
Three branching anecdotes
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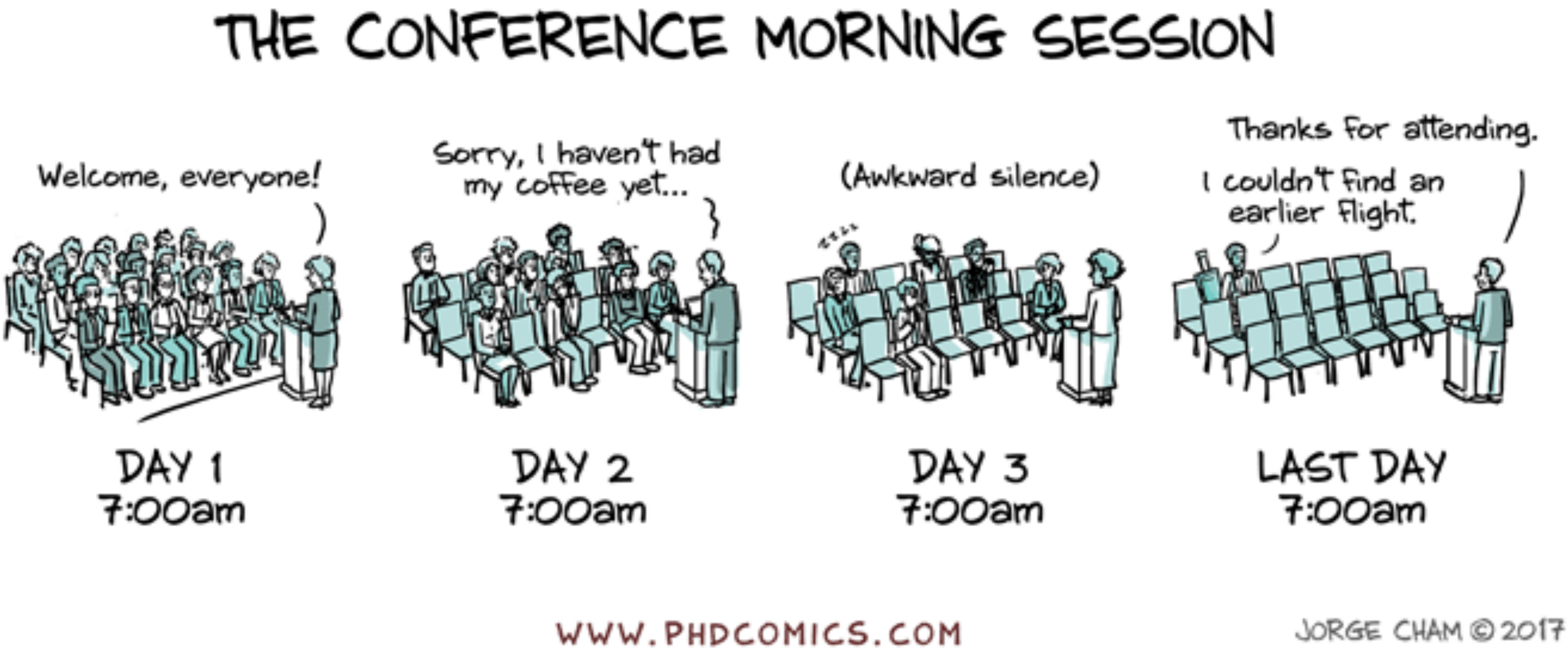
Dynamics on Random Graphs and Random Maps
October 23 - 27, 2017
CIRM
CENTRE INTERNATIONAL DE RENCONTRES MATHÉMATIQUES


THE CONFERENCE MORNING SESSION
Welcome, everyone!
DAY 1
7:00am
Sorry, I haven't had my coffee yet...
DAY 2
7:00am
(Awkward silence)
DAY 3
7:00am
Thanks for attending.
I couldn't find an earlier flight.
LAST DAY
7:00am
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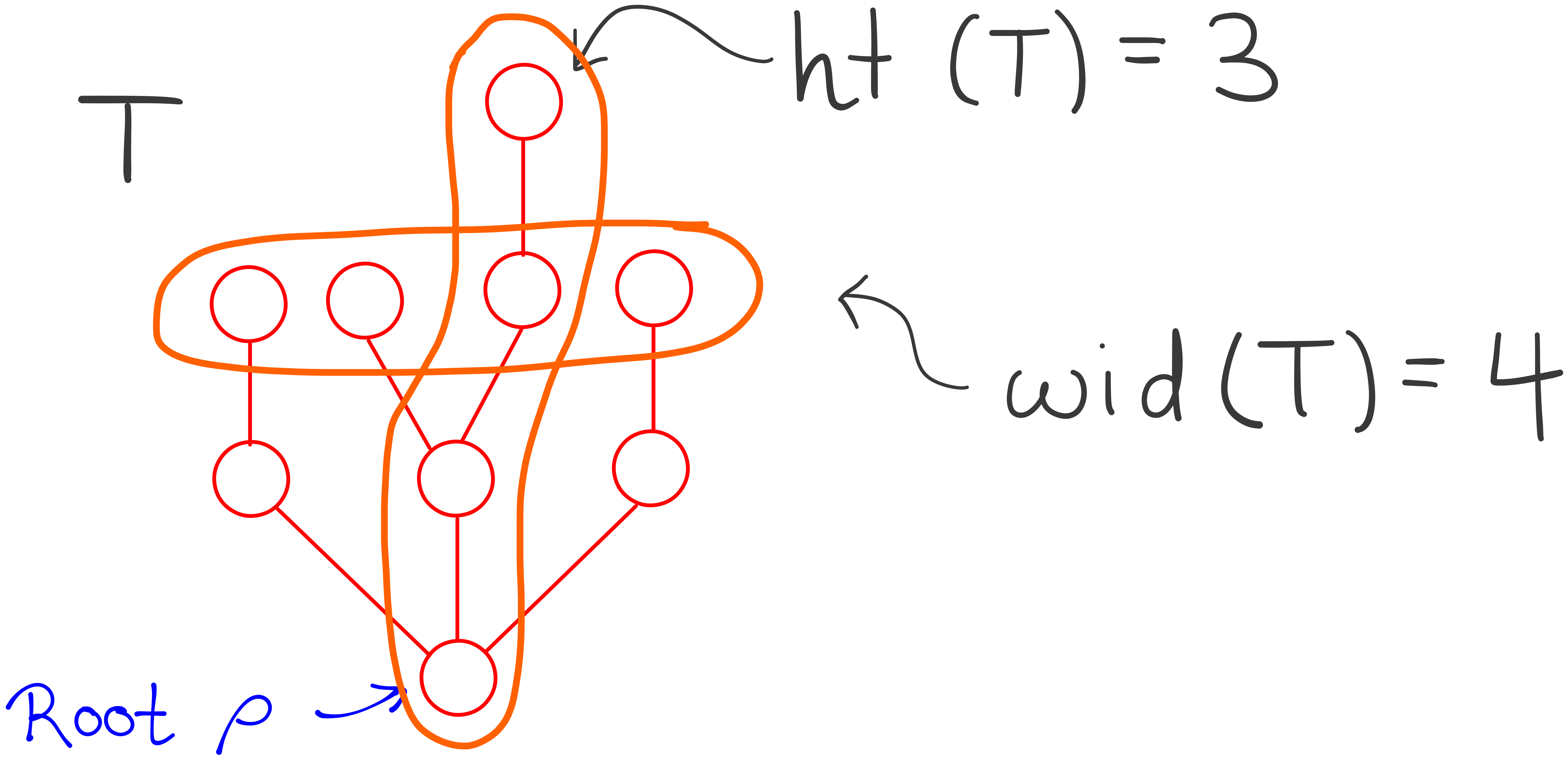
1| Most trees are short and fat.

Trees:



Height: Greatest distance from any node to the root } ht(T)

Width: Greatest # nodes on a single level. } wid(T)

Main Results

Fix any r.v. C with $\sum_{k\geq 0}\mathbb{P}(C=k)=1$, write $p_k=\mathbb{P}(C=k)$.

Let T be $GW(C)$ distributed.

Theorem ("Most trees are short & fat")

There is a universal constant $\delta>0$ s.t.

$$\mathbb{P}\left(ht(T)\geq\frac{k}{1-p_1}\cdot wid(T)\right)\leq\exp(-\delta k).$$

Remark: If $\mathbb{E}C>1$ then $\mathbb{P}(\sigma=\infty)>0$, and

$$\mathbb{P}(ht(T)=wid(T)=\infty\,|\,\sigma=\infty)=1.$$

Also, given that $\sigma<\infty$, the cond. dist. of T is $GW(\hat{C})$ where $\mathbb{P}(\hat{C}=1)=p_1$,

$\mathbb{E}(\hat{C})<1$ so can assume $\mathbb{E}C\leq 1$.

Heuristic: GW trees satisfy $wid(T)\cdot ht(T)\simeq vol(T):=\sigma$

Implies "$ht>C^2 wid$" $\cong$ "$ht^2\geq C^2\cdot vol$" so $\mathbb{P}\left(ht(T)>\frac{k}{\sqrt{1-p_1}}\sqrt{vol(T)}\right)\leq\exp(-\delta k^2)$

Theorem: $$\mathbb{P}\left(ht(T)\geq\frac{k}{\sqrt{1-p_1}}vol(T)^{1/2}\right)\leq\exp\left(-\delta k^2\right)$$

Galton-Watson Trees

- Each node has random # of children
- Nodes reproduce independently

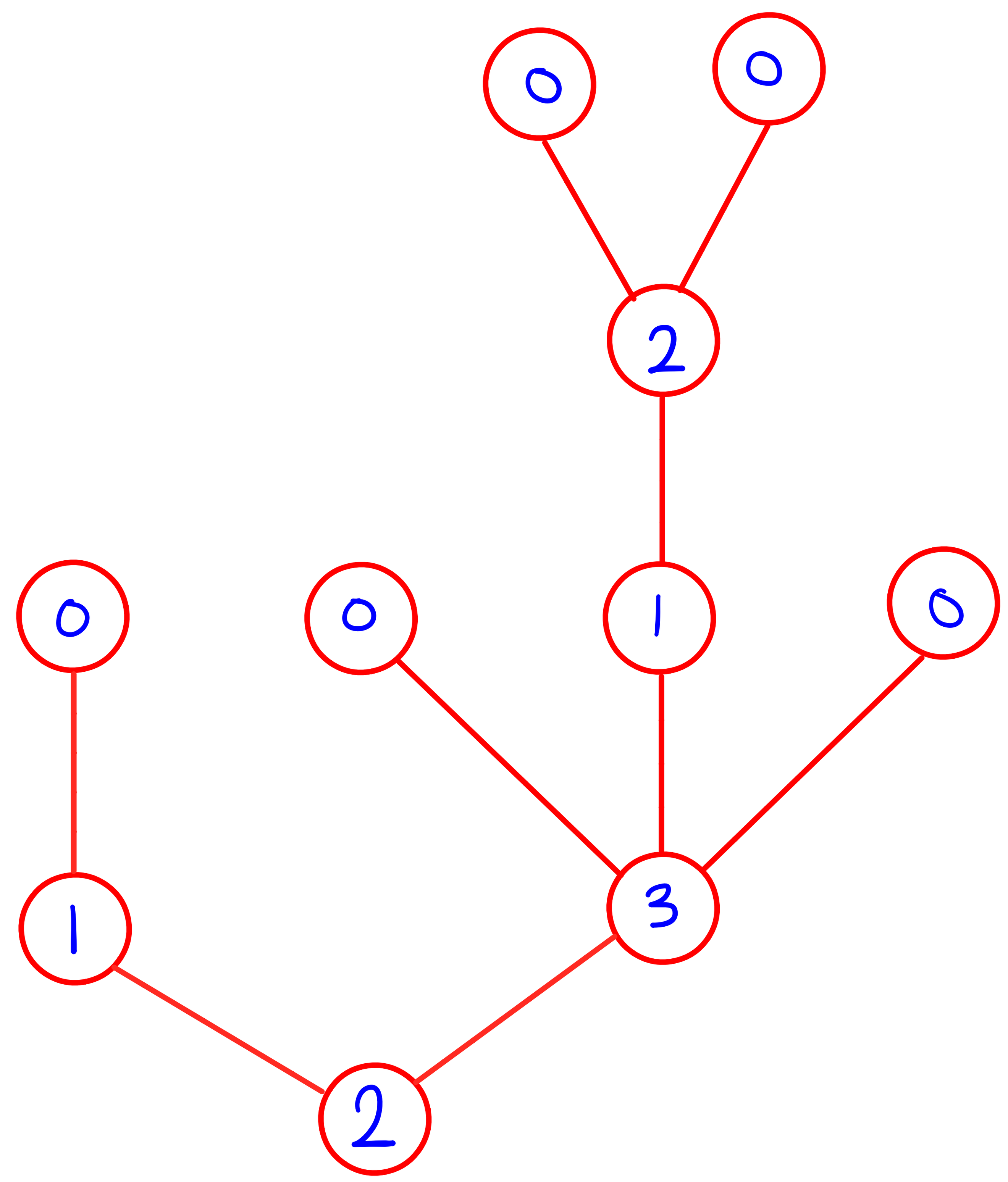
Construction

- $(C_i, i \geq 1)$ independent copies of a random variable C with $\sum_{k \geq 0} \mathbb{P}(C = k) = 1$.

Rule

The sequence $(C_i, i \geq 1)$ gives # children of nodes, in breadth-first search order

Example: $(2, 1, 3, 0, 0, 1, 0, 2, 0, 0, 1, 4, 0, \ldots)$



Halting Condition

Trees with n nodes have $n-1$ edges.

For $n \geq 0$

#nodes discovered by $\underbrace{\text{time } n}_{n \text{ steps of BFS}}$.

$$= 1 + \sum_{i=1}^{n} C_i$$

vertices explored by time n

$$= n$$

Total # vertices $= \sigma$

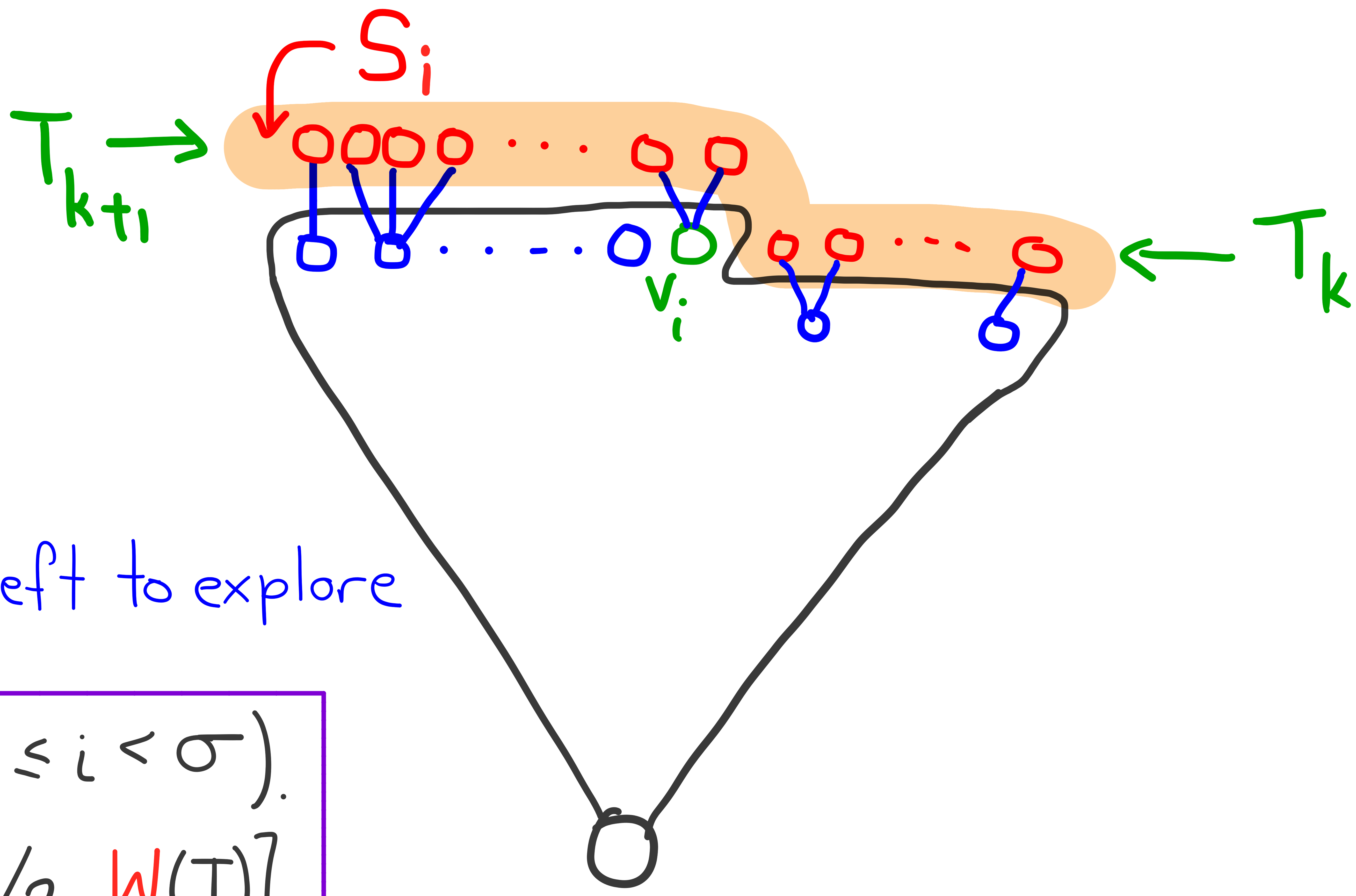
$$= \inf\left\{n : 1 + \sum_{i=1}^{n} C_i = n\right\}.$$

Setup

$1+\sum_{j=1}^{i} C_j$ = # nodes discovered by time i.

Let $S_i = 1+\sum_{j=1}^{i}(C_j-1)$

= # nodes in "BFS queue" at time i



$$\mathbb{E}C \le 1 \Rightarrow \mathbb{E}S_n = 1+n(\mathbb{E}C-1) \le 1.$$

$\sigma = \inf\{t : S_t = 0\}$ = first time no nodes left to explore

Prop: Let $W(T) = \max(S_i, 0 \le i < \sigma)$.
Then $\mathrm{wid}(T) \in (W(T)/2, W(T)]$

Proof: During BFS on level k, "exploration queue" $\subset T_k \cup T_{k+1}$; and $= T_k$ at start of level k. ∎

Idea: $ht(T) = \sum_{k=1}^{ht(T)} 1 = \sum_{k=1}^{ht(T)} \sum_{v \in T_k} \frac{1}{|T_k|}$.

When $v_i \in T_k$ then $S_i \approx |T_k|$ so perhaps

$$ht(T) \approx \sum_{k=1}^{ht(T)} \sum_{v_i \in T_k} \frac{1}{S_i} = \sum_{i=1}^{\sigma} \frac{1}{S_i} =: H(T)\ ?$$

[False; consider a star with n leaves. But...]

Prop: $ht(T) \le 3H(T)$.

Corollary Suffices to prove $\mathbb{P}\left(H(T) \ge \frac{k}{1-p_1} W(T)\right) \le e^{-\delta k}$; thm. follows.

$W(\sigma) = \max(S_i, 0 \le i < \sigma)$ $H(\sigma) = \sum_{i=1}^{\sigma} \frac{1}{S_i}$ **Aim**: $\mathbb{P}\left(H(\sigma) \ge \frac{k}{1-p_1} W(\sigma)\right) \le e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j \approx 2^\ell$ ("scale 2^ℓ") for $j \in \{i, \ldots, i+2^\ell\}$, have

$$H(i+2^\ell) - H(i) = \sum_{j=i+1}^{i+2^\ell} \frac{1}{S_j} \simeq 2^\ell \cdot \frac{1}{2^\ell} = 1.$$

So bound (a) time to change scales,

(b) "# visits to scales" $= (M(\ell), \ell \ge 1)$

(a) Thm (Lévy; Doeblin; Kolmogorov; Rogozin; Le Cam; Esséen; Kesten):

With $p = \max p_i$, have $\boxed{\max_k \mathbb{P}(S_n = k) \le \frac{A\, p}{\sqrt{n(1-p)}}}$ $A > 0$ universal.

"Any random walk spreads out over $\ge \sqrt{n}$ values by time n." Here $\sqrt{n} \simeq 2^\ell$.

So leave scale 2^ℓ after time $\tau = O_p(4^\ell)$.

Contribution to $H \le C \frac{\tau}{2^\ell} = O_p(2^\ell)$

(b) $\boxed{\begin{array}{l}\text{Fact: Given that } M(\ell) \neq 0,\ M(\ell) \text{ dominated by} \\ \text{sum of 2 Geom}(\tfrac{1}{2}) \text{ r.v.s;} \Rightarrow \mathbb{P}(M(\ell) > k \mid M(\ell) > 0) \le 2^{-k/2}.\end{array}}$

Proof via upcrossings. ∎

With $\ell_{max} = \max(\ell : M(\ell) \neq 0)$ get $W \approx 2^{\ell_{max}}$, $H = O_p(1) \cdot \sum_{\ell \le \ell_{max}} 2^\ell M(\ell) = O_p(2^{\ell_{max}})$

Remarks

- Stronger results if add info. about tails of degrees.

Ex: • If $\mathbb{P}(C \geq k) = \Theta(t^{-\alpha})$, $\alpha \in (1,2)$,

$$\text{then } \mathbb{P}(ht(T) > A \cdot m \cdot wid(T)^{\alpha-1}) \leq 2^{-\delta m}$$

• If $Var(C) = \infty$ then $\forall \varepsilon > 0\ \exists n_0$ s.t. for $x > 0$, $n > x^2 n_0$,

$$\mathbb{P}\left(ht(T) > A \cdot m \cdot wid(T),\ |T| > n\right) \leq \frac{x}{n^{1/2}} e^{-x/\varepsilon}.$$

- Conjecture: All this works even conditional on size of tree: $\mathbb{P}(ht(T) > A \cdot m \cdot wid(T) \mid \sigma = n) \leq 2^{-\delta m}$.

- Conjecture: Binary trees are the tallest.

Consider random trees $T_{\vec{n}}$ with a fixed degree seq $\vec{n} = (n_i, i \geq 1)$.

Here n_i = # nodes of $\underbrace{\text{deg}}_{\text{\# children}}$ i.

To stochastically maximize $ht(T_{\vec{n}})$ among sequences with $n_0 = n$, $n_1 = 0$,

choose the seq. $(n, 0, n-1, 0, \ldots)$

2) Comparing arbitrary trees to binary trees

Conjecture: Binary trees are the tallest.

Consider random trees $T_{\vec{n}}$ with a fixed degree seq $\vec{n} = (n_i, i \geq 1)$.

Here n_i = # nodes of $\underbrace{\text{deg}}_{\text{\# children}}$ i.

To stochastically maximize $ht(T_{\vec{n}})$ among sequences with $n_0 = n$, $n_1 = 0$,

choose the seq. $bin(n) = (n, 0, n-1, 0, \ldots)$

"Evidence".

Prop: Let $\vec{n}$ have $n_0 = n$, $n_1 = 0$.

Let $(T_{\vec{n}}, V)$ be random $\overbrace{\text{marked}}^{\text{distinguished node}}$ tree with deg. sequence $\vec{n}$

Let $(T_{bin(n)}, W)$ be random marked binary tree with n leaves.

Then $\text{height}(V) \preceq_{st} \text{height}(W)$.

Warm-up:

binary trees, n leaves $= \frac{1}{2n-1}\binom{2n-1}{n}$

trees with degree seq. $\vec{n}$. With $|\vec{n}| := \sum n_i$ then get

$$\frac{1}{|\vec{n}|}\binom{|\vec{n}|}{n_i, i\geq 0} = \frac{1}{|\vec{n}|}\frac{|\vec{n}|!}{n_0!\,n_1!\cdots}$$

$= \frac{1}{|\vec{n}|} \cdot$ # lattice walks, n_i steps of size $i-1$,

binary forests, n leaves, k connected components $= \frac{k}{2n-k}\binom{2n-k}{n}$

forests with degree seq. $\vec{n}$. With k trees, then get

$$\frac{k}{|\vec{n}|}\binom{|\vec{n}|}{n_i, i\geq 0}$$

$= \frac{k}{|\vec{n}|} \cdot$ # lattice walks, n_i steps of size $i-1$,

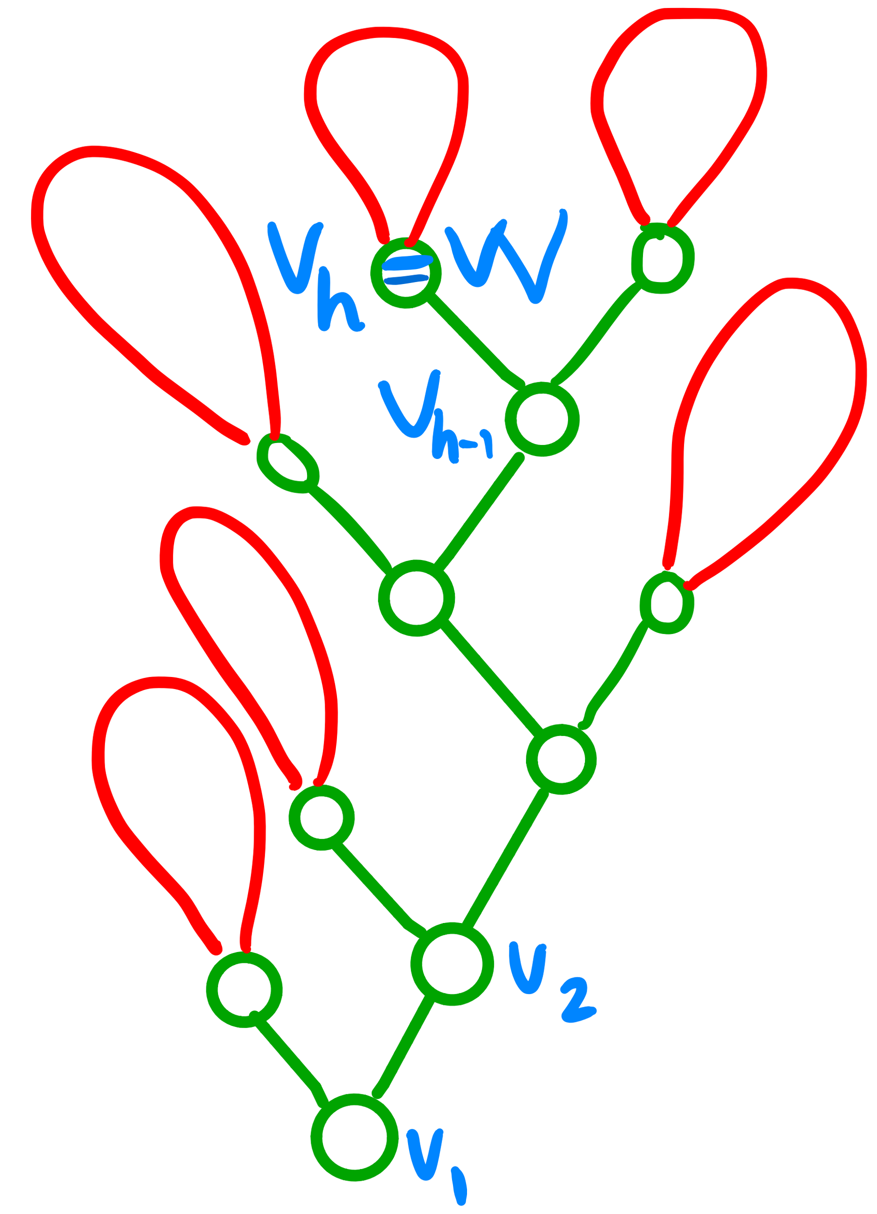
marked forests with degree seq. $\vec{n}$, mark in last tree

distinguished vertex

$= |\vec{n}| \cdot \frac{1}{k} \cdot$ # forests with degree seq. $\vec{n}$

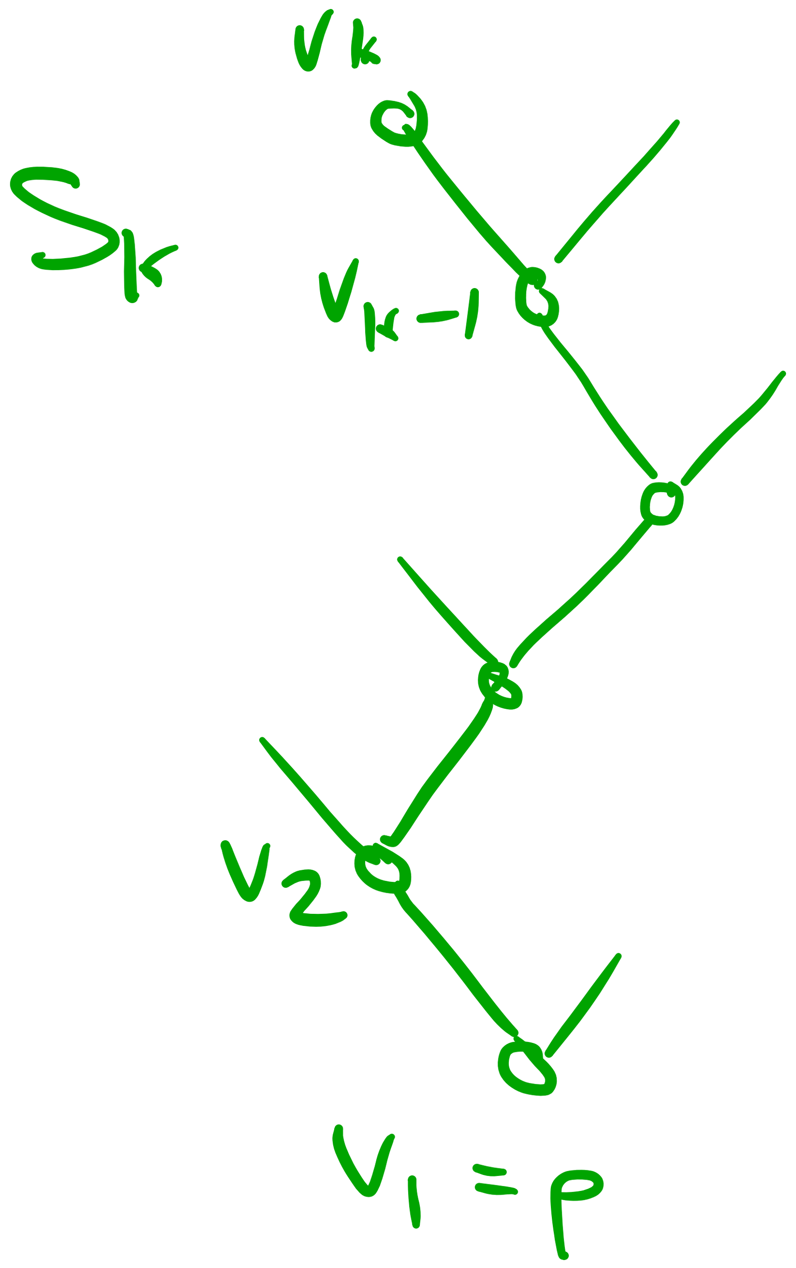
$= \binom{|\vec{n}|}{n_i, i\geq 0} =$ # lattice walks, n_i steps of size $i-1$

Trunks of trees



Let (T, W) be a random marked binary tree, n leaves.

Trunk = path from root to marked vertex, together with children of path vertices

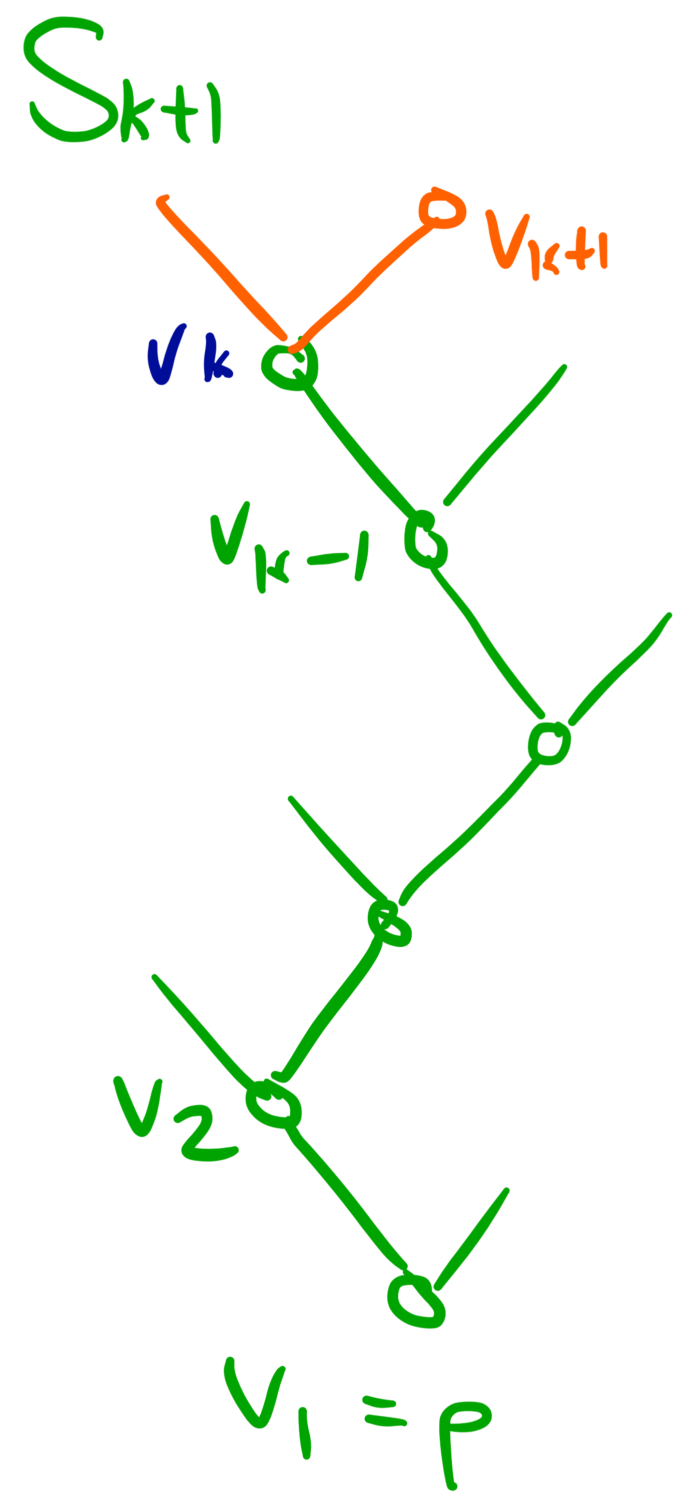


\# Marked binary trees with n leaves, trunk containing S_k,

(marked node in subtree rooted at v_k)

$= \#$ binary forests with n leaves, k trees, mark in last tree $= \binom{2n-k}{n}$

\# Marked binary trees with n leaves, trunk containing S_{k+1}, $= \binom{2n-k-1}{n}$

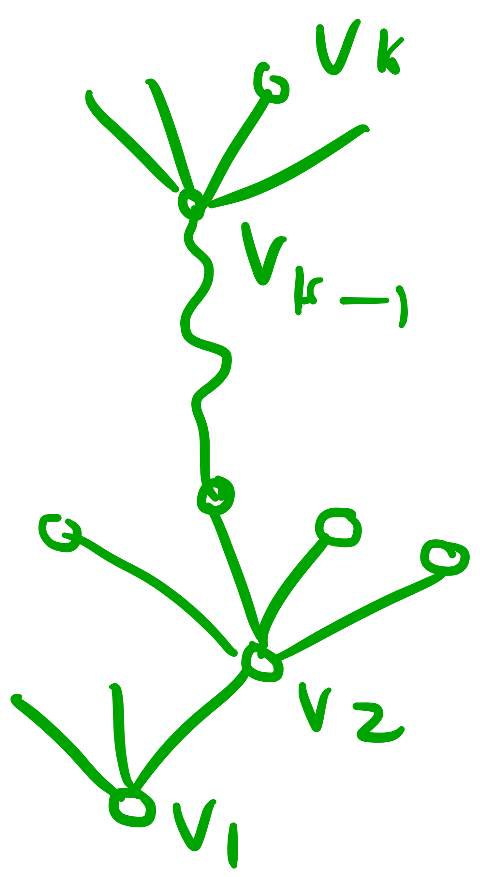


Ratio is $\frac{(2n-k-1)!}{n!(n-k-1)!} \cdot \frac{n!\,(n-k)!}{(2n-k)!} = \frac{n-k}{2n-k}$

Two possible choices for v_{k+1} (left or right)

So $\mathbb{P}(W = v_k \mid \text{Spine contains } S_k) = 1 - 2 \cdot \frac{n-k}{2n-k} = \boxed{\frac{k}{2n-k}}$

Trunks of trees



Let (T,V) be a random marked tree with degree sequence $\vec{n}$

\# Marked forests with degree sequence $\vec{n}=(n,0,n_2,n_3,\dots)$, trunk containing S_k?

Let $\vec{m}=(m_0,m_1,m_2,\dots)$ be obtained from $\vec{n}$ by removing degrees of $v_1,\dots,v_{k-1}$:

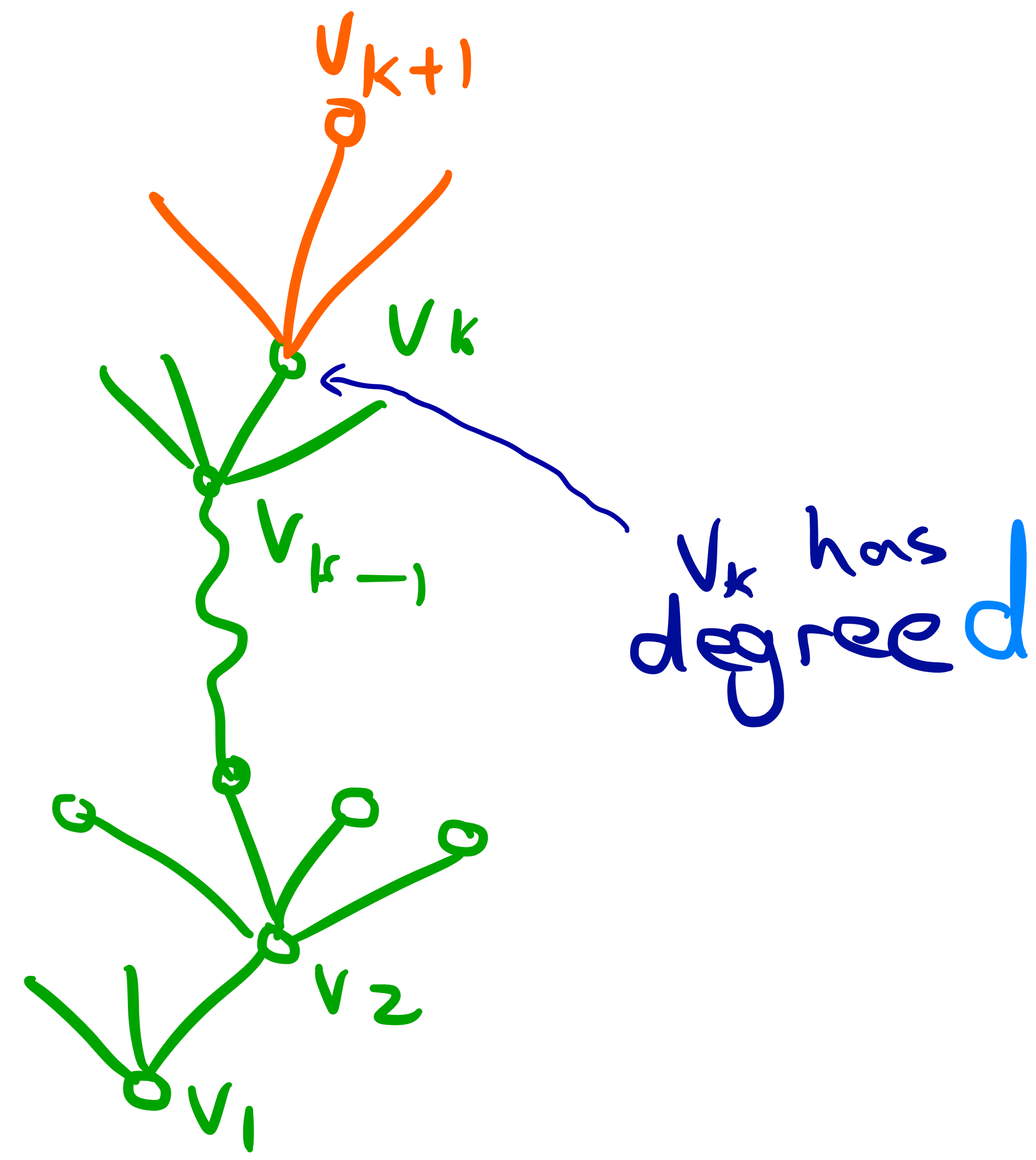
(if $\#\{1\le i\le k-1 : \deg(v_i)=d\}=j$ then $m_d=n_d-j$)

Then \# is $\binom{|\vec{m}|}{\vec{m}}$

\# Marked forests with degree sequence $\vec{n}$, trunk containing S_{k+1}

New degree seq. $\vec{m}(d)=(m_0,\dots m_{d-1}, m_d-1, m_{d+1},\dots)$

\# is $\binom{|\vec{m}|-1}{\vec{m}(d)}$. Ratio is $\binom{|\vec{m}|-1}{\vec{m}(d)} \Big/ \binom{|\vec{m}|}{\vec{m}} = \frac{m_d}{|\vec{m}|}$.



d possible choices for v_{k+1} (which child)

$\rightarrow \mathbb{P}(V\neq v_k, \deg(v_k)=d \mid \text{Trunk contains } S_k) = \frac{d\, m_d}{|\vec{m}|}$

So $\mathbb{P}(V=v_k \mid \text{Trunk contains } S_k) = 1-\frac{\sum d\, m_d}{|\vec{m}|} = \frac{\overbrace{1+\sum_{i=1}^{k-1}(\deg(v_i)-1)}^{\#\text{ subtrees hanging from trunk}}}{\underbrace{|\vec{n}|-(k-1)}_{\#\text{ vertices in these subtrees.}}} \ge \boxed{\frac{k}{2(n-k)}}$

distinguished node

Prop: Let $\vec{n}$ have $n_0 = n$, $n_1 = 0$.
Let $(T_{\vec{n}}, V)$ be random $\underbrace{\text{marked}}$ tree with deg. sequence $\vec{n}$
Let $(T_{bin(n)}, W)$ be random marked binary tree with n leaves.

Then $\mathrm{height}(V) \preccurlyeq_{st} \mathrm{height}(W)$.

Proof:

In marked binary tree $\mathbb{P}(W = v_k \mid \text{Trunk contains } S_k) = \dfrac{k}{2n-k}$

In marked tree with degree sequence $\vec{n}$, $\quad \mathbb{P}(V = v_k \mid \text{Trunk contains } S_k) \geq \dfrac{k}{2n-k}$

So can couple so that $W \leq V$.

∎

This construction of a randomly sampled node seems useful.

Svante Janson: Simply Generated Trees, conditioned Galton-Watson Trees, Random Allocations and Condensation.

Prob. Surveys Vol 9 (2012) 103-252

Conjecture 21.5. *If $\nu = 1$ and $\sigma^2 = \infty$, then $H(\mathcal{T}_n)/\sqrt{n} \xrightarrow{\text{p}} 0$.*

Conjecture 21.6. *If $\nu = 1$ and $\sigma^2 = \infty$, then $W(\mathcal{T}_n)/\sqrt{n} \xrightarrow{\text{p}} \infty$.* ← ✓

Problem 21.7. *Does $\nu < 1$ imply that $H(\mathcal{T}_n)/\sqrt{n} \xrightarrow{\text{p}} 0$?*

Problem 21.8. *Does $\nu < 1$ imply that $W(\mathcal{T}_n)/\sqrt{n} \xrightarrow{\text{p}} \infty$?* ← ✓ (I think)

Furthermore, still in the case $\nu \geqslant 1$, $\sigma^2 < \infty$, Addario-Berry, Devroye and Janson [1] have shown sub-Gaussian tail estimates for the height and width

$$\mathbb{P}(H(\mathcal{T}_n) \geqslant x\sqrt{n}) \leqslant Ce^{-cx^2}, \tag{21.12}$$

$$\mathbb{P}(W(\mathcal{T}_n) \geqslant x\sqrt{n}) \leqslant Ce^{-cx^2}, \tag{21.13}$$

uniformly in all $x \geqslant 0$ and $n \geqslant 1$ (with some positive constants C and c depending on $\boldsymbol{\pi}$ and thus on $\mathbf{w}$). In view of (21.11), we cannot expect (21.13) to hold when $\sigma^2 = \infty$ (or when $\nu < 1$), but we see no reason why (21.12) cannot hold; (21.10) suggests that $H(\mathcal{T}_n)$ typically is smaller when $\sigma^2 = \infty$.

Problem 21.9. *Does (21.12) hold for any weight sequence $\mathbf{w}$ (with C and c depending on $\mathbf{w}$, but not on x or n)?*

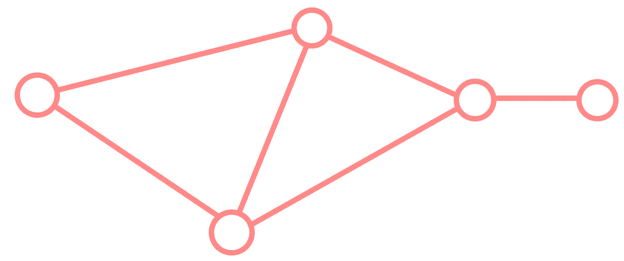
It follows from (21.10)–(21.11) and (21.12)–(21.13) that $\mathbb{E}\,H(\mathcal{T}_n)/\sqrt{n}$ and $\mathbb{E}\,W(\mathcal{T}_n)/\sqrt{n}$ converge to positive numbers. (In fact, the limits are $\sqrt{2\pi}/\sigma$ and $\sqrt{\pi/2}\,\sigma$, see e.g. Janson [61], where also joint moments are computed.)

Problem 21.10. *What are the growth rates of $\mathbb{E}\,H(\mathcal{T}_n)$ and $\mathbb{E}\,W(\mathcal{T}_n)$ when $\sigma^2 = \infty$ or $\nu < 1$?*

Can we use it to prove binary trees are tallest???

3 The MST of random 3-regular graphs

The MST problem



$G=(V,E)$ finite connected graph.

A spanning forest is a subgraph $F=(V,E')$, $E'\subseteq E$, with no cycles. It is a **spanning tree** if it is also connected.

For fixed positive weights $(U_e)_{e\in E}$, the weight of a spanning tree T of G is $\omega(T)=\sum_{e\in E(T)} U_e$

Say T is the MST of G if

$$\omega(T)=\min\{\omega(T'): T' \text{ a spanning tree of } G\}$$

If weights U_e are all distinct then MST is unique.

Old thm (A-B, Broutin, Goldschmidt, Miermont 2017):

Give K_n IID exchangeable edge weights, then

$n^{-1/3} \cdot \mathrm{MST}(K_n) \xrightarrow{d} \mathcal{T}$ $\mathcal{T}$ = Some random $\mathbb{R}$-tree.

New thm

Random 3-regular graph, n vertices.

Give $G(n,3)$ IID exchangeable edge weights, then

$(6n)^{-1/3} \cdot \mathrm{MST}(K_n) \xrightarrow{d} \mathcal{T}$ $\mathcal{T}$ = SAme random $\mathbb{R}$-tree.

Cycle breaking

Alg. for computing MST of $G=(V,E)$

- Let $G_0 = G$.
- Order edges as $e(1), \dots, e(m)$ so $U(e(1)) > \dots > U(e(m))$

For $0 \le i < m$

If $G_i - e(i+1)$ is connected set $G_{i+1} = G_i - e(i+1)$

Else set $G_{i+1} = G_i$.

NB: If edge weights are exchangeable then $e(i+1)$ is a u. rand. edge of G_i.

One way to remove random edges: give them IID lengths, run a Poisson process on G according to length measure, cut where points fall.

Key Idea :

① A-BBGM showed T is well-approximated by T_λ = MST of largest component of $G(n, p_\lambda)$

$p_\lambda = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ = "λ-barely supercritical"

when λ is large

② $T_\lambda \approx T_\lambda^{CORE}$:= MST of CORE of largest component of $G(n, p_\lambda)$

CORE = Graph remaining after removing all pendant subtrees.

③ CORE(λ) $\approx$ Random 3-regular graph with $2\lambda^3/3$ vertices, exponential edge lengths

[Can use cycle breaking via Poisson cuts to build MST of comp.; this is done in A-BBGM]

④ MST without edge lengths and with lengths related by a scaling factor

T

$\downarrow$

T_λ

$\downarrow$

T_λ^{CORE}

$\downarrow$

$T_{2\lambda^3/3}^{3,exp}$

$\downarrow$

$cT_{2\lambda^3/3}^{3}$

Universality, extensions

The tree $\mathcal{T}$ is expected to be the MST scaling limit on any "high-dimensional" graph, e.g.

- random d-regular graphs
- the hypercube
- the lattice torus $(\mathbb{Z}/n\mathbb{Z})^d$ for $d > 8$

For low dimensions the picture is blurry.

$d=2$: The MST scaling limit (as an embedded object) exists and has dimension $\in (1+\varepsilon, 7/4)$. (Garban, Pete, Schramm 20??)

Numerics suggest dimension 1.22... no non-numerical predictions in physics literature.

$3 \leq d \leq 6$, Wide open.

$6 < d \leq 8$ Opinions vary.