Probabilistic aspects of minimum spanning trees

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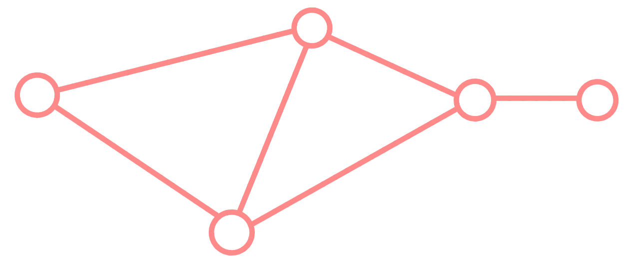
I. The MST problem



$G=(V,E)$ finite connected graph.

A spanning forest is a subgraph $F=(V,E')$, $E'\subseteq E$, with no cycles. It is a spanning tree if it is also connected.

For fixed positive weights $(U_e)_{e\in E}$, the weight of a spanning tree T of G is $\omega(T)=\sum_{e\in E(T)} U_e$

Say T is the MST of G if

$$\omega(T)=\min\{\omega(T') : T' \text{ a spanning tree of } G\}$$

If weights U_e are all distinct then MST is unique.

II. MST Algorithms.

Greedy: Kruskal's algorithm

Order $E = \{e_1, e_2, \dots, e_m\}$ s.t. $U_{e_1} < U_{e_2} < \cdots < U_{e_m}$

Set $F_0 = (V, \phi)$; empty forest.

Aim: Maintain that $F^i = (V, E^i)$ is the minimum spanning forest of $G^i = (V, \{e_1, \dots, e_i\})$

componentwise MST

Let $E^{i+1} \begin{cases} E^i \cup \{e_{i+1}\} & \text{if this creates no cycle} \\ E^i & \text{otherwise.} \end{cases}$

Output: $F^m = (V, E^m)$ which is the minimum spanning tree.

[Aside: Kruskal's algorithm finds a min-weight basis for any matroid, and this characterizes matroids.]

II. MST Algorithms.

Local: Prim's algorithm

Fix a starting vertex (a root) v.

Set $T_1 = (\{v\}, \phi)$; a tree with 1 vertex.

Aim: Maintain that $T_i = (V_i, E_i)$ is the minimum spanning tree of $G|_{V_i} = (V_i, \{uv \in E : u \in V_i, v \in V_i\})$

For $1 \leq i < |V|$, let $e_{i+1} = uv$ be the smallest weight edge from V_i to $V \setminus V_i$.

Let $V_{i+1} = V_i \cup \{u, v\}$, $E_{i+1} = E_i \cup \{e_{i+1}\}$.

Then $T_{|V|}$ is the MST.

Interesting hard question: when do local algorithms yield (near)-optimal outputs?

III. Randomization.

Average-case analysis: understand behaviour of an algorithm when averaged over all possible inputs.

[Smoothed analysis: understand behaviour on random perturbation of input data.]

Algorithmic complexity

$\neq$

Average-case complexity

$\neq$

Probabilistic difficulty

MST is algorithmically easy but probabilistically challenging.

III. Randomization.

Mean-field

Graph = K_n = complete graph, n vertices

Edge weights: independent, identically distributed, Uniform $[0,1]$

[Kruskal's algorithm only uses edge ordering; so any IID continuous weights give same law for MST viewed as an unweighted graph.]

Theorem (Frieze, 1985):

Let $T_n = \mathrm{MST}(K_n)$.

Then $w(T_n) \xrightarrow{P} \zeta(3)$.

Euclidean

$\mathbb{Z}^d$, nearest-neighbour edges.

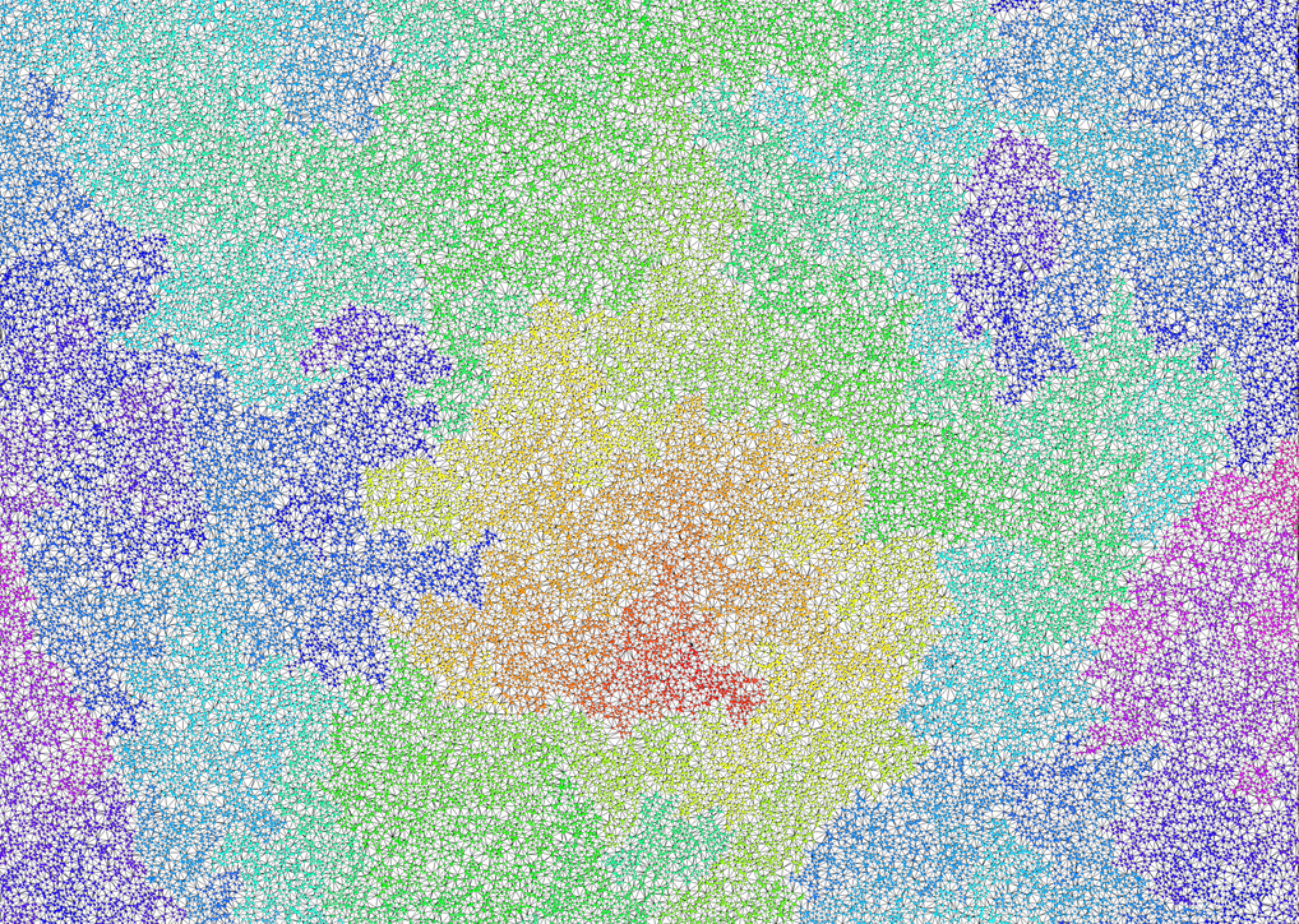
Edge weights: independent, identically distributed, Uniform $[0,1]$

[On ∞ graphs some care is needed: Kruskal not obviously well-defined; Prim well-defined but T_∞ may not reach all vertices]

"Invasion Percolation"

Theorem (Newman, 1995):

T_∞ has 0 density $\Longleftrightarrow$ No percolation at criticality.

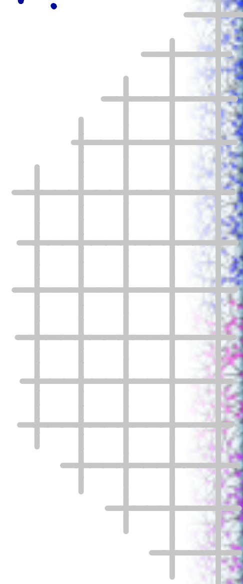


IV. Percolation

Fix connected graph $G=(V,E)$, IID Uniform$[0,1]$ edge weights $(U_e, e\in E)$.

Set $E_p = \{e\in E : U_e \le p\}$, $G_p = (V, E_p)$.

Assume G transitive: $\forall\, u,v\in V\ \exists$ an automorphism φ of G with $\varphi(u)=v$.



Percolation threshold (for G infinite):

$$p_c(G) = \inf\{p\in[0,1] : \mathbb{P}(G_p \text{ has an infinite connected component}) > 0\}$$

$p_c(\mathbb{Z}) = 1$; $p_c(\mathbb{Z}^2) = \frac{1}{2}$; $p_c(\mathbb{Z}^d) = \frac{1}{2d} + O\left(\frac{1}{d^2}\right)$, d large.

For $v\in V$ write $v \xleftrightarrow{G_p} \infty$ if v in an infinite component of G_p.

Let $\Theta_p(G) = \mathbb{P}(o \xleftrightarrow{G_p} \infty)$. Doesn't depend on root/origin $o\in V$.

Definition: G percolates at criticality if $\Theta_{p_c}(G) > 0$.

Conjecture: No percolation at criticality in $\mathbb{Z}^d$, all $d\ge 2$. Known for $d=2, d>10$.

V. Percolation on finite graphs.

The percolation process $(G_p, 0 \le p \le 1)$ still makes sense but all components are finite for every p.

Many possible definitions of percolation threshold.

One possibility which "works" for many geometries:

Order components of G_p in decreasing order of size as $(C_p(i), i \ge 1)$.

Let $p_c = \inf\{p : \mathbb{E}[|C_p(2)|/|C_p(1)|] \le 1/2\}$.

(Intuition: Once giant component appears, all other components much smaller.)

<u>Mean-field</u>: $G = K_n$. Then percolation on G gives the Erdős-Rényi random graph process, typically denoted $(G_{n,p}, 0 \le p \le 1)$.

VI. Percolation and MSTs.

Assume $G=(V,E)$ finite.

IID Uniform$[0,1]$ edge weights $(U_e, e\in E)$ now and forever after.

Recall With $E=\{e_1,e_2,\dots,e_m\}$ ordered s.t. $U_{e_1}<U_{e_2}<\cdots<U_{e_m}$,

Kruskal's algorithm maintains that $F^i=(V,E^i)$ is the minimum spanning forest of $G^i=(V,\{e_1,\dots,e_i\})$; $F^m=(V,E^m)$ is MST.

With $p_i=U_{e_i}$, then $G^i=G_{p_i}$ for all i

Set $E_p^m=\{e\in E_m : U_e\le p\}$, $F_p=(V,E_p^m)$.

Then the vertex sets of connected components of F_p and G_p are identical.

VII Dimension and volume growth

Distances determined by major routes, so known by time p_c

Heuristic [Percolation threshold p_c
=
point at which long-range correlations / global structure appear.

At times $p \neq p_c$, all finite/nongiant components are small:

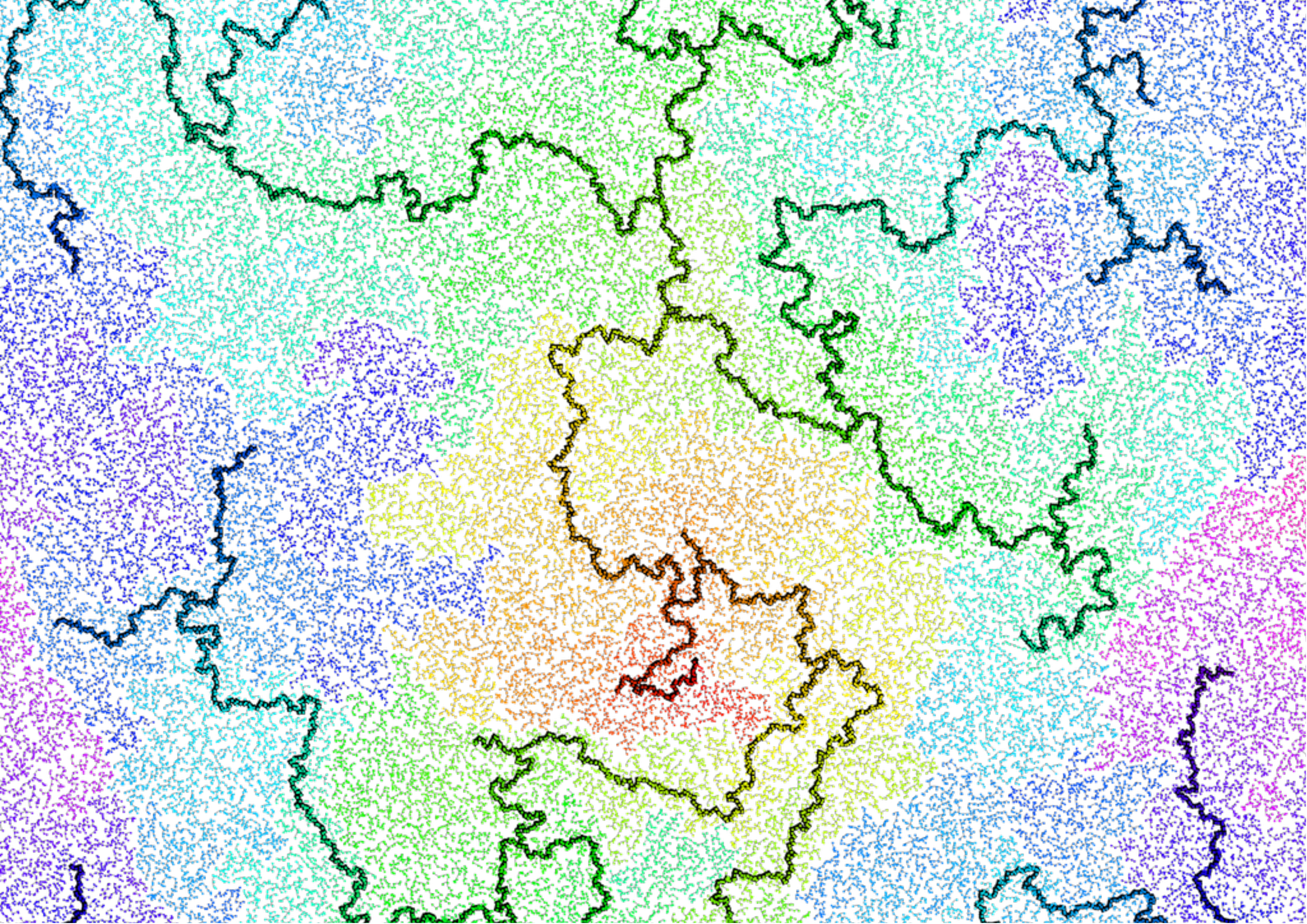
$$\mathbb{E}[\text{Size of component } C \text{ of } G_p \text{ containing } v \mid C \neq C_p(1)] = O(1)$$

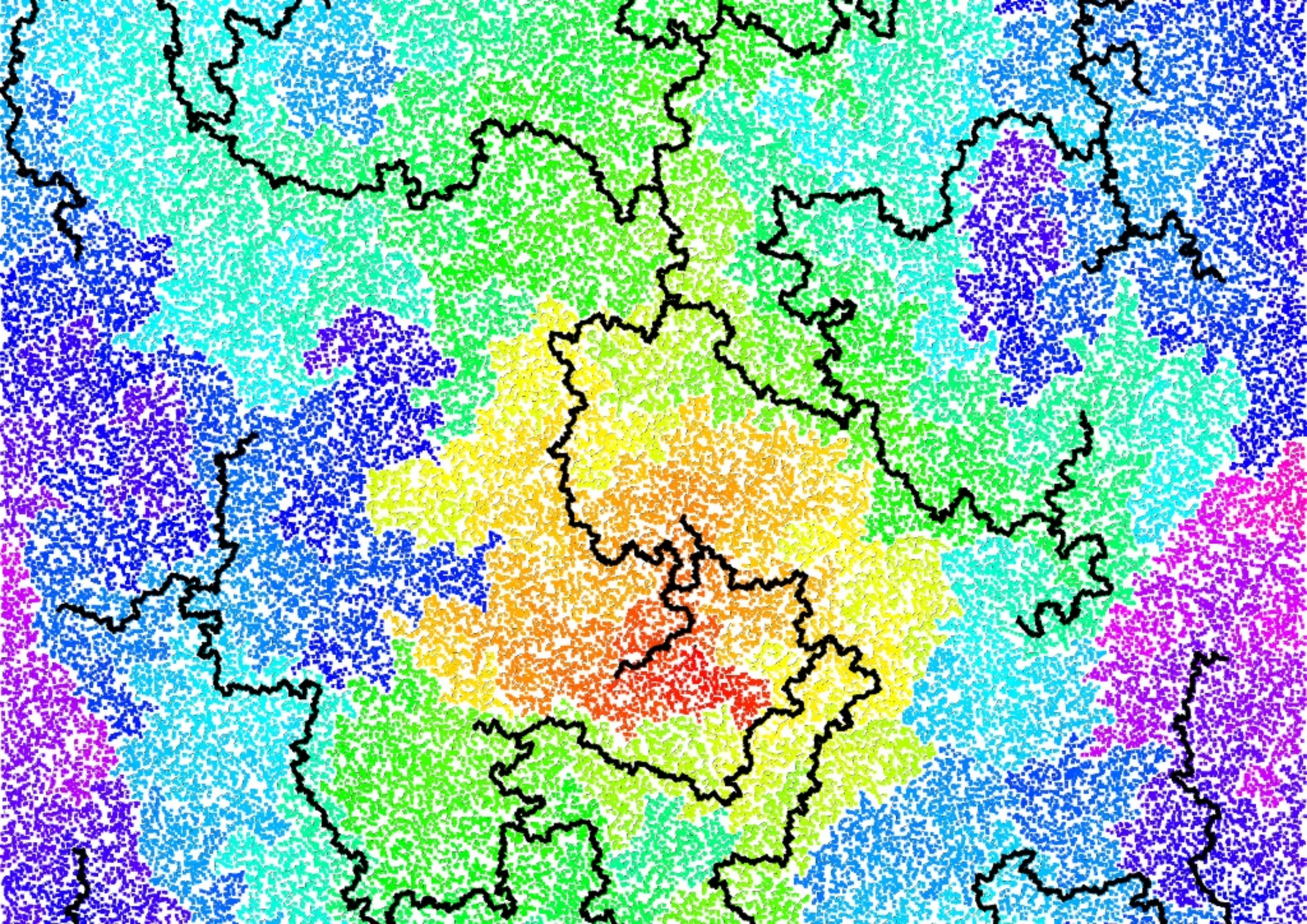
Percolation-MST coupling then suggests:

Construction of major routes of MST completed by time $\approx p_c$
Rural routes connect to nearby highways.

If $d_{MST}(v,w) :=$ Graph distance between v and w in MST
$d_{HW}(v,w) :=$ # highway edges on v-w path (edges of $C_{p_c}(1)$, say)

Then $\sup\{\mathbb{E}(d_{MST}(v,w)/d_{HW}(v,w)) : v,w \in V\} < \infty.$





VII Dimension and volume growth

On n-vertex graph $G=(V,E)$ with percolation threshold p_c,

$$\sup\{d_{MST}(v,w) : v,w \in C_p(1)\} \asymp n^{\beta}$$

Then expect MST to have

- diameter $O(n^{\beta})$
- volume growth $\frac{1}{\beta}$ $\quad \mathbb{E}\left[\#\{w\in V : d_{MST}(v,w)\leq k\}\right] \asymp k^{1/\beta}$
- scaling limit with "intrinsic dimension" $1/\beta$.

VIII Results.

$\xrightarrow{d}$: converges in distribution

Let T_n be MST of K_n. Then

- $\dfrac{\mathrm{diam}(T_n)}{n^{1/3}} \xrightarrow{d} D$, for some continuous r.v. D, and $\mathbb{E}[\mathrm{diam}(T_n)/n^{1/3}] \longrightarrow \mathbb{E}D$.

- There is an infinite random rooted tree (T,ρ) such that $(T_n,\rho_n) \xrightarrow{d} (T,\rho)$ in the Benjamini-Schramm sense (convergence of local graph statistics) and $\mathbb{E}|B(\rho,k)| \asymp k^3$

- There is a random compact measured metric space $\mathcal{T} = (\mathcal{T}, d, \mu)$ with Minkowski dimension 3, such that the following holds.

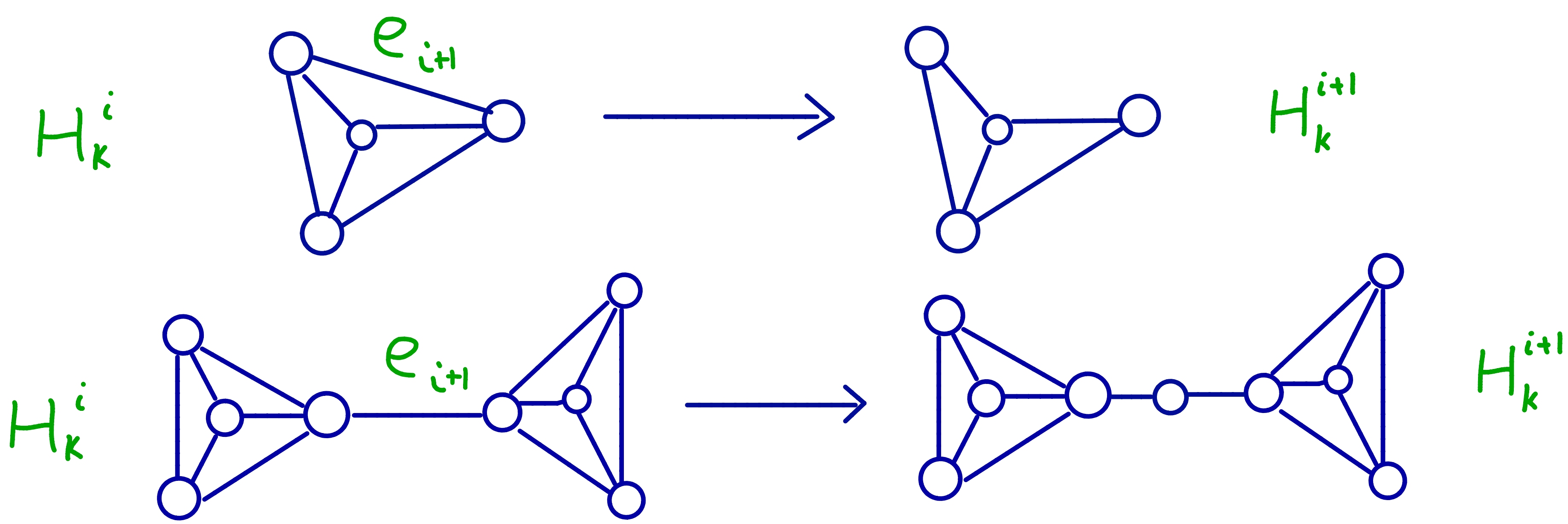
  Write V_n = vertices of T_n = vertices of K_n.

  Let $d_n = n^{-1/3} \cdot$ (graph distance in T_n), $\mu_n = \frac{1}{n} \sum_{v \in V_n} \delta_v$ = empirical measure on V_n

  Then $(V_n, d_n, \mu_n) \xrightarrow{d} \mathcal{T}$ in the Gromov-Hausdorff sense.

IX Alternate description of scaling limit $\mathcal{T}$

- Let H_k^0 be a random 3-regular graph with k vertices.
- Given H_k^i, choose an edge e_{i+1} of H_k^i uniformly at random.
  - Remove e_{i+1} to create H_k^{i+1} unless this would disconnect H_k^i



- Repeat until no more cycles; call result H_k.
- Scale distances by $c \cdot k^{-1/3}$, let $k \to \infty$. Result has same law as $\mathcal{T}$.

<u>Corollary</u>: $\mathcal{T}$ is also the scaling limit of the MST of random 3-regular graphs.

X Universality, extensions

The tree $\mathcal{T}$ is expected to be the MST scaling limit on any "high-dimensional" graph, e.g.

- random d-regular graphs
- the hypercube
- the lattice torus $(\mathbb{Z}/n\mathbb{Z})^d$ for $d > 8$

For low dimensions the picture is blurry.

$d=2$: The MST scaling limit (as an embedded object) exists and has dimension $\in (1+\varepsilon, 7/4)$. (Garban, Pete, Schramm 20??)

Numerics suggest dimension 1.22... no non-numerical predictions in physics literature.

$3 \leq d \leq 6$, Wide open. $6 < d \leq 8$ Opinions vary.

XI References

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