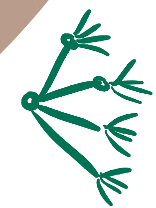
Critical first-passage percolation

Part 1: background and behaviour on regular trees



Joint work w/ Jack Hanson

OWPS

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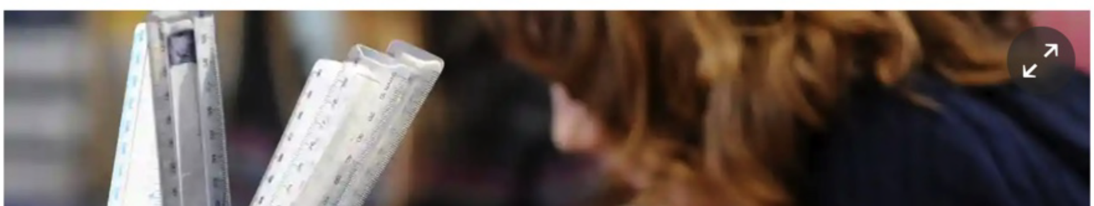

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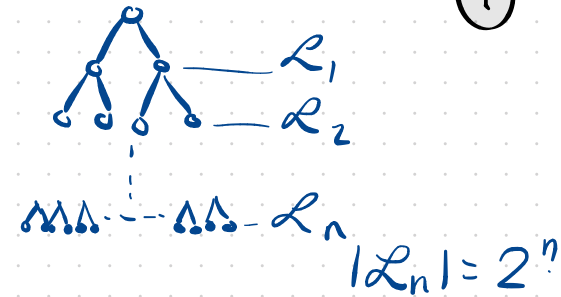
Contributing equally to your domestic life is far more important than hearing me talk about critical first-passage percolation on trees.

Feel free to disconnect and go take some family time.

Branching processes (percolation on trees)

Let B be the infinite rooted binary tree



For $p \in (0,1)$, form a subgraph B_p of B as follows:

Let $(X(e),\ e \in e(T))$ IID $X(e) = \begin{cases} 0 & \text{w.p. } p \\ 1 & \text{w.prob. } 1-p \end{cases}$

Let B_p have edges $(e \in e(T) : X(e) = 0)$.

Write $\mathcal{C}(v) = \mathcal{C}(v, B_p) \equiv$ subtree of B_p rooted at v.

("Cluster" of v in B_p)

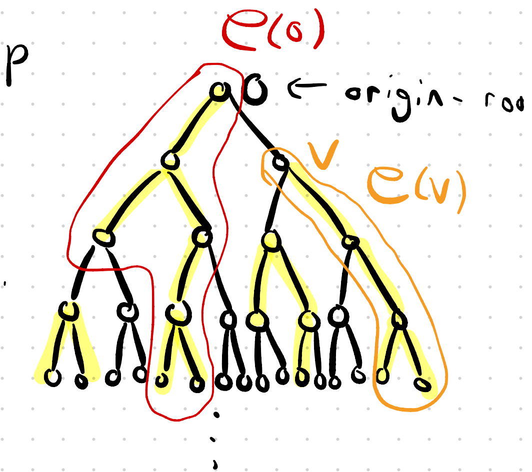


Figure 1

Define $\theta(p) = \theta(p, T) : \mathbb{P}(|\mathcal{C}(v, B_p)| = \infty) = \mathbb{P}(0 \xleftrightarrow{B_p} \infty)$

Theorem (Fund. thm of branching processes): $\Theta(p)=0 \iff p \leq \frac{1}{2}$. (2)

More strongly,

$$p > \tfrac{1}{2} \implies \mathbb{P}(|\mathcal{C}(v, B_p)| = \infty) > 0$$

$$p < \tfrac{1}{2} \implies \mathbb{P}(|\mathcal{C}(v, B_p)| > n) \asymp e^{-c_p \cdot n}$$

$$p = \tfrac{1}{2} \implies \mathbb{P}(|\mathcal{C}(v, B_p)| > n) \asymp \frac{1}{n^{1/2}}$$ "cluster size exponent" $\delta = \frac{1}{2}$

and

$$\mathbb{P}(0 \xleftrightarrow{B_p} \mathcal{L}_n) = \mathbb{P}(\mathcal{C}(v, B_p) \cap \mathcal{L}_n \neq \emptyset) \asymp \frac{1}{n}$$

"diameter exponent" $\rho = 1$.

First-passage percolation on trees

Let $T(x,y) := \sum_{e \text{ on } x,y \text{ path}} X_e$

$= \#$ ones on $x-y$ path

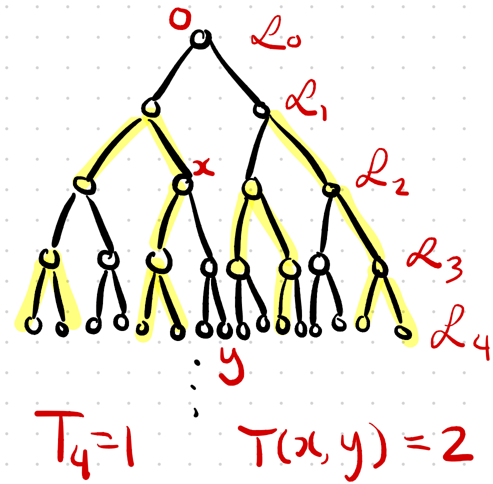


Figure 2

$T_n := \min(T(0,v), v \in \mathcal{L}_n)$

Phase transition in behaviour of T_n at $p = \frac{1}{2}$

$p > \frac{1}{2}$ If $p > \frac{1}{2}$ then $\mathbb{P}(0 \xleftrightarrow{B_p} \infty) > 0$ so

$$\inf_n \mathbb{P}(T_n = 0) = \inf_n \mathbb{P}(0 \xleftrightarrow{B_p} \mathcal{L}_n) > 0.$$

In this case $T_n \xrightarrow{a.s.} T_\infty \overset{a.s.}{<} \infty$

$p<\frac{1}{2}$ First moment method predicts first-passage times to first order: 4

For $c \in (0,1)$ approximate

$$\circledast = \mathbb{E}[\#\{v \in \mathcal{L}_n : T(0,v) = \lfloor cn \rfloor\}] = 2^n \,\mathbb{P}(\mathrm{Bin}(n,1-p) = \lfloor cn \rfloor)$$

$$\cong 2^n \binom{n}{\lfloor cn \rfloor} (1-p)^{cn}\, p^{(1-c)n}$$

$$\cong \exp\Big(n\Big(\log 2 + \underbrace{c\log\tfrac{1}{c} + (1-c)\log\tfrac{1}{1-c} + c\log(1-p) + (1-c)\log p}_{x(p,c)}\Big)\Big)$$

When $p<\frac{1}{2}$:

for c near 1, $\log 2 + x(p,c) > 0$ and $\circledast$ grows exp in n,

for c near 0 $\log 2 + x(p,c) < 0$ and $\circledast$ decays exp. in n.

Let $c^* = \inf\{c > 0 : \log 2 + x(p,c) > 0\}$

$$= \inf\{c>0 : \mathbb{E}[\#\{v \in \mathcal{L}_n : T(o,v) \le cn\}] \to \infty \text{ as } n \to \infty\}.$$

NB $c^* > 0$ iff $p < \frac{1}{2}$

Theorem (Hammersley–Kingman–Biggins, 1970's)

$$\frac{T_n}{n} \to c^* \text{ almost surely as } n \to \infty.$$

In fact, there is $c^{**} = c^{**}(p) > 0$ s.t.

$$T_n - (c^* n - c^{**} \log n) \xrightarrow{\text{dist}} T_\infty \text{ for some a.s. finite r.v. } T_\infty$$

(Bramson–Zeitouni, Addario-Berry–Reed, Aïdekon, ...)

Zhan Shi's book on branching random walks:

https://www.lpsm.paris/pageperso/zhan/brw.html

$p=\frac{1}{2}$ Not hard to see $\frac{T_n}{n} \xrightarrow{a.s.} 0$. Can we say anything finer?

Thm (Bramson 1978, Dekking & Host, 1991) $\frac{T_n}{\log_2 \log n} \xrightarrow{a.s.} 1$ (in fact they show $T_n - \log_2 \log n = O_p(1)$)

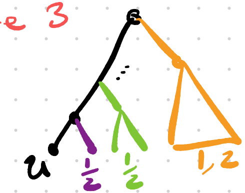
Proof idea:

Step 1 First moment. Let $P_n = \{v \in \mathcal{L}_n : T(0,v) = 0\} = \mathcal{C}(0) \cap \mathcal{L}_n$

$$\mathbb{E}\,|P_n| = 2^n \cdot \mathbb{P}(T(0,v) = 0) = 2^n \cdot 2^{-n} = 1.$$

any fixed $v \in \mathcal{L}_n$

Figure 3



Step 2 Second moment. For any $u \in \mathcal{L}_n$, $\mathbb{E}[|P_n| \mid u \in P_n] = 1 + \frac{n-1}{2} = \frac{n+1}{2}$

so
$$\mathbb{E}[|P_n|^2] = \sum_{u,v \in \mathcal{L}_n} \mathbb{P}(T(0,u)=0, T(0,v)=0) = \sum_{u \in \mathcal{L}_n} 2^{-n} \cdot \mathbb{E}[|P_n| \mid u \in \mathcal{L}_n]$$
$$= \sum_{u \in \mathcal{L}_n} 2^{-n} \cdot \frac{n+1}{2} = \frac{n+1}{2}.$$

NB: Also $\mathbb{E}[\,|P_m| \mid u \in P_n] = \frac{m+1}{2}$

so $$\mathbb{E}[\,|P_{\leq n}| \mid u \in P_n] = \sum_{m=0}^{n} \frac{m+1}{2} = \frac{1}{2}\binom{n+2}{2},$$

which leads to

Theorem (Kolmogorov):

$$\mathbb{P}(T(o,v)=0) = \mathbb{P}(o \xleftrightarrow{B_p} n) = \frac{2+o(1)}{n} \qquad \rho = 1$$

and $$\mathbb{P}(|\mathcal{C}(o,B_p)| \geq n^2) \asymp \frac{1}{n}. \qquad \delta = \frac{1}{2}$$

Step 3 Amplify.

v_k = ancestor of v in $\mathcal{L}_k$.

For $0 \leq k \leq n$, let $P_n(k) = \{ v \in \mathcal{L}_n \;\; T(v_k, v) = 0 \}$

$=$ # nodes in $\mathcal{L}_n$ with a zero-path to level k.

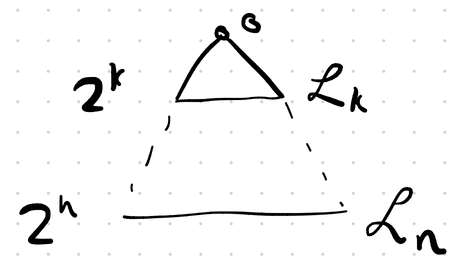


Figure 4

$$\mathbb{E}|P_n(k)| = 2^k \quad \text{(one in each tree)}$$

$$\mathbb{E}\left[|P_n(k)|^2\right] = \sum_{u,v \in \mathcal{L}_n} \mathbb{P}\left(u \in P_n(k),\, v \in P_n(k)\right)$$

$$= \underbrace{\sum_{u \in \mathcal{L}_n}}_{2^n} \frac{1}{2^{n-k}} \cdot \Big(\underbrace{\sum_{v:\, u_k = v_k} \mathbb{P}(v \in P_n(k) \mid u \in P_n(k))}_{\frac{(n-k)+1}{2}} + \underbrace{\sum_{v:\, v_k \neq u_k} \mathbb{P}(v \in P_n(k) \mid u \in P_n(k))}_{2^k - 1} \Big)$$

$$= 2^k \left(\frac{(n-k)+1}{2} + 2^k - 1 \right) = \left(\mathbb{E}\, |P_n(k)| \right)^2 + 2^k \left(\frac{n-k-3}{2} \right)$$

So $\mathrm{Var}(|P_n(k)|) = \begin{cases} O((\mathbb{E}|P_n(k)|)^2) & \text{if } n = O(2^k) \\ o(\mathbb{E}|P_n(k)|^2) & \text{if } n = o(2^k) \end{cases}$ $\quad \log n \le k + O(1)$, $\quad k - \log n \gg 1$

Chebyshev gives that $\mathbb{P}\left(|P_n(k)| / \mathbb{E}|P_n(k)| \in (\tfrac{1}{2}, 2) \right) > \delta > 0$ whenever $n = O(2^k)$

$|P_n(k)| / \mathbb{E}|P_n(k)| \xrightarrow{\text{prob}} 1$ whenever $n = o(2^k)$

Corollary: $\mathbb{P}(T(0,n) \le (1+\varepsilon)\log n) \ge \mathbb{P}(|P_n((1+\varepsilon)\log n)| > 0) \xrightarrow{P} 1$ so $T(0,n) \le (1 + o_p(1)) \log n$.

Step 4 Iterate. Step 3 generalizes as follows.

Fix a set of m incomparable nodes $\mathcal{U} = \{u_1, \ldots, u_m\}$ in B,

For $n \geq 0$ let

$$P_n(\mathcal{U}) = \{ v \text{ a } n\text{'th gen. desc. of some } u \in \mathcal{U},\ T(u,v) = 0 \}$$

then

for $n = O(m)$, $\mathbb{P}(\, |P_n(\mathcal{U})| \in (\frac{m}{2}, 2m)\,) > \delta > 0.$

for $n = o(m)$, $|P_n(\mathcal{U})| = (1 + o_p(1))\, m$

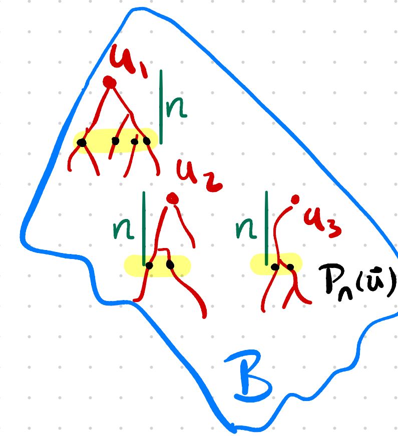


Figure 5

$$\Rightarrow \quad P_{\leq m}(\mathcal{U}) := \bigcup_{n=1}^{m} P_n(\mathcal{U}) \text{ has size } \Theta_p(1) \cdot m^2.$$

Now use the external boundary

$$\partial(P_{\leq m}(\mathcal{U})) = \{ v \notin P_{\leq m}(\mathcal{U}) : \text{parent}(v) \in P_{\leq m}(\mathcal{U}) \}$$

as a new "starting set" $\mathcal{U}'$. In the binary tree $|\partial S| \simeq |S|$, so $|\mathcal{U}'| = \Theta_p(m^2)$.

This gives

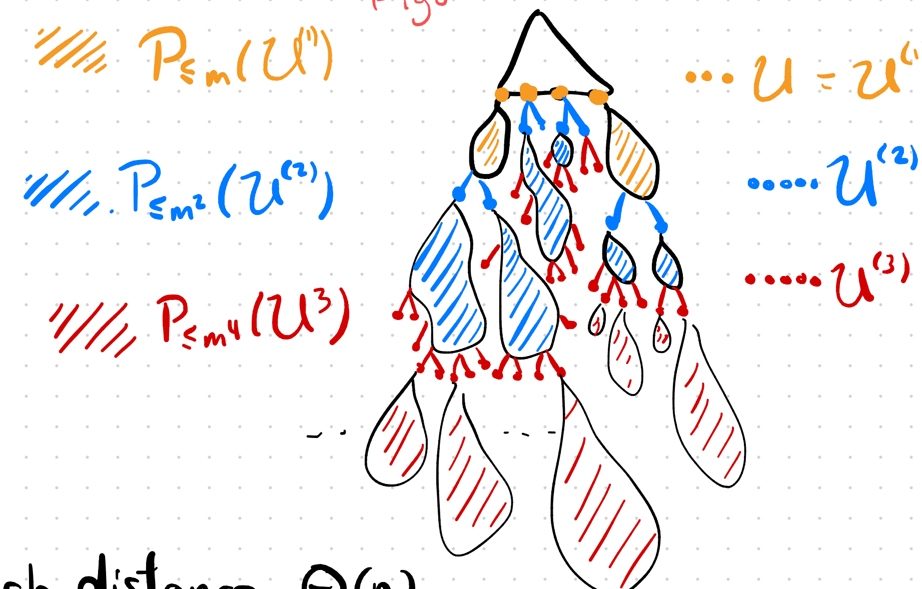
$$|P_{\le m^2}(\mathcal{U}')| = \Theta_p(1)\cdot|\mathcal{U}'|^2 = \Theta_p(1)\cdot m^4.$$

Repeat this sequentially:

$$\mathcal{U}'' := \partial\big(P_{\le m^2}(\mathcal{U}')\big)$$

$$|P_{\le m^4}(\mathcal{U}'')| = \Theta_p(m^8) \ldots$$

Figure 6



After k iterations, have a set of size $\asymp m^{2^{k+1}}$, reaching distance m^{2^k} in the tree.

After loglog$n$ iterations we reach distance $\Theta(n)$. ($2^{2^k} = n$ when $k = \log\log n + o(1)$)

This gives the upper bound of the Dekking-Host thm.

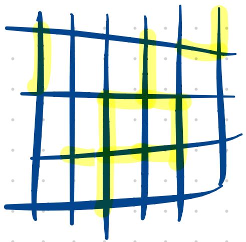
For a matching lower bound, run the same argument but using full percolation clusters instead of "height-truncated" clusters; vol. and diam. growth behave same way. $\square$

① Percolation phase transitions.

$\mathbb{Z}^d$

(d=2)

— -> in $\mathbb{Z}^d_p$



Fix $p\in(0,1)$, keep each edge indep. with prob. p.

This gives a random subgraph $\mathbb{Z}^d_p$

Q: Does $\mathbb{Z}^d_p$ have an ∞ connected component.

Let $\underbrace{\Theta(p)}_{\Theta(p,\,\mathbb{Z}^d)} := \mathbb{P}_p(0 \longleftrightarrow \infty) := \mathbb{P}($in $\mathbb{Z}^d_p$, the cluster containing the origin is infinite$)$

$\underbrace{p_c}_{p_c(\mathbb{Z}^d)} := \inf\{p : \Theta(p)>0\}$ (or just $p_c(d)$)

Fact: $\Theta(p)>0 \iff \mathbb{P}_p(\exists$ an infinite cluster$) = 1$.

Thm $\Theta(p_c) = 0$ when $\begin{cases} d = 2 & ??? \\ d \geq 11 & \text{Fitzner + van der Hofstad} \\ d \geq 6 & \text{Hara - Slade} \end{cases}$

(spread out models)

(Time-permitting: mention cluster size exponents in mean field setting:

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \asymp \frac{1}{n^{1/2}}$$

$$\mathbb{P}_{p_c}(\mathrm{diam}_{\mathrm{int}} |\mathcal{C}(0)| \geq n) \asymp \frac{1}{n}.$$