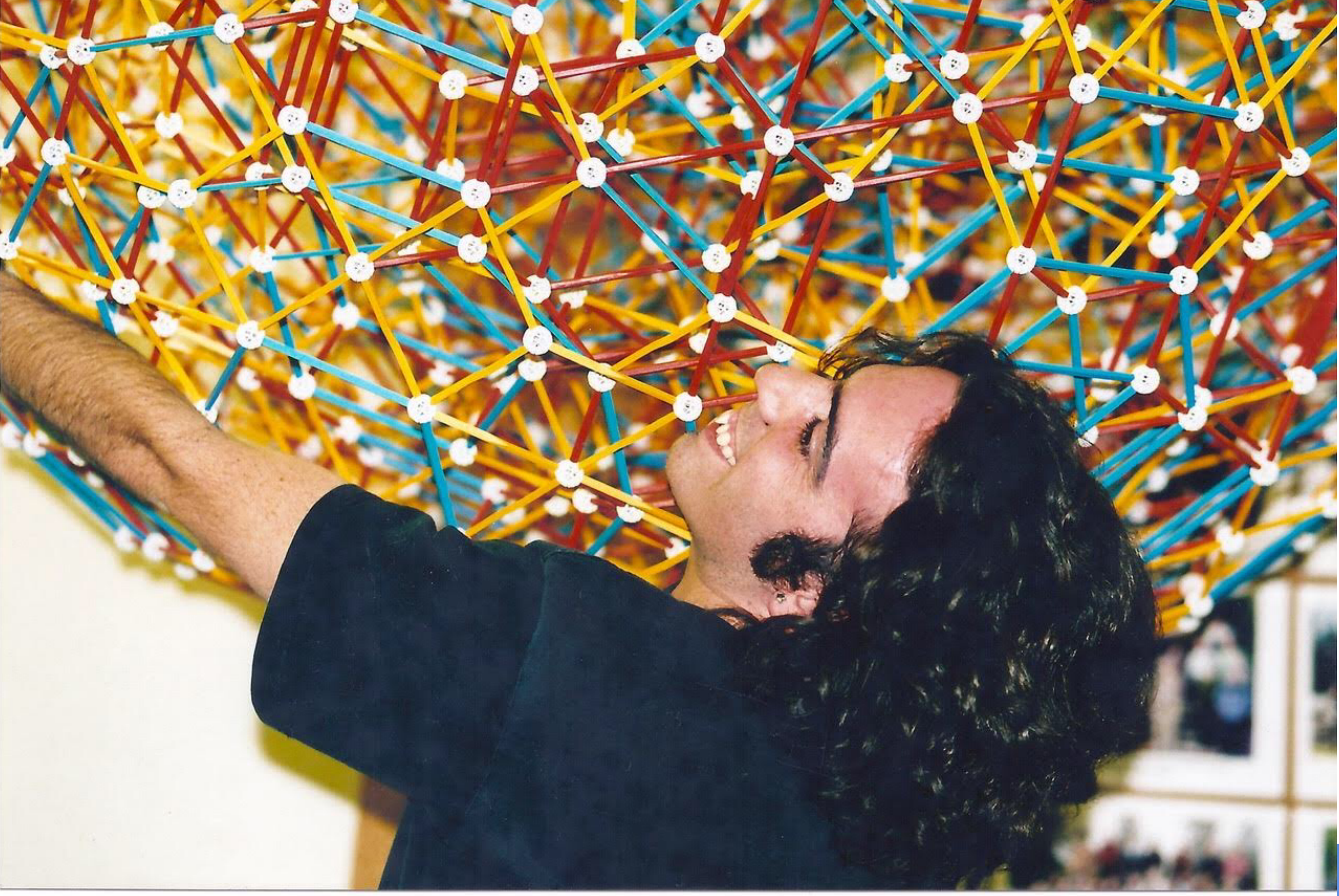

The algorithmic hardness threshold for the continuous random energy model

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Branching Structures
Zurich, Sep 3 2018

Branching random walk

Generation n: T_n (assume at least binary so $|T_n| \geq 2^n$)

Position of node v: X_v

Minimum position in generation n: M_n

Fairly generic fact: $\exists\, c \in \mathbb{R}$ s.t. $n^{-1} M_n \xrightarrow{a.s.} c$

and moreover $\mathbb{E} M_n = (1 + o(1))\, c n$

Fairly generic proof of a.s. conv.: a) lower bound:

$\forall\, c_- < c$, $\mathbb{E}\, \#\{v \in T_n : X_v \leq c_- n\} = O(e^{-\delta n})$, some $\delta = \delta(c_-) > 0$.

Then use Borel-Cantelli.

b) upper bound: Fix $c^+ > c$, then $\exists\, k \in \mathbb{N}$ s.t.

$\mathbb{E}\, \#\{v \in T_k : \forall j < k,\ \underbrace{X_{a(v,j)}}_{\text{gen. } j \text{ ancestor of } v} \leq c^+ k\} > 1$

Define a renormalized BRW

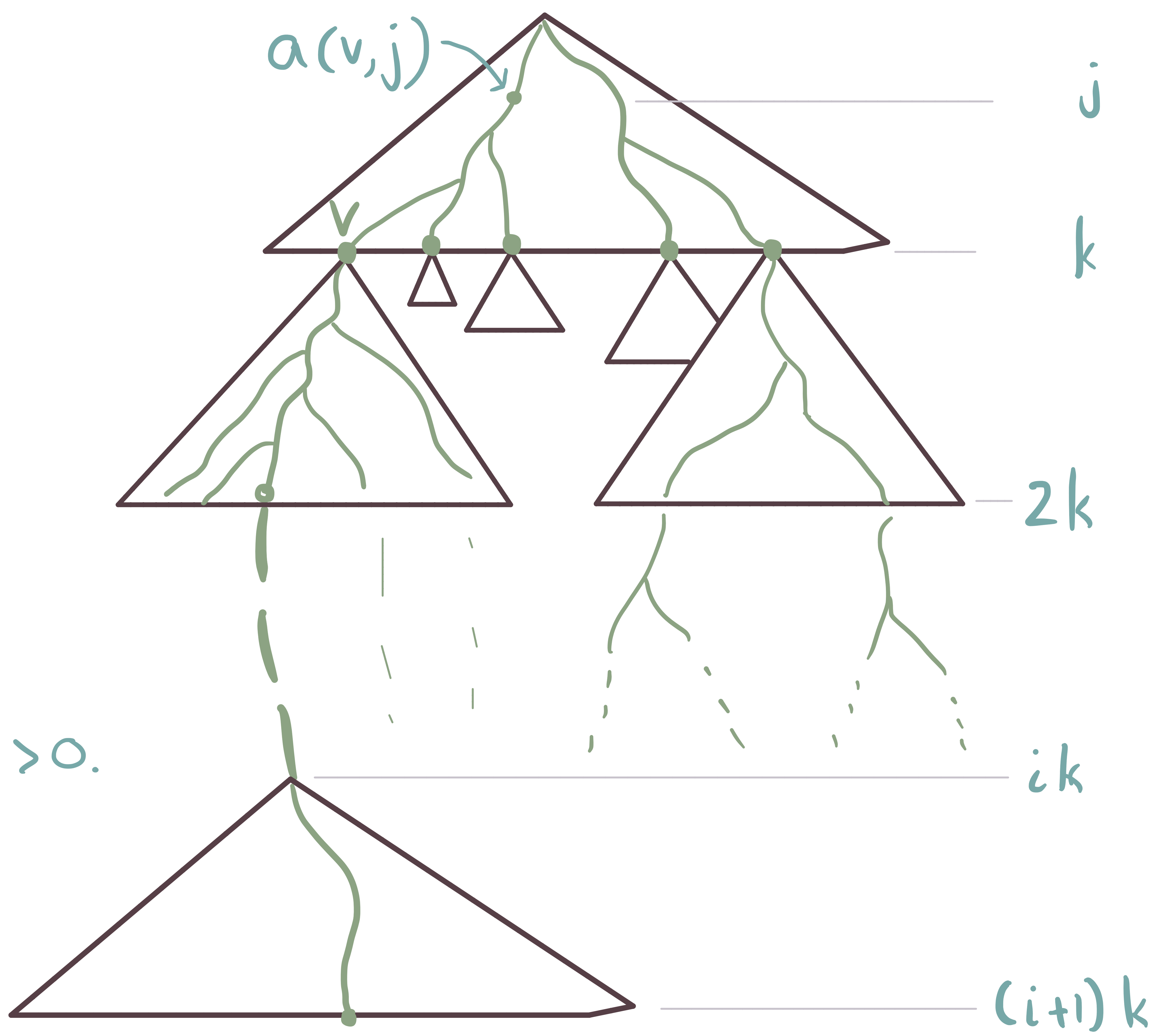
$T^{(1)} = \{v \in T_k : \forall j < k,\ X_{a(v,j)} \leq c^+ k\}$ $\quad \mathbb{E}\, \# T^{(1)} > 1$.

$T^{(i+1)} = \{v \in T_{(i+1)k} : \forall j \in [ik, (i+1)k],\ X_{a(v,j)} - X_{a(v,ik)} \leq c^+ k\}$

Then $\hat{T} := (T^{(i)}, i \geq 1)$ is a supercritical branching process

and on the event $\{\hat{T} \text{ survives}\}$, have $\limsup_{n\to\infty} n^{-1} M_n \leq c^+$.

So $\mathbb{P}\{\limsup_{n\to\infty} n^{-1} M_n \leq c^+\} \geq \mathbb{P}\{\hat{T} \text{ survives}\} =: p(c^+) > 0$.



Now amplify: fix m large.

For any $u \in T_m$, by the branching property,

$\mathbb{P}\{\limsup_{n\to\infty} n^{-1} \underbrace{M_n^{(u)}}_{\min\{X_v - X_u :\ v \in T_{n+m},\ a(v,m) = u\}} \leq c^+\} \geq p(c^+)$

And $\forall u \in T_m$, $\limsup_{n\to\infty} n^{-1} M_n^{(u)} \geq \limsup_{n\to\infty} M_n$, so

$\mathbb{P}\{\limsup_{n\to\infty} n^{-1} M_n \leq c^+\}$

$\geq \mathbb{P}\{\exists\, u \in T_m : \limsup_{n\to\infty} n^{-1} M_n^{(u)} \leq c^+\}$

$\geq 1 - (1 - p(c^+))^{|T_m|} \geq 1 - (1 - p(c^+))^{2^m}$

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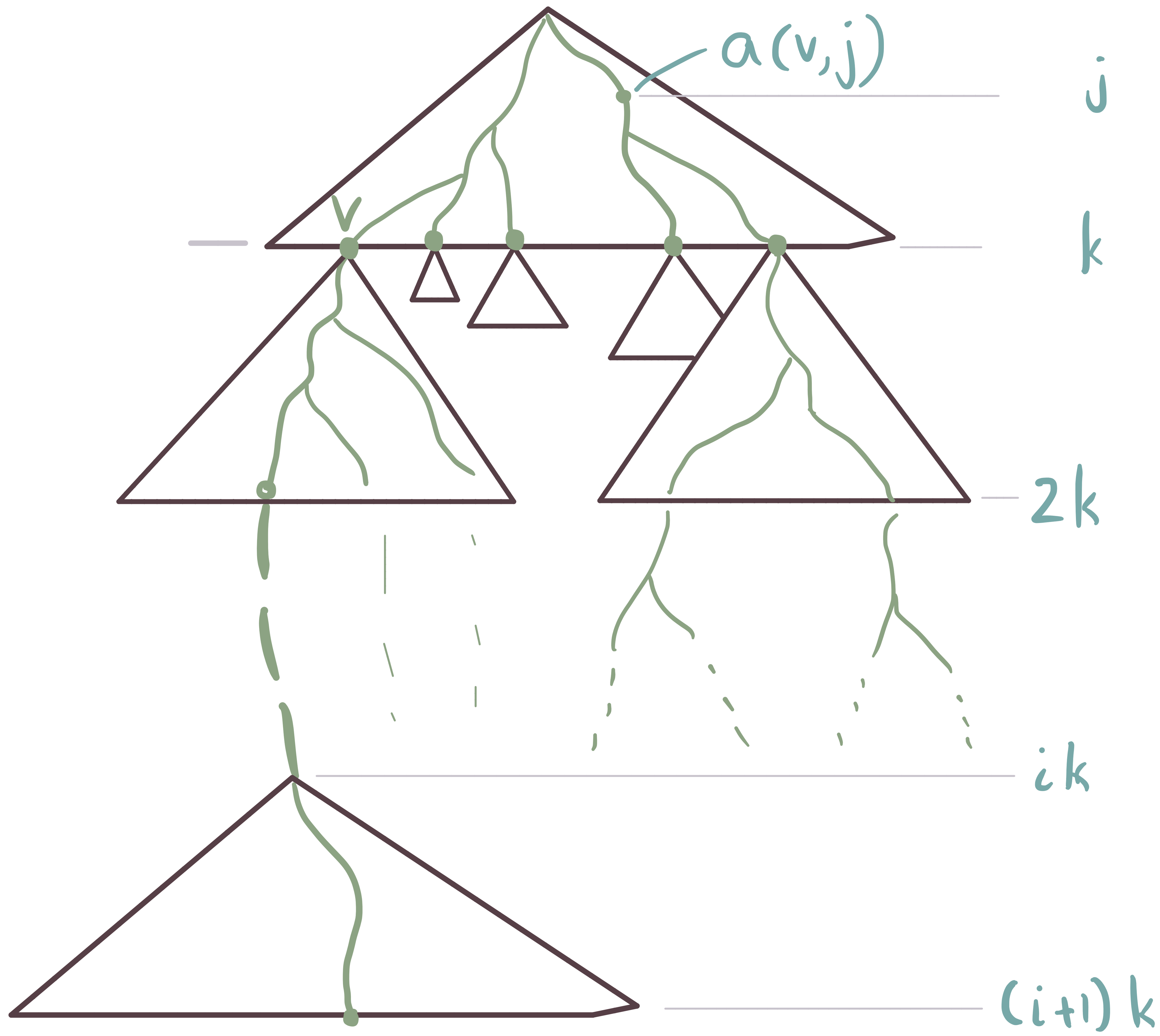
Fairly generic fact: $\exists\, c \in \mathbb{R}$ s.t. $n^{-1} \cdot M_n \xrightarrow{a.s.} c$

and moreover $\mathbb{E} M_n = (1+o(1))\, c\, n$

Remark: In fair generality, c is also the expectation threshold, in that

$$\lim_{n\to\infty} n^{-1} \log \mathbb{E}\, \#\{v \in T_n : X_v \leq x n\} \text{ is } \begin{cases} >0, & x > c \\ =0, & x = c \\ <0, & x < c \end{cases}$$



Finding near-minimal states

Hereafter assume binary branching, sub-Gaussian displacements:

$$\exists\, c, C>0 \text{ s.t. } \mathbb{P}(|X_v - X_{\text{parent}(v)}| > y) \leq C e^{-cy^2},$$

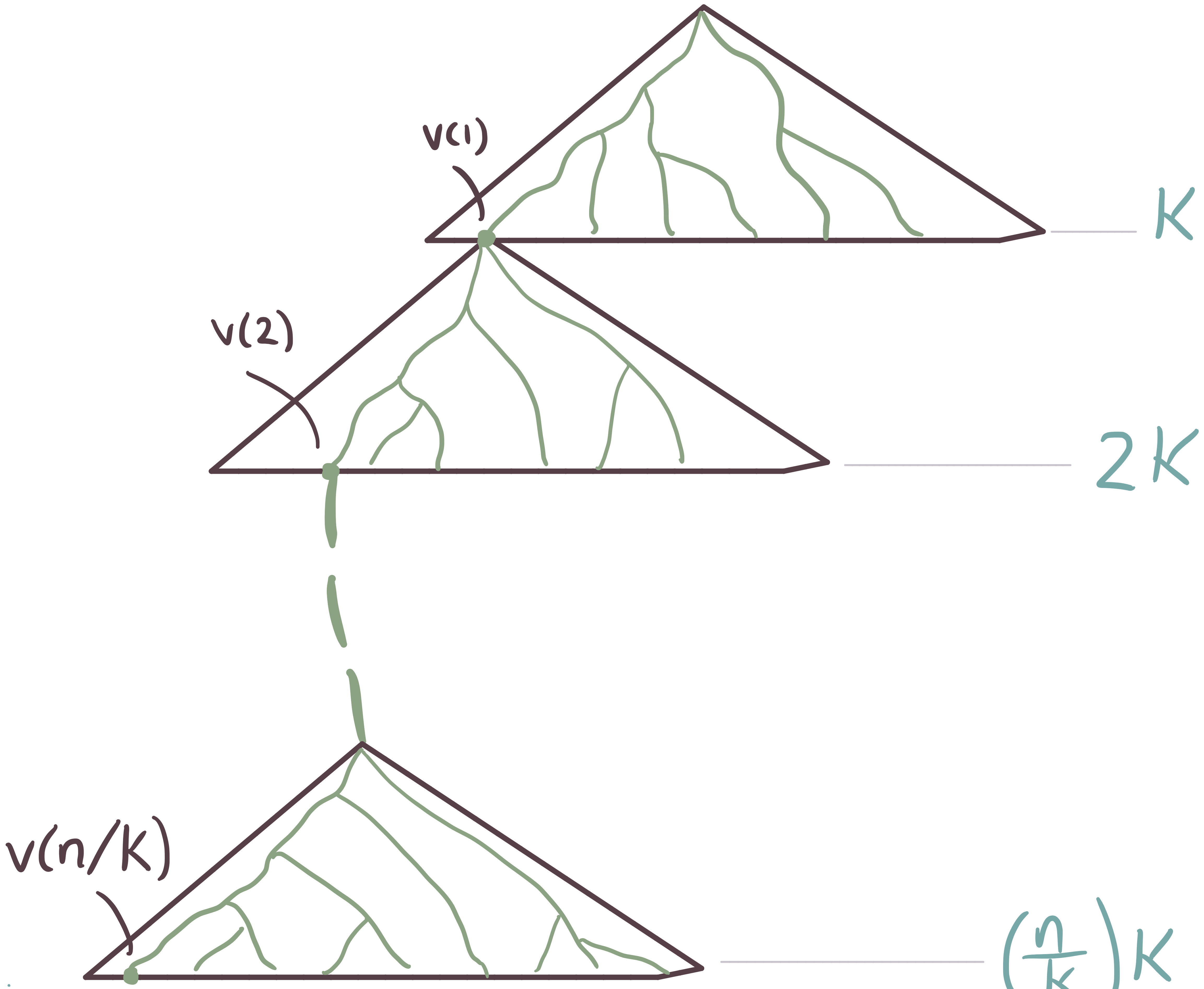
for all nodes v.

Write σ^2 for offspring variance.

How can one find nodes $v \in T_n$ with $X_v \approx cn$?

Bootstrap the law of large numbers:

- Given $\varepsilon > 0$, fix $K = K(\varepsilon)$ large enough that $\mathbb{E} M_K < (c+\varepsilon) K$
- Let $v(1) \in T_K$ have minimal position among depth-K nodes:
  $$X_{v(1)} = M_K$$
- For $j = 2, \ldots, n/K$, let $v(j) \in T_{jK}$ have minimal position among depth-jK descendants of $v(j-1)$:
  $$X_{v(j)} - X_{v(j-1)} = M_K^{(v(j-1))}$$
- Then $\mathbb{E} X_{v(n/K)} = \frac{n}{K} \mathbb{E} X_{v(1)} < (c+\varepsilon) n$



Follows that $X_{v(n/K)} \leq (c+2\varepsilon) n$ with probability $(1-o(1))$ as $n \to \infty$.

Claim $\operatorname{Var} X_{v(n/K)} \leq n \cdot 2^K \sigma^2$.

Proof: By the branching property,

$$\operatorname{Var} X_{v(n/K)} = \frac{n}{K} \operatorname{Var}(X_{v(1)}),$$

and

$$\begin{aligned} \operatorname{Var} X_{v(1)} &= \operatorname{Var} \min\{X_v : v \in T_K\} \\ &\leq \sum_{v \in T_K} \operatorname{Var} X_v \\ &= \sum_{v \in T_K} K\sigma^2 = 2^K \cdot K\sigma^2. \end{aligned}$$ $\square$

node value queries $= 2^K \cdot n/K = O_\varepsilon(n)$: linear-time algorithm.

CREM and its minima

Setting: continuous random energy model (CREM)
CREM(A,n)

A: Cumulative dist. f$^{\underline{n}}$ of a finite measure on $[0,1]$; so $A(0)=0$, $A(1)\in(0,\infty)$.
n: number of levels

Gaussian process $(X_v, v\in \mathbb{T}_n)$ indexed by
$\mathbb{T}_n := T_1 \cup T_2 \cup \cdots \cup T_n$

Displacement laws:
If $v\in T_k$ then $X_v - X_{\text{parent}(v)}$ is

$$\mathcal{N}\left(0, n\left(A\left(\tfrac{k}{n}\right)-A\left(\tfrac{k-1}{n}\right)\right)\right).$$

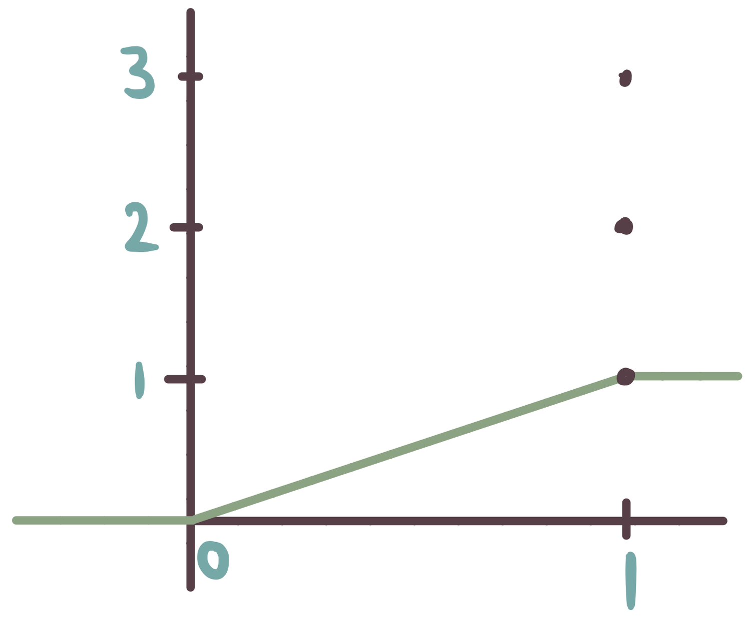
Displacements mutually independent.

Idea: along any root-to-leaf path, observe an inhomogeneous Brownian motion whose infinitesimal variance at time $\approx zn$ is $A'(z)$.

CREM and its minima : Examples

Standard (binary) Gaussian BRW.

$$A(z)=\begin{cases}0, & z\le 0\\ z, & z\in(0,1)\\ 1, & z\ge 1\end{cases}$$

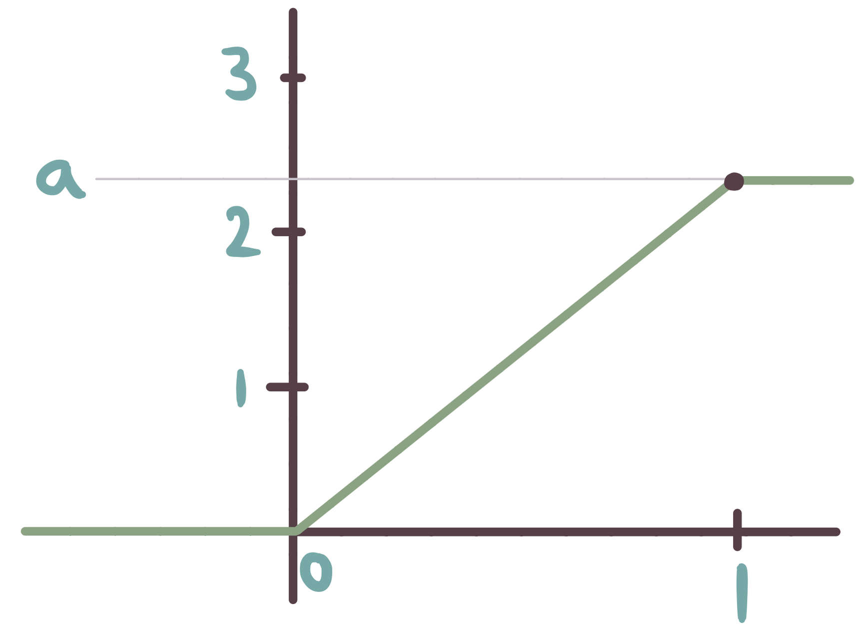


$$\mathbb{E}\,\#\{v\in T_n : X_v\le -xn\}\approx 2^n\exp\left(-\frac{x^2}{2n}\right)$$

$$n^{-1}M_n \xrightarrow{a.s.} -(2\log 2)^{\frac12}$$

Speed-a Gaussian BRW

$$A(z)=\begin{cases}0, & z\le 0\\ a, & z\in(0,1)\\ a, & z\ge 1\end{cases}$$

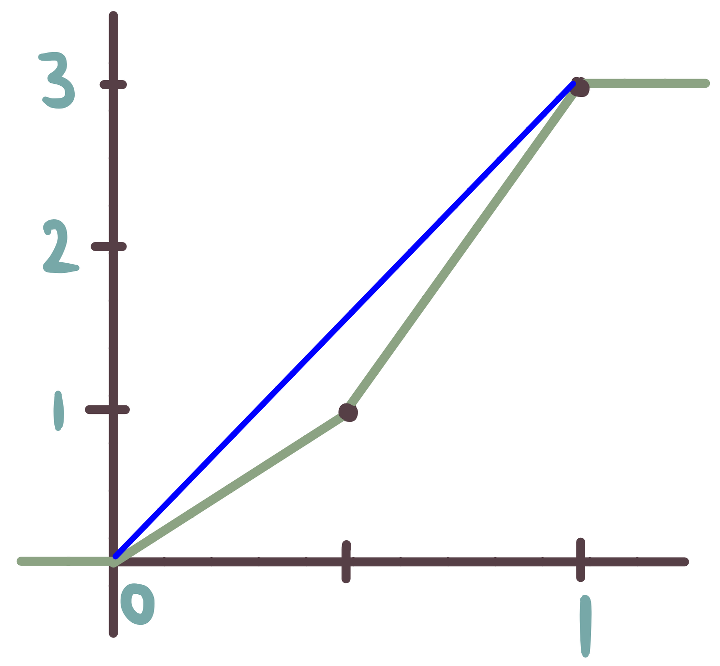


$$\mathbb{E}\,\#\{v\in T_n : X_v\le -xn\}\approx 2^n\exp\left(-\frac{x^2}{a\cdot 2n}\right)$$

$$n^{-1}M_n \xrightarrow{a.s.} -(a\cdot 2\log 2)^{\frac12}$$

Two-speed concave Gaussian BRW

$$A(z)=\begin{cases}0, & z\le 0\\ 2z & z\in(0,\tfrac12)\\ 1+4z, & z\in(\tfrac12,1)\\ 3, & z\ge 1\end{cases}$$



$$\mathbb{E}\,\#\{v\in T_{n/2} : X_v\le -x\tfrac{n}{2}\}\approx 2^{n/2}\exp\left(-\frac{x^2}{2\cdot n}\right)\quad [2n=2\cdot 2\cdot n/2]$$

$$\mathbb{E}\,\#\{v\in T_n : X_v-X_{a(v,n/2)}\le -y\tfrac{n}{2}\}\approx 2^{n/2}\exp\left(-\frac{y^2}{4n}\right)$$

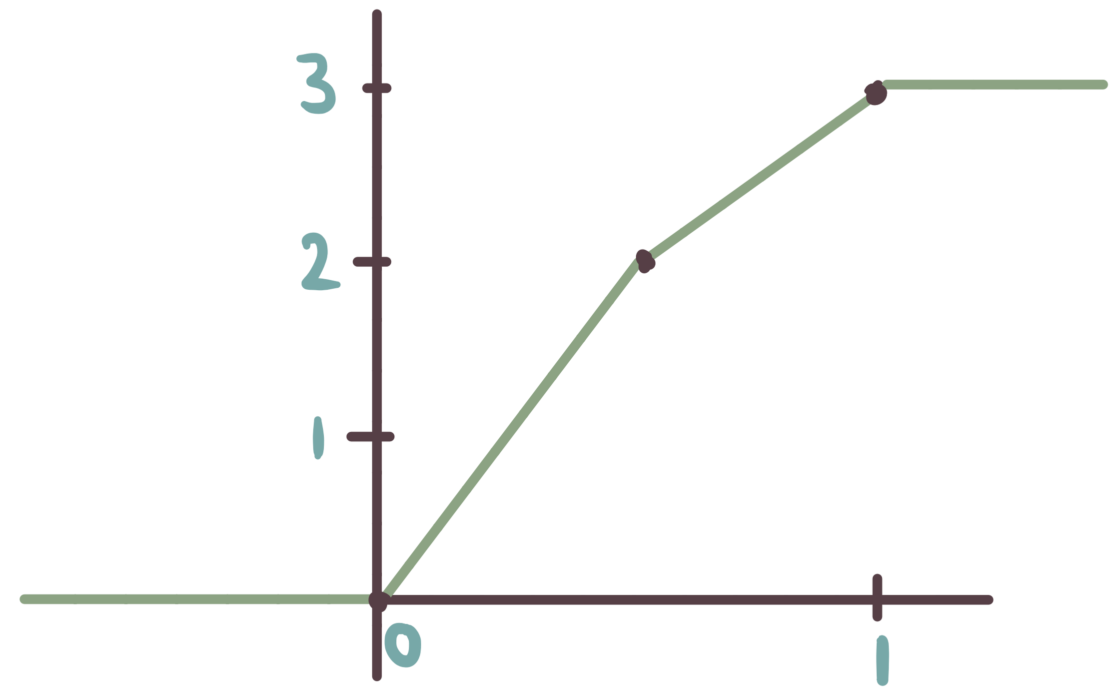
$$\mathbb{E}\,\#\left\{v\in T_n : X_{a(v,n/2)}\le -\left(\frac{2\cdot\log 2}{3}\right)^{\frac12}n,\ X_v-X_{a(v,n/2)}\le -2\left(\frac{2\log 2}{3}\right)^{\frac12}n\right\}$$

$$\asymp 2^n\exp\left(-\frac{\log 2}{3}n-\frac{2\log 2}{3}n\right)\asymp 1$$

$$n^{-1}M_n\to -\left(\frac{2\cdot\log 2}{3}\right)^{\frac12}n-2\left(\frac{2\log 2}{3}\right)^{\frac12}n=-(3\cdot 2\log 2)^{\frac12}$$

Two-speed convex Gaussian BRW.

$$A(z)=\begin{cases}0, & z\le 0\\ 4z, & z\in(0,\tfrac12)\\ 2+2z, & z\in(\tfrac12,1)\\ 3, & z\ge 1\end{cases}$$



$$\mathbb{E}\,\#\{v\in T_{n/2} : X_v\le -x\tfrac{n}{2}\}\approx 2^{n/2}\exp\left(-\frac{x^2}{4n}\right)$$

$$\mathbb{E}\,\#\{v\in T_n : X_v-X_{a(v,n/2)}\le -y\tfrac{n}{2}\}\approx 2^{n/2}\exp\left(-\frac{y^2}{2\cdot n}\right)$$

$$\mathbb{E}\,\#\left\{v\in T_n : X_{a(v,n/2)}\le -2\left(\frac{2\cdot\log 2}{3}\right)^{\frac12}n,\ X_v-X_{a(v,n/2)}\le -\left(\frac{2\log 2}{3}\right)^{\frac12}n\right\}$$

$$\asymp 2^n\exp\left(-\frac{2\log 2}{3}n-\frac{\log 2}{3}n\right)\asymp 1$$

But $n^{-1}M_n \not\to -(3\cdot 2\log 2)^{\frac12}$, because

$$\mathbb{E}\,\#\left\{v\in T_n : X_{a(v,n/2)}\le -2\left(\frac{2\cdot\log 2}{3}\right)^{\frac12}n\right\}\approx 2^{n/2}\exp\left(-\frac{2\log 2}{3}n\right)\approx 2^{-n/6};$$

the needed trajectories do not exist.

In fact, here $n^{-1}M_n\to(\sqrt{2}+1)\log 2$.

CREM: The minimum position

Proposition (Bovier-Kurkova; Mallein; LAB-Mallard)

Suppose A is absolutely continuous wrt Lebesgue measure, and has a Riemann-integrable derivative a. Let $\hat{A}$ be the concave hull of A, let $\hat{a}$ be the left-derivative of $\hat{A}$.

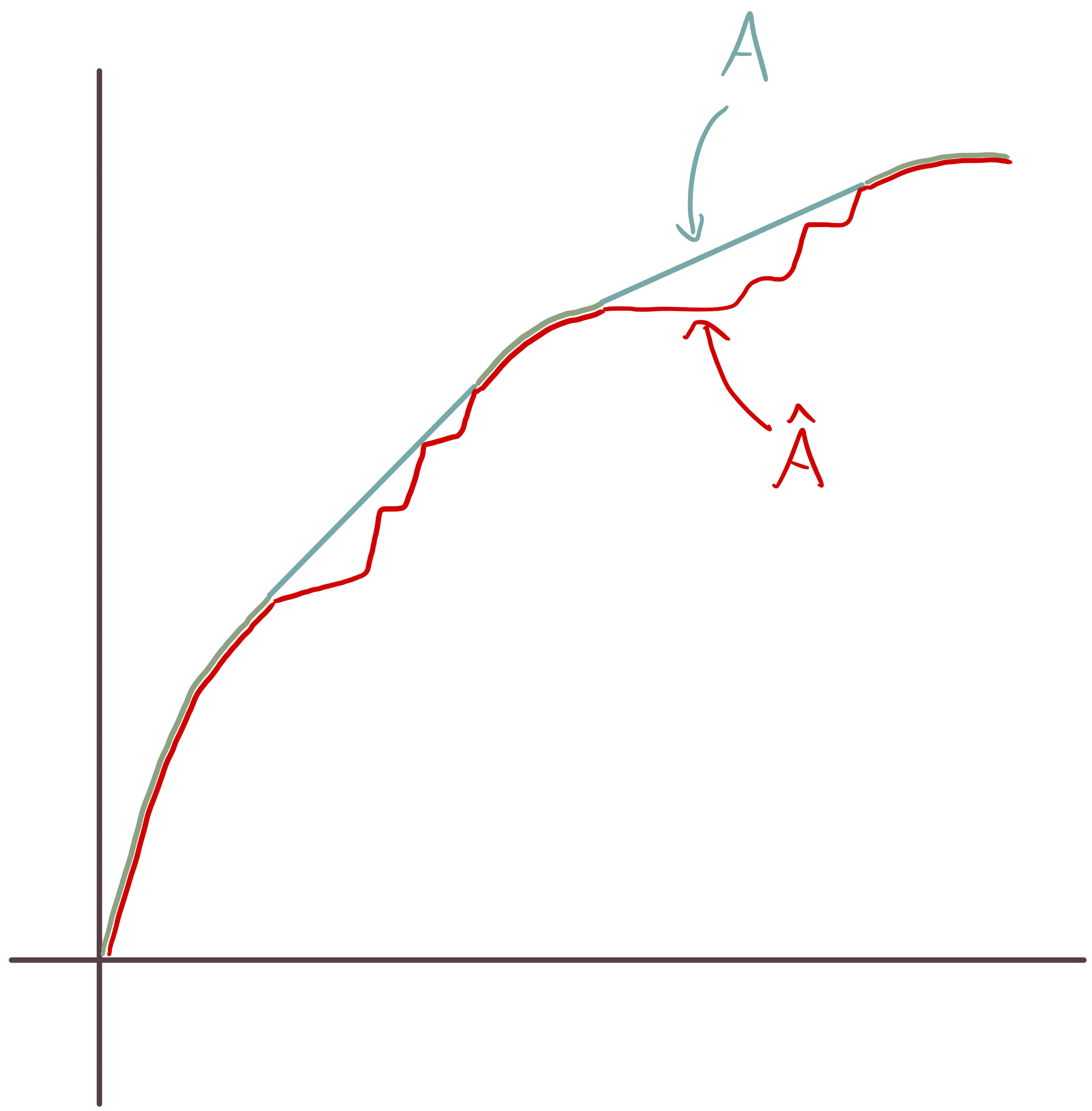
Then $n^{-1}M_n \xrightarrow{a.s.} \int_0^1 \sqrt{\hat{a}(t)}\,dt =: -c$

Moreover, $c = \sup\left\{\int_0^1 v(s)ds : v:[0,1]\to\mathbb{R} \text{ measurable}, \right.$

$$\left.\forall t\in(0,1],\ \int_0^t \frac{v(s)^2}{2a(s)}ds \leq t\log 2\right\}.$$

The supremum is attained via the f^n $v_{max}:[0,1]\to\mathbb{R}$ with

$$v_{max}(s) = a(s)\cdot\left(\frac{2\log 2}{\hat{a}(s)}\right)^{1/2}.$$



N.B.: The value c only depends on A through $\hat{A}$; but the trajectory v_{max} followed to reach $-cn$ within $\mathbb{T}$ depends sensitively on A.

CREM and its minima: The algorithmic barrier

Def (Mallein): The natural speed path for A is the function $Z(t) = Z_A(t) = \int_0^t (2\log 2)^{\frac{1}{2}} a(t)^{\frac{1}{2}} dt$

Theorem: (LAB, Maillard 2018+)

If A abs. continuous, $a = A'$ Riemann-integrable, then with $z_* = Z_A(1)$, we have:

1. For all $x < z_*$, there is a linear-time algorithm that finds $v \in T_n$ with $X_v \leq -xn$ with high probability.
2. For all $x > z_*$, there is $\gamma = \gamma(A, x) > 0$ s.t. for n large, for any algorithm, the expected # of queries before finding a node $v \in T_n$ with $X_v \leq -xn$ stochastically dominates a Geometric$(\exp(-\gamma n))$ random variable.

Proof Idea:

1) In the inhomogeneous setting, the renormalization search follows the natural speed path.

2) For every node $v \in T_n$ with $X_v \leq -xn$, there is a linear-size subsection of the ancestral trajectory of v along which the slope is unnatural (different from the slope of the natural speed path).

The branching property + Gaussian tail estimates
$\Rightarrow$ exponentially unlikely to find such a segment on any single query, even conditionally given past queries. $\square$