive answer would constitute substantial progress towards proving an asymptotic formula, conjectured by Tutte [27, Section 9], for the number of perfect squarings with n squares.

(10) Let $\hat{S}_n$ be uniformly distributed over squarings of a rectangle with n squares. Does $\hat{S}_n$ converge in distribution to S_∞ for the Hausdorff distance? This follows if the laws of S_n and $\hat{S}_n$ are close, which would itself follow from a positive answer to the previous question.

(11) The behaviour of the simple random walk on G_∞ is also of interest. How do quantities such as $\mathbb{P}(X_t = X_0)$, $d_{G_\infty}(X_0, X_t)$, and $\#\{X_s, 0 \le s \le t\}$ scale in t?

(12) It seems likely that $R(S_\infty)$ is recurrent; is it? Here is one tempting argument for recurrence; its incorrectness was pointed out to us by Ori Gurel-Gurevich. By Lemma 4.5, $R(S_\infty)$ may be viewed as a subgraph of D^2_∞. Since D_∞ is recurrent, so is D^2_∞; then conclude via Rayleigh monotonicity. The problem with the argument

Bonus Question

Le Gall's Question (arXiv 1403.7943, to appear in ICM 2014 proceedings)

Next suppose that, for every even integer $n \geq 2$, we have constructed a circle-packing embedding $\mathcal{C}_n$ of a uniformly distributed simple triangulation with n faces. Write $V(\mathcal{C}_n)$ for the vertex set of $\mathcal{C}_n$ and d^n_{gr} for the graph distance on $V(\mathcal{C}_n)$. By Theorem 3.1, or more precisely by the extension of Theorem 3.1 to simple triangulations found in [2], we have

$$\Big(V(\mathcal{C}_n), (\frac{3}{2})^{1/4} n^{-1/4}\, d^n_{\mathrm{gr}}\Big) \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{m}_\infty, D)$$

in the Gromov-Hausdorff sense. Note that the constant $(3/2)^{1/4}$ is different from the constant $6^{1/4}$ in (1) or in Theorem 3.1, because we are dealing with simple triangulations.

Conjecture. *One can construct the circle packing embeddings $\mathcal{C}_n$ in such a way that*

$$\sup_{x\in\mathbb{S}^2} \Big(\min_{y\in V(\mathcal{C}_n)} |x-y|\Big) \xrightarrow[n\to\infty]{} 0$$

in probability, and there exists a continuous random process $(\Delta(x,y))_{x,y\in\mathbb{S}^2}$, which is nonzero outside the diagonal and such that

$$\sup_{x,y\in V(\mathcal{C}_n)} \Big|\Delta(x,y) - (\frac{3}{2})^{1/4} n^{-1/4}\, d^n_{\mathrm{gr}}(x,y)\Big| \xrightarrow[n\to\infty]{} 0$$

in probability.

(I have another similar procedure that grows random 3-connected triangulations via local modifications. This gives a.s. convergence to a "random ∞ circle packing in $\mathbb{R}^2$" Can it be used to tackle Le Gall's ICM conjecture?)