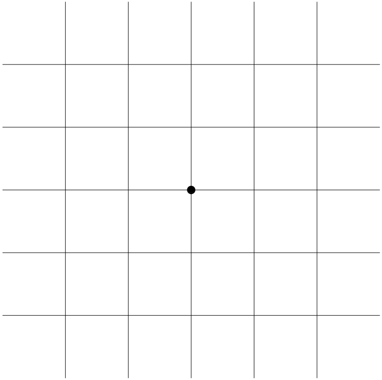
Colloque panquébécois des étudiants et étudiantes de l'ISM
Sherbrooke

Critical random graphs: a collaborative history

Louigi Addario-Berry (McGill)
and
Christina Goldschmidt (Oxford)

I. Prehistory. A) Euclidean percolation

- Example: Nearest neighbour graph on $\mathbb{Z}^2$.



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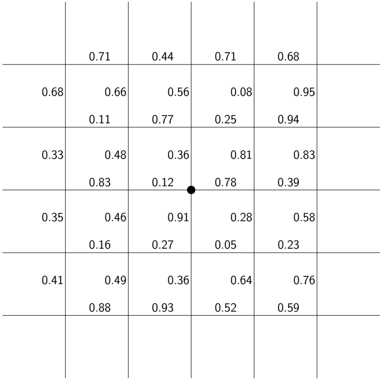

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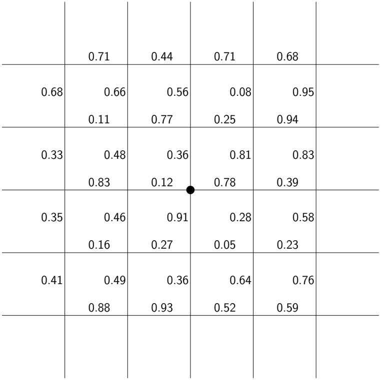

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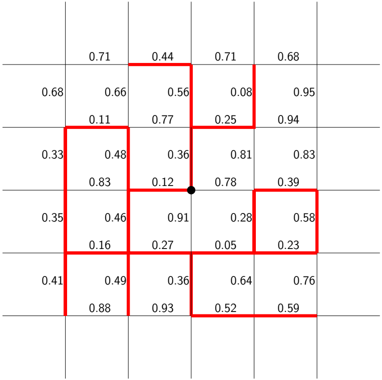

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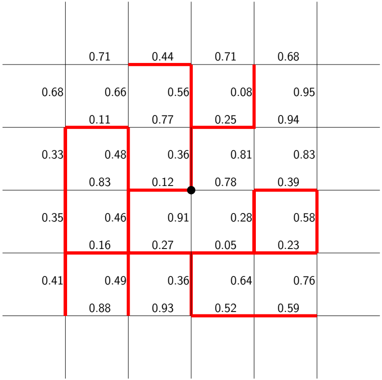
- Here $p = 0.6$.



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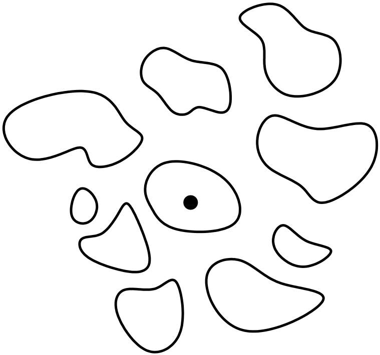

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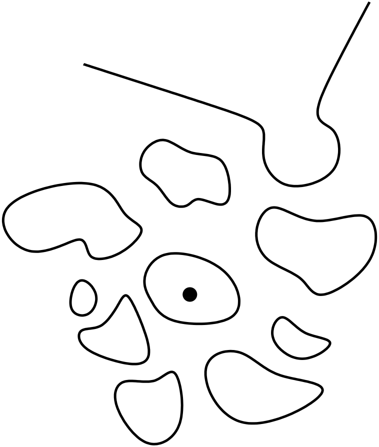

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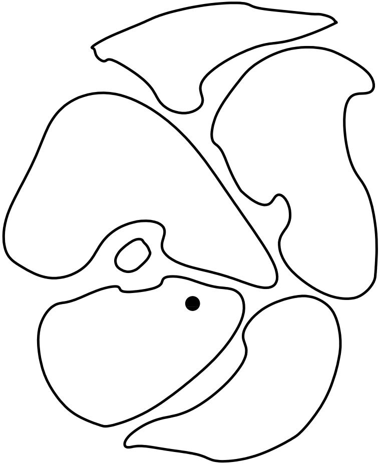
- $p = 1/2$: there is no infinite component* but $\mathbb{E}|\mathcal{C}_p| = \infty$.

  (Critical case)



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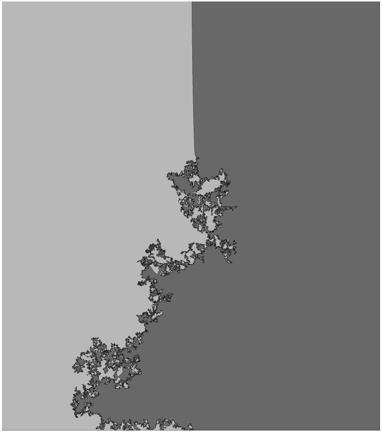

Image by Vincent Beffara

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Animation by Jason Miller

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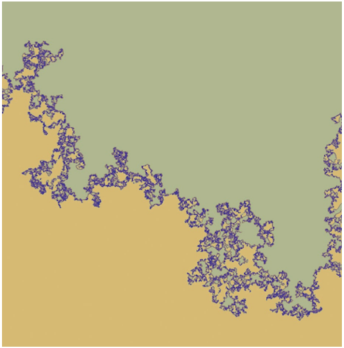

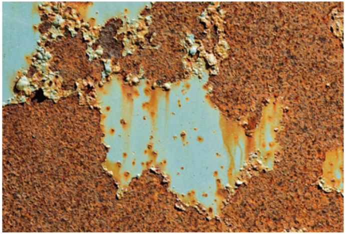

Image from `http://images.math.cnrs.fr/La-percolation-jeu-de-pavages-aleatoires.html`.

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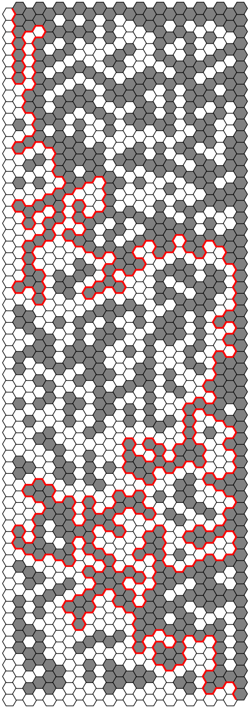
- Universality: choice of lattice should not matter
- But so far we only know the SLE(6) story for the hexagonal lattice, which has special "conformal" structure.



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- In all dimensions $d \geq 2$ there is a critical probability $p_c = p_c(\mathbb{Z}^d) \in (0, 1)$ such that for $p < p_c$ there is no infinite component and for $p > p_c$ there is an infinite component.

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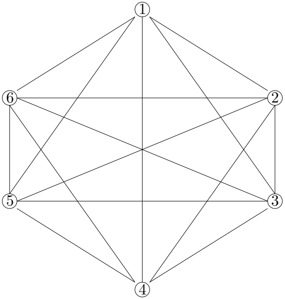
- Probably the most famous open problem in probability: **Prove (or disprove) that**

$$\mathbb{P}(\mathbb{Z}^d_{p_c} \text{ contains an infinite component}) = 0\,.$$

I. Prehistory. B) The Erdős–Rényi random graph

$G(n, p)$: Take vertices labelled $1, 2, \ldots, n$ and fix $p \in [0, 1]$. For each pair of vertices, put an edge between them with probability p, independently for different pairs.

This is percolation on the complete graph at parameter p.



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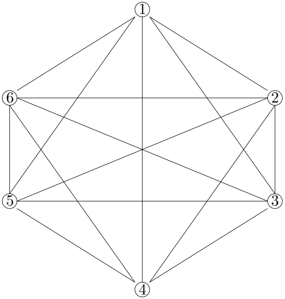
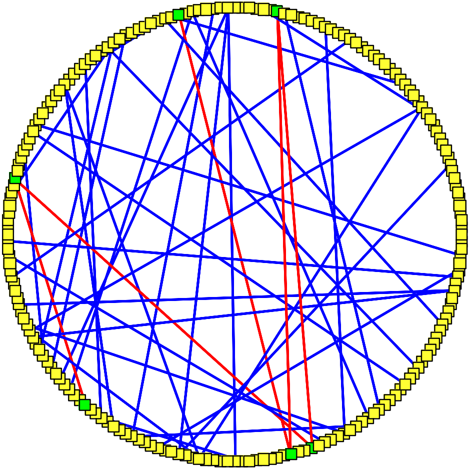


We will consider the case where $p = c/n$ for a constant $c > 0$, so that a single vertex has a Binomial$(n-1, c/n)$ number of neighbours, with mean approximately c for large n.

Erdős–Rényi random graph

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

$n = 200$, $c = 0.4$



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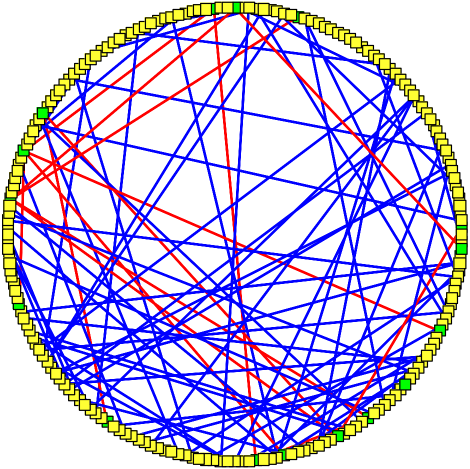
$n = 200$, $c = 0.8$



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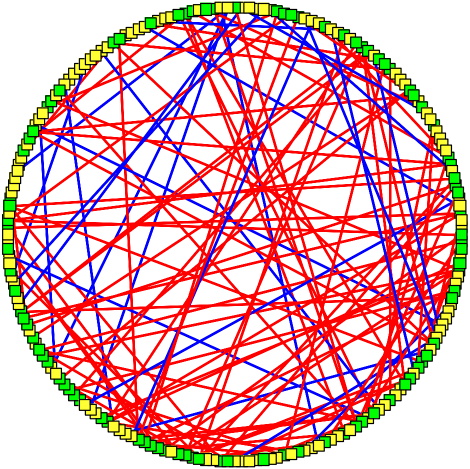

The phase transition

Theorem. (Erdős and Rényi (1960))
For $G(n, c/n)$ the following statements hold with probability tending to 1 as $n \to \infty$.

- If $c < 1$, the largest component contains $O(\log n)$ vertices.
- If $c > 1$, the largest component contains $\Theta(n)$ vertices and the rest contain $O(\log n)$ vertices each.

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Both of us worked on problems related to the Erdős-Rényi phase transition during our doctoral degrees.

II. First contact: Fields Institute, Toronto

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FIELDS

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May 23, 2018

May 15-19, 2006

Workshop on Random Walks in Random Environments
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Christina Goldschmidt, Cambridge:
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Unimodularity and Stochastic Processes

Stochastic processes on vertex-transitive graphs, especially Cayley graphs of groups, have been studied for 50 years (not counting the special case of integer lattices, which goes back hundreds of years). The assumption of invariance under graph automorphisms plays a key role, but investigations of the last 15 years have shown that an additional assumption is also extremely useful. This newer assumption is the property of unimodularity, which is equivalen to the Mass-Transport Principle. We shall review some well-known applications and also discuss recent work with David Aldous. This includes three theorems on RWRE.

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(Neither of us worked on random walks in random environments.)

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- Important: Nicolas Broutin was also there (much of our joint work is with Nic).

Nicolas Broutin

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2012

2005

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First interaction.

There were several open problem sessions. I presented an open problem I had heard about earlier in the term. From this paper:

Combinatorics, Probability and Computing (2006) 15, 815–822. © 2006 Cambridge University Press
doi:10.1017/S0963548306007565 First published online 14 August 2006 Printed in the United Kingdom

Simulating a Random Walk with Constant Error

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Received 19 February 2004; revised 28 May 2004

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The open problem: For $z = (z_1, z_2, z_3) \in \mathbb{Z}^3$, write $p_n(z)$ for the probability that a random walk starting at the origin is at z after n steps. If $z_1 + z_2 + z_3$ is even then $(p_{2n}(z), n \geq 1)$ is a unimodal sequence.

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Everyone basically ignored me, except Christina. She came over and suggested trying to prove that the sequence was in fact log-concave. I didn't know what that meant so I ran away. (The paper actually addresses this precise question: "... attempting to show unimodality via log-concavity is futile, since these functions are often (perhaps always?) log-convex in the tail".)

III. Oxford

[By Diliff - Own work, CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=1284604]

Excitement!

```
LAB to NB: I just got the exciting news that
Christina Goldschmidt (we met her in Toronto) will be
coming to Oxford for a(nother) postdoc starting in
March, which is exciting.
-- December 5, 2006.
```

Medawar

Culture clash...

```
LAB to CG: It was good to see you the other day.
Looking forward to more lunches together and
hopefully also to spending frustrating but ultimately
productive hours in front of a blackboard together.
-- April 23, 2007
```

A seminar by Jean-François Le Gall

22 October 2007
15:45

Oxford-Man Institute

The continuous limit of random planar maps

Professor Jean Francois Le Gall

(ENS, France)

Stochastic Analysis Seminar

View all

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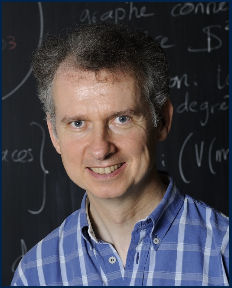

Abstract

We discuss the convergence in distribution of rescaled random planar maps viewed as random metric spaces. More precisely, we consider a random planar map M(n), which is uniformly distributed over the set of all planar maps with n faces in a certain class. We equip the set of vertices of M(n) with the graph distance rescaled by the factor n to the power 1/4. We then discuss the convergence in distribution of the resulting random metric spaces as n tends to infinity, in the sense of the Gromov-Hausdorff distance between compact metric spaces. This problem was stated by Oded Schramm in his plenary address paper at the 2006 ICM, in the special case of triangulations. In the case of bipartite planar maps, we first establish a compactness result showing that a limit exists along a suitable subsequence. Furthermore this limit can be written as a quotient space of the Continuum Random Tree (CRT) for an equivalence relation which has a simple definition in terms of Brownian labels attached to the vertices of the CRT. Finally we show that any possible limiting metric space is almost surely homomorphic to the 2-sphere. As a key tool, we use bijections between planar maps and various classes of labelled trees.

Back in Paris in 2004...

Probability Surveys
Vol. 2 (2005) 245–311
ISSN: 1549-5787
DOI: 10.1214/154957805100000140

Random trees and applications*

Jean-François Le Gall

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e-mail: legall@dma.ens.fr

Abstract: We discuss several connections between discrete and continuous random trees. In the discrete setting, we focus on Galton-Watson trees under various conditionings. In particular, we present a simple approach to Aldous' theorem giving the convergence in distribution of the contour process of conditioned Galton-Watson trees towards the normalized Brownian excursion. We also briefly discuss applications to combinatorial trees. In the continuous setting, we use the formalism of real trees, which yields an elegant formulation of the convergence of rescaled discrete trees towards continuous objects. We explain the coding of real trees by functions, which is a continuous version of the well-known coding of discrete trees by Dyck paths. We pay special attention to random real trees coded by Brownian excursions, and in a particular we provide a simple derivation of the marginal distributions of the CRT. The last section is an introduction to the theory of the Brownian snake, which combines the genealogical structure of random real trees with independent spatial motions. We introduce exit measures for the Brownian snake and we present some applications to a class of semilinear partial differential equations.

AMS 2000 subject classifications: Primary 60J80; secondary 05C05, 35J65, 60C05, 60J65.
Keywords and phrases: random tree, contour process, conditioned tree, Brownian motion, Brownian excursion, real tree, coding of trees, CRT, Brownian snake, exit measure, partial differential equation.

Received September 2005.

IV. Convergence of metric measure spaces

Let $(M_n, d_n, \mu_n)_{n\geq 1}$ be a sequence of (random) metric spaces (M_n, d_n) each endowed with a probability measure μ_n on (the Borel sets of) M_n.

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We want to give a notion of convergence for such a sequence. In order to do this, we will sample points from M_n according to μ_n and then look at the matrix of pairwise distances between them.

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Idea: as we select more and more points, we get better and better (finite) approximations to the true space (M_n, d_n). If each sequence of finite approximations converges, we say that the spaces do.

IV. Convergence of metric measure spaces

Fix $k \geq 2$ and let $U_1^n, U_2^n, \ldots, U_k^n$ be independent random variables, each with distribution μ_n. For $1 \leq i, j \leq k$, let

$$D_{i,j}^{n,k} = d_n(U_i^n, U_j^n)$$

and let $D^{n,k} = (D_{i,j}^{n,k})_{1 \leq i,j \leq k}$.

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We will say

$$(M_n, d_n, \mu_n) \longrightarrow (M, d, \mu)$$

as $n \to \infty$ if, for each $k \geq 2$,

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in distribution as $n \to \infty$. **N.B.** If the metric spaces (M_n, d_n, μ_n) are random then the matrices $D^{n,k}$ have "two levels" of randomness – this requires care to set up formally.

A key example: random trees

Let $\mathbb{T}_n$ be the set of trees with vertices labelled $1, 2, \ldots, n$.
Cayley's formula: $|\mathbb{T}_n| = n^{n-2}$.

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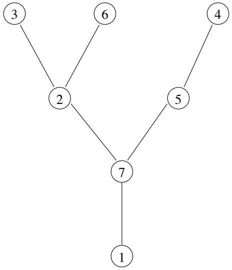

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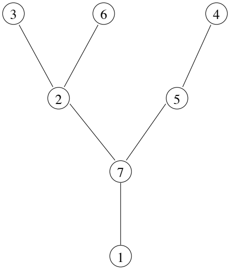


What does the tree look like as n gets large?

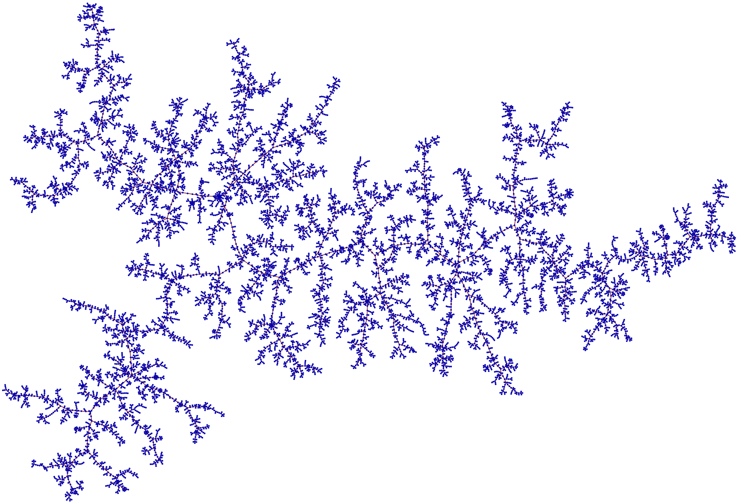
Convergence of the uniform random tree

Theorem. (Aldous (1991))
As $n \to \infty$,

$$\frac{1}{\sqrt{n}} T_n \xrightarrow{d} \mathcal{T}$$

where $\mathcal{T}$ is the Brownian continuum random tree (CRT).

The Brownian continuum random tree

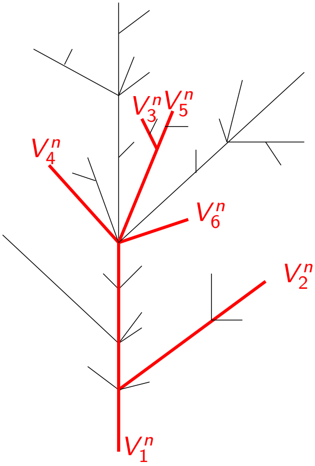


[Picture by Igor Kortchemski]

Convergence of the uniform random tree

Fix $k \geq 2$, take the uniform random tree T_n and pick k independent uniform random vertices, $V_1^n, V_2^n, \ldots, V_k^n$.

$k = 6$:



Convergence of the uniform random tree

The matrix of pairwise distances between these independent uniform points is

$$D^{n,k} = \begin{pmatrix} 0 & \mathrm{dist}(V_1^n, V_2^n) & \cdots & \mathrm{dist}(V_1^n, V_k^n) \\ \mathrm{dist}(V_2^n, V_1^n) & 0 & \cdots & \mathrm{dist}(V_2^n, V_k^n) \\ \vdots & \vdots & & \vdots \\ \mathrm{dist}(V_k^n, V_1^n) & \mathrm{dist}(V_k^n, V_2^n) & \cdots & 0 \end{pmatrix}$$

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Aldous' theorem says that for each $k \geq 2$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} D^{n,k} \xrightarrow{d} D^k$$

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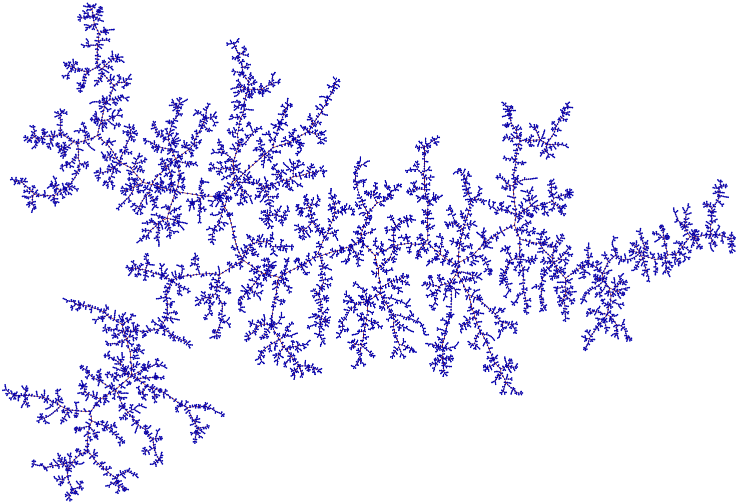
The $1/\sqrt{n}$ scaling says T_n is in some sense 2-dimensional: n points, diameter of order $\sqrt{n}$.

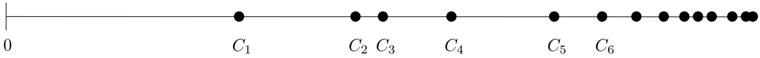
The Brownian continuum random tree



[Picture by Igor Kortchemski]

Line-breaking

Take $E_1, E_2, \ldots, E_k$ independent and identically distributed Exponential(1/2) random variables, i.e. $\mathbb{P}(E_1 > x) = e^{-x/2}$. For $j \geq 1$, let $C_j = \sqrt{\sum_{i=1}^{j} E_i}$.



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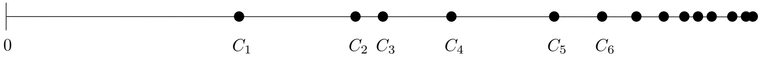

Consider the line-segments $[0, C_1)$, $[C_1, C_2), \ldots, [C_{k-1}, C_k)$.

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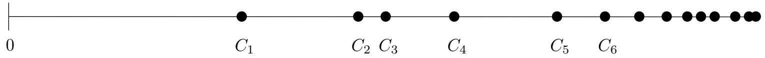

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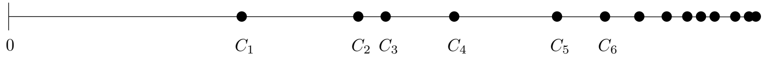

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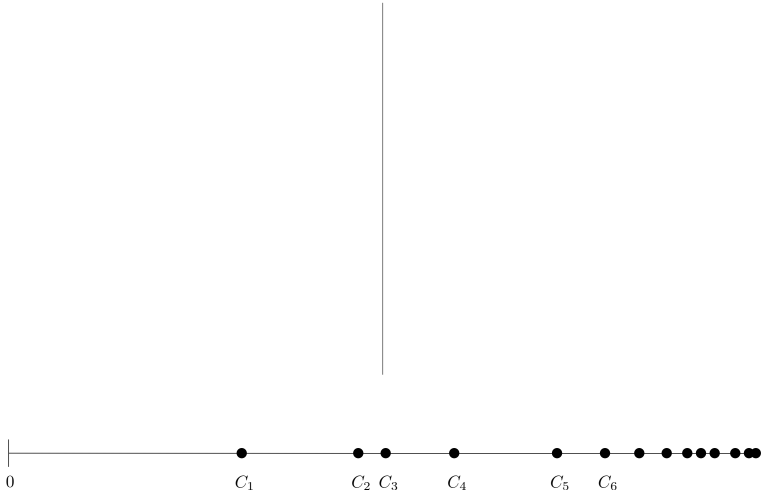
Start from $[0, C_1)$ and proceed inductively.

For $2 \leq i \leq k$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.

Line-breaking



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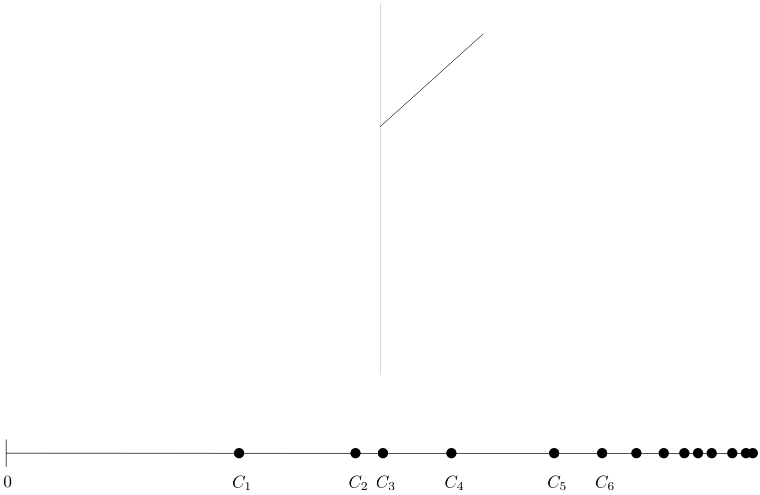

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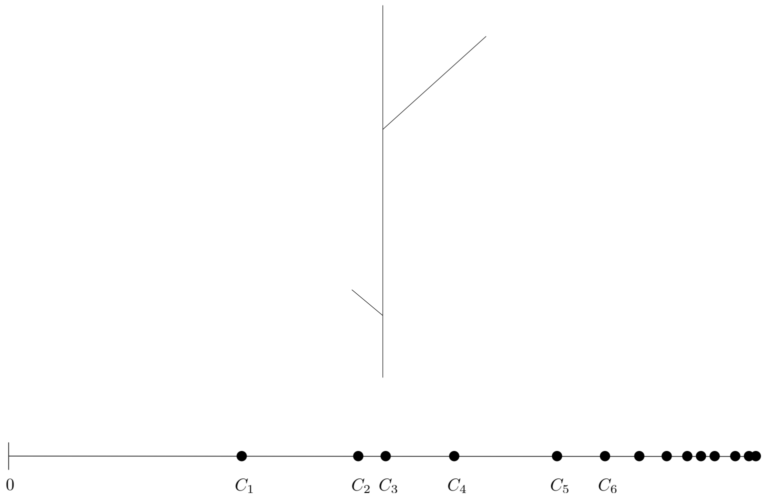

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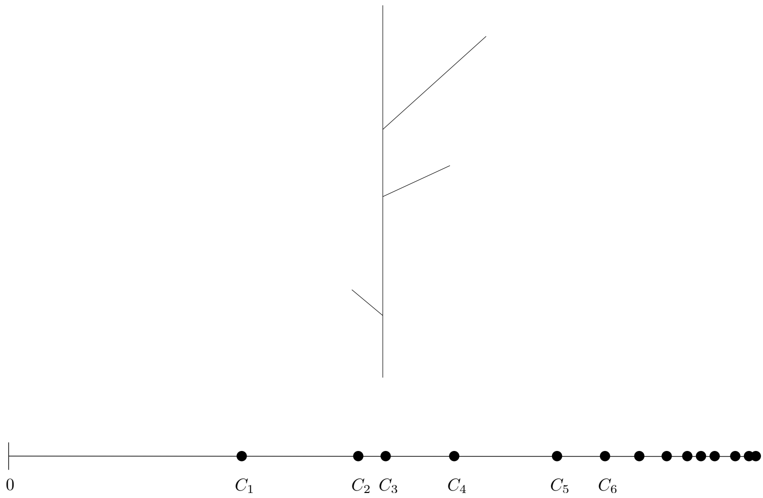

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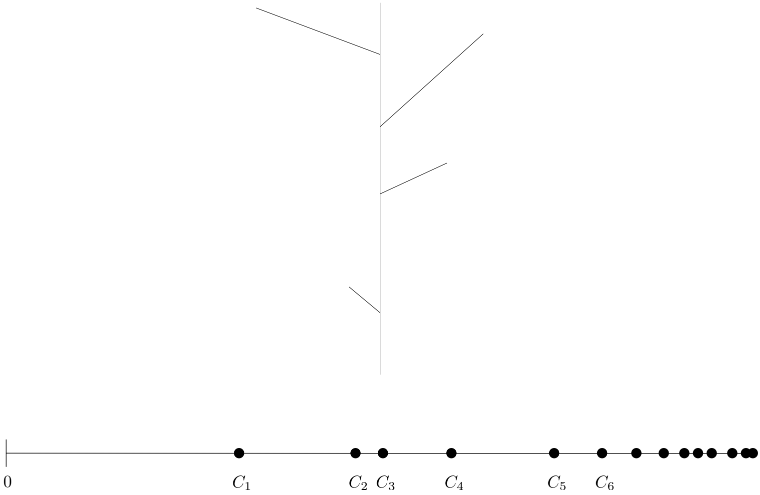

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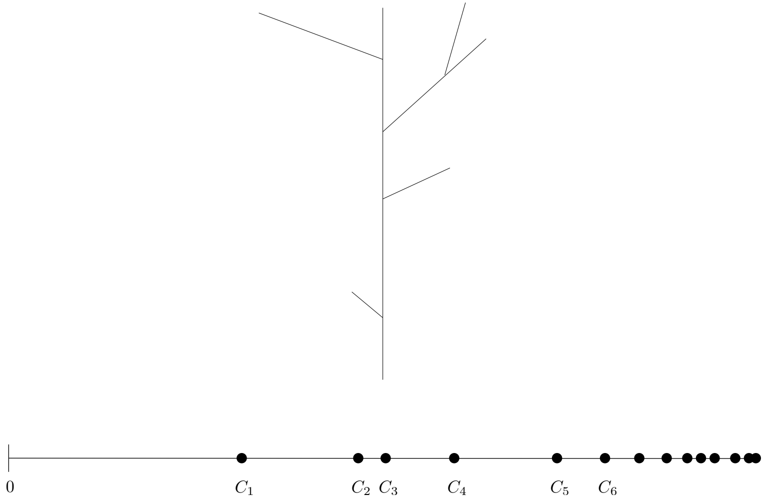

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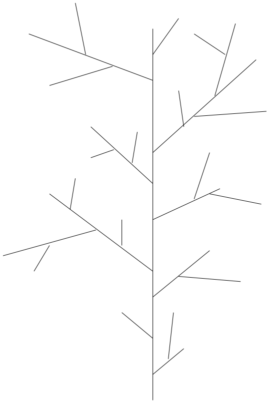

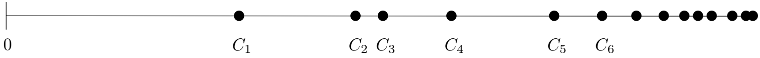

Line-breaking

D^k is the matrix of pairwise distances between the k leaves in the line-breaking construction.

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For increasing k, the line-breaking construction gives a sequence of better and better approximations to the Brownian CRT. One way to define the tree is then as the metric space completion of the increasing limit as $k \to \infty$ of the sequence of approximations.

Universality

The convergence result is, in fact, much more general, in that (up to a constant scaling factor) very many random trees have the Brownian CRT as their scaling limit.

- uniform unordered unlabelled rooted trees
- uniform unordered unlabelled unrooted trees
- Galton–Watson branching trees with offspring mean 1 and finite offspring variance, conditioned to have size n
- random trees with a prescribed degree sequence satisfying certain conditions
- random dissections
- random graphs from subcritical classes.

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In particular, all these objects are in some sense "2-dimensional".

Back to Oxford: the starting-point of our project

We realised we could apply this theory to the setting of the critical Erdős–Rényi random graph.

V. The critical Erdős–Rényi random graph

We saw earlier that the Erdős–Rényi random graph $G(n, p)$ undergoes a phase transition when $p = 1/n$. What happens exactly at the critical point?

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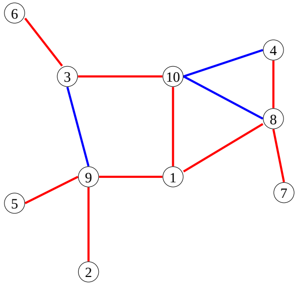
We saw earlier that the Erdős–Rényi random graph $G(n, p)$ undergoes a phase transition when $p = 1/n$. What happens exactly at the critical point?

We start by considering the sizes of the components. We will also be interested in the surplus of a component, that is the number of edges more than a tree that it has.

A component with surplus 3:



V. The critical Erdős–Rényi random graph

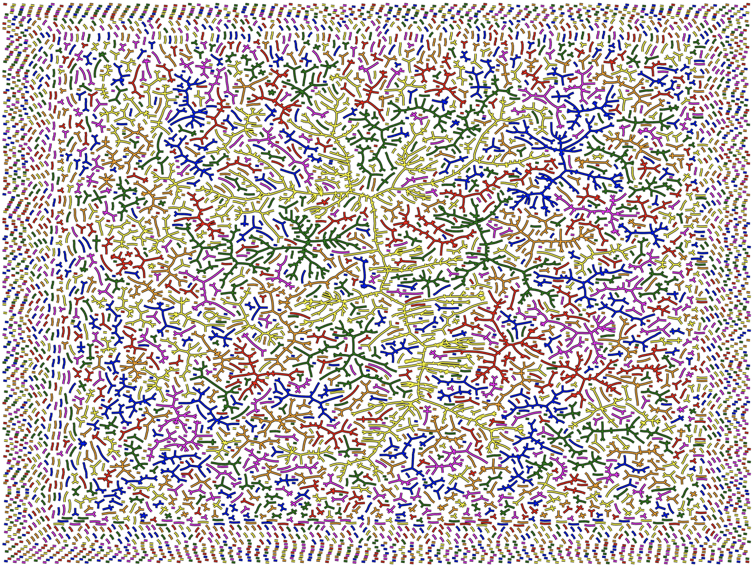


Image by Nicolas Broutin.

V. The critical Erdős–Rényi random graph

Let $C_1^n, C_2^n, \ldots$ be the ordered component sizes of $G(n, 1/n)$.
Let $S_1^n, S_2^n, \ldots$ be the corresponding surpluses.

Theorem. (Aldous (1997))
As $n \to \infty$, we have jointly that

$$\frac{1}{n^{2/3}}(C_1^n, C_2^n, \ldots) \xrightarrow{d} (C_1, C_2, \ldots)$$

and

$$(S_1^n, S_2^n, \ldots) \xrightarrow{d} (S_1, S_2, \ldots),$$

where $C_1, C_2, \ldots \in \mathbb{R}_+$ and $S_1, S_2, \ldots \in \mathbb{Z}_+$ are random variables with an explicit (but complicated) distribution, such that $\mathbb{P}(S_k = 0) \to 1$ as $k \to \infty$.

Component convergence

This theorem tells us the limiting rescaled sizes of the components, and that they are quite close to being trees. Indeed, most of them really are trees!

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Suppose, for the moment, we know that the vertices $v_1, v_2, \ldots, v_m$ form a tree-component of $G(n, 1/n)$. Then that tree is uniform its vertex-set (which we may as well think of as $\{1, 2, \ldots, m\}$).

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So we should be thinking about rescaling by $m^{-1/2}$ in order to see a limit, which will be the Brownian CRT.

Since components have sizes on the order of $n^{2/3}$, we should rescale by $n^{-1/3}$.

Component convergence

What about components which aren't trees? Again, if we know that $v_1, v_2, \ldots, v_m$ form a component of $G(n, 1/n)$ with surplus $s \geq 1$, that component is uniform among the possibilities.

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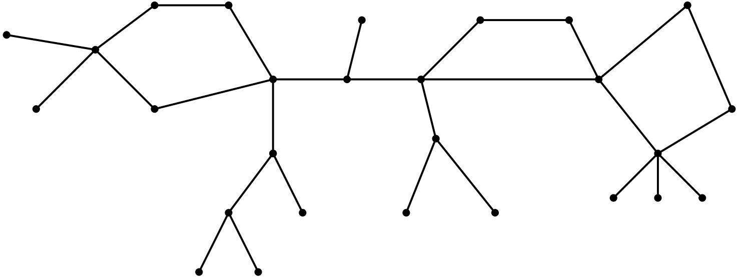
So we need to understand the large-m behaviour of a uniform random connected graph on m vertices with surplus s.

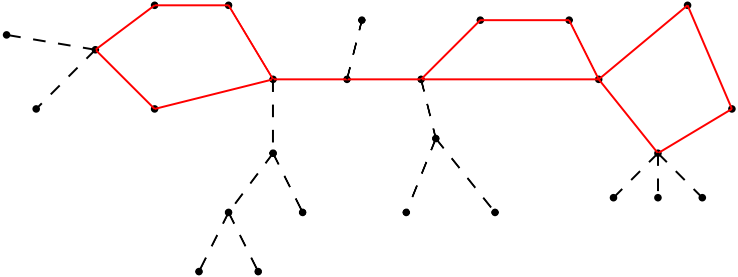
Cycle structure

Let's first look into the cycle structure.



Cycle structure

Core $C(G)$



Cycle structure

The kernel $K(G)$ is the multigraph which gives the "shape of the core": take the vertices of the core of degree 3 or more; contract the paths between them to a single edge.

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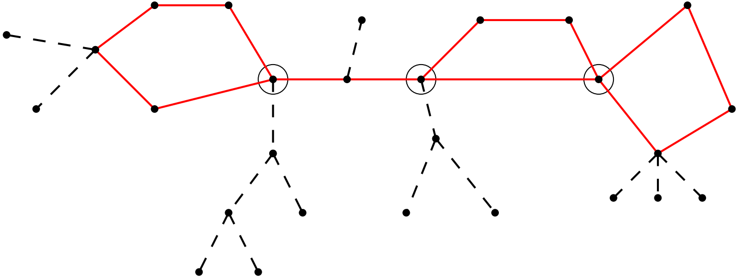
(By convention, the kernel of a tree or unicyclic component is empty.)

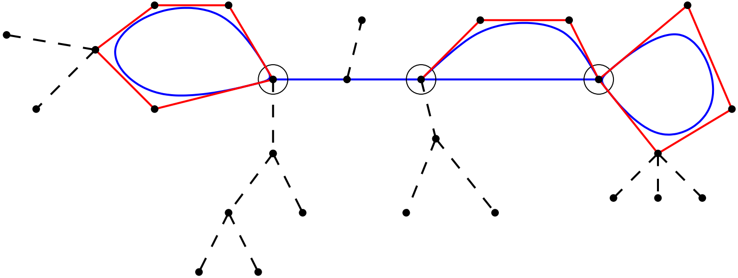
Cycle structure

Vertices of degree at least 3 in the core



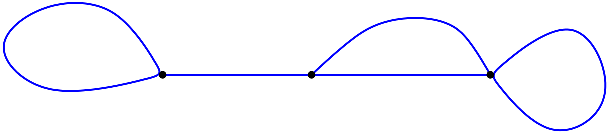
Cycle structure

Contract paths between them



Cycle structure

Kernel $K(G)$



Cycle structure

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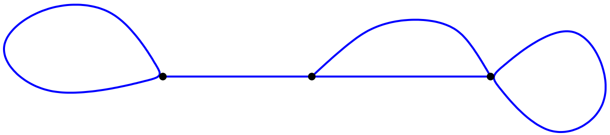

Note that the kernel has the same surplus as the original graph.

Scaling limits for uniform connected graphs

Theorem. (A.-B., Broutin and G. (2010, 2012))
Let U_m^s be a uniform connected graph with vertices labelled by $1, 2, \ldots, m$ and surplus $s \geq 1$. Then

$$\frac{1}{\sqrt{m}} U_m^s \xrightarrow{d} \mathcal{U}^s,$$

for a random limit object we will construct in a moment.

Scaling limits for uniform connected graphs

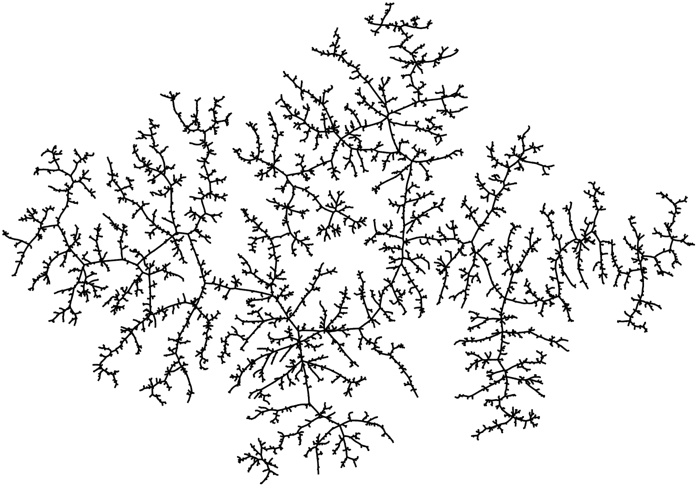

Image by Nicolas Broutin.

Scaling limits for uniform connected graphs

Again, this convergence means that if we sample k uniform random points and look at the matrix of pairwise distances between them, that matrix converges on rescaling by $m^{-1/2}$.

Construction of $\mathcal{U}^s$ for $s \geq 2$. Step 1: The core.

- Take $2s - 2$ labelled vertices and assign each one 3 half-edges.

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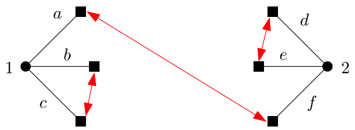

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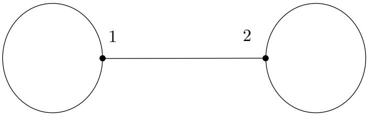

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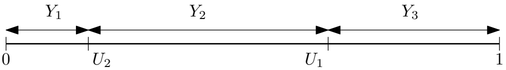

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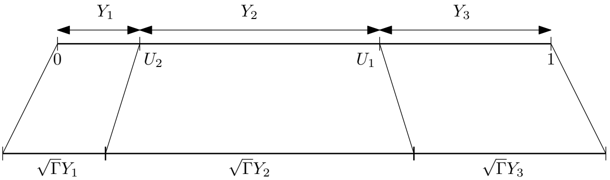

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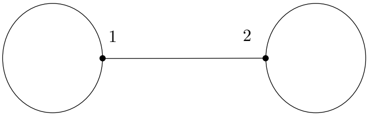

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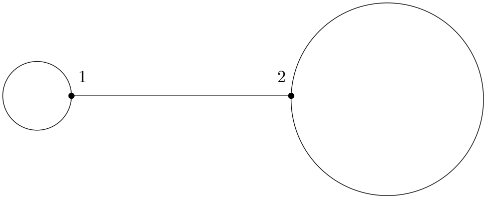

Construction of $\mathcal{U}^s$ for $s \geq 2$. Step 2: Line-breaking.

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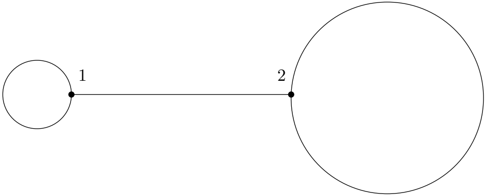

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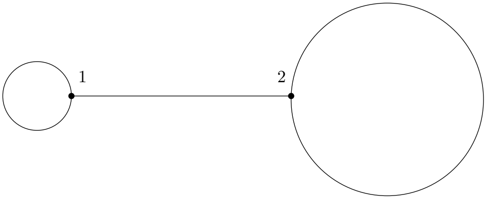

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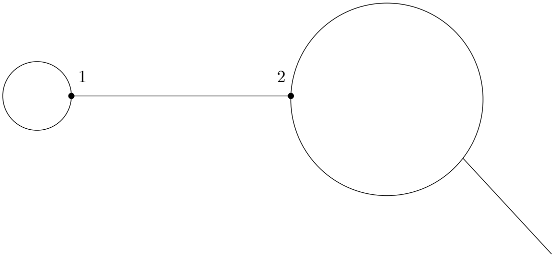

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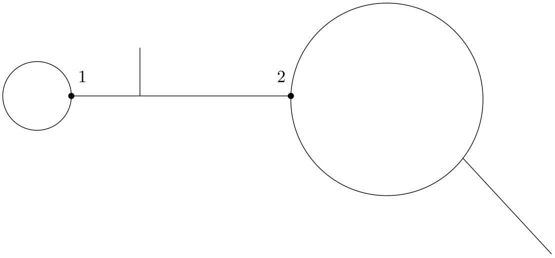

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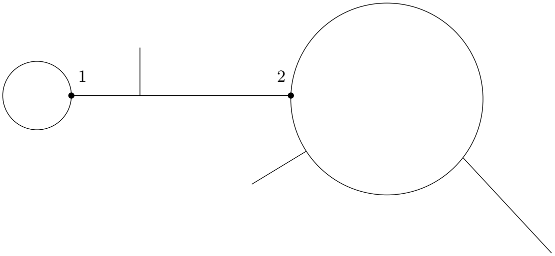

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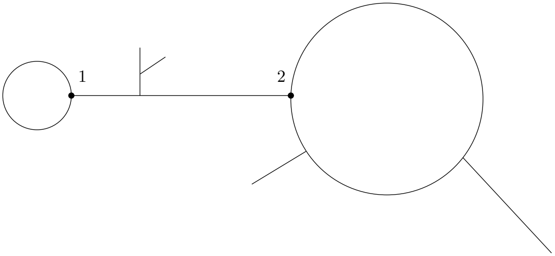

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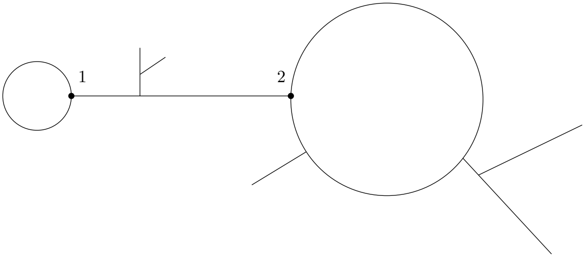

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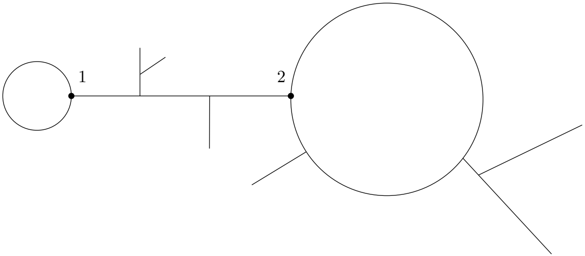

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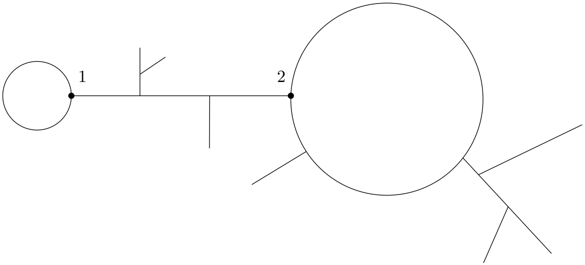

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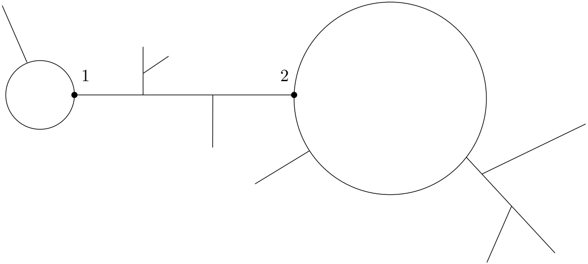

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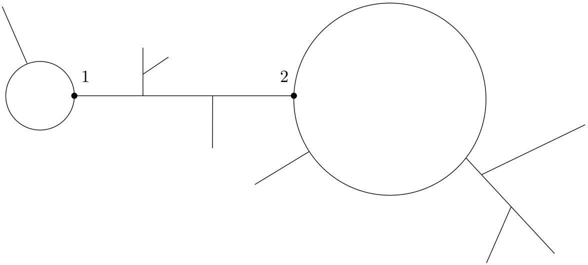
- Now perform line-breaking starting from the core, and recursively gluing line-segments $[C_j, C_{j+1}]$ uniformly over the structure built so far.
- Do this forever, then take the completion of the limit; the result is $\mathcal{U}^s$.



Construction of $\mathcal{U}_s$ for $s = 1$.

Needs to be handled separately because the kernel is empty.

Step 1: The core

- Start from a lollipop shape  of total length $\sqrt{\Gamma}$ where $\Gamma \sim \text{Gamma}(\frac{3}{2}, \frac{1}{2})$.
- Split that length uniformly between the cycle and the stick.

Step 2: Line-breaking

- This step works exactly as before.

Scaling limit

Putting this together with Aldous' theorem yields a scaling limit for the whole critical Erdős–Rényi random graph.

Let $G_1^n, G_2^n, \ldots$ be the components of $G(n, 1/n)$ in decreasing order of size.

Theorem. (A.-B., Broutin and G. (2010,2012))
As $n \to \infty$,

$$\frac{1}{n^{1/3}}(G_1^n, G_2^n, \ldots) \xrightarrow{d} (\mathcal{G}_1, \mathcal{G}_2, \ldots)$$

where $\mathcal{G}_1, \mathcal{G}_2, \ldots$ are conditionally independent given the sizes $(C_1, C_2, \ldots)$ and surpluses $(S_1, S_2, \ldots)$ with $\mathcal{G}_i \stackrel{d}{=} \sqrt{C_i}\, \mathcal{U}^{S_i}$.

Probab. Theory Relat. Fields (2012) 152:367–406
DOI 10.1007/s00440-010-0325-4

The continuum limit of critical random graphs

L. Addario-Berry · N. Broutin · C. Goldschmidt

Received: 30 November 2009 / Revised: 30 June 2010 / Published online: 12 October 2010

Abstract We consider the Erdős–Rényi random graph $G(n, p)$ inside the critical window, that is when $p = 1/n + \lambda n^{-4/3}$, for some fixed $\lambda \in \mathbb{R}$. We prove that the sequence of connected components of $G(n, p)$, considered as metric spaces using the graph distance rescaled by $n^{-1/3}$, converges towards a sequence of continuous compact metric spaces. The result relies on a bijection between graphs and certain marked random walks, and the theory of continuum random trees. Our result gives access to the answers to a great many questions about distances in critical random graphs. In particular, we deduce that the diameter of $G(n, p)$ rescaled by $n^{-1/3}$ converges in distribution to an absolutely continuous random variable with finite mean.

Keywords Random graphs · Gromov–Hausdorff distance · Scaling limits · Continuum random tree · Diameter

Mathematics Subject Classification (2000) 05C80 · 60C05

Electronic Journal of Probability

Vol. 15 (2010), Paper no. 25, pages 741–775.

Journal URL
http://www.math.washington.edu/~ejpecp/

Critical random graphs: limiting constructions and distributional properties*†

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Abstract

We consider the Erdős–Rényi random graph $G(n,p)$ inside the critical window, where $p = 1/n + \lambda n^{-4/3}$ for some $\lambda \in \mathbb{R}$. We proved in [1] that considering the connected components of $G(n,p)$ as a sequence of metric spaces with the graph distance rescaled by $n^{-1/3}$ and letting $n \to \infty$ yields a non-trivial sequence of limit metric spaces $\mathscr{C} = (\mathscr{C}_1, \mathscr{C}_2, \ldots)$. These limit metric spaces can be constructed from certain random real trees with vertex-identifications. For a single such metric space, we give here two equivalent constructions, both of which are in terms of more standard probabilistic objects. The first is a global construction using Dirichlet random variables and Aldous' Brownian continuum random tree. The second is a recursive construction from an inhomogeneous Poisson point process on $\mathbb{R}_+$. These constructions allow us to characterize the distributions of the masses and lengths in the constituent parts of a limit component when it is decomposed according to its cycle structure. In particular, this strengthens results of Łuczak et al. [29] by providing precise distributional convergence for the lengths of paths between kernel vertices and the length of a shortest cycle, within any fixed limit component.

MSC 2000 subject classifications: primary 05C80; secondary 60C05.

†L.A.B. was supported by an NSERC Discovery Grant throughout the research and writing of this paper. C.G. was funded by EPSRC Postdoctoral Fellowship EP/D065755/1.

Timeline

- Jean-François Le Gall's seminar in October 2007.
- Project crystallized in spring 2008.
- LAB moved back to Montréal in summer 2008.
- Major progress in September 2008 during a visit of NB and CG to LAB at Université de Montréal.
- Finished first paper in March 2009. Less than 2 years! Math is not fast, but this actually feels like it did come together quickly.
- (On the other hand, that paper didn't appear in print in a journal until 2012!)
- Finished follow-up paper in March 2010.

Universality

Our scaling limit has subsequently been shown to be universal, in that a whole host of different critical random graph models have essentially the same limit.

VI. Minimum Spanning Trees

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- For *random edge weights*, the structure of the minimum spanning tree of K_n is intimately connected to that of the critical Erdős-Renyi random graph.
- We used this connection to study the metric space limit of the minimum spanning tree.

The metric space limit of the minimum spanning tree

- Set-up: K_n, independent Uniform$[0, 1]$ edge weights $\{U_e, e \in E(K_n)\}$.

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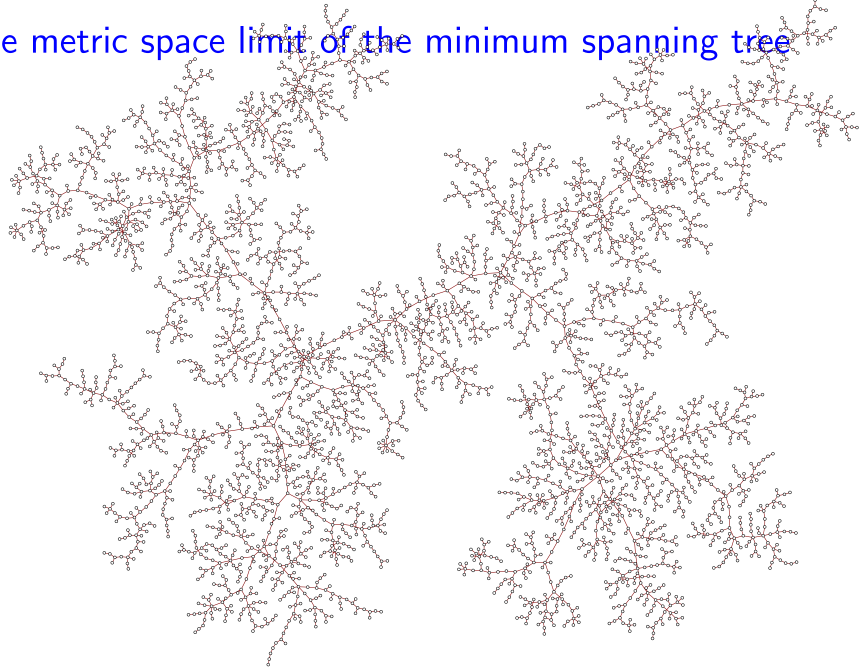

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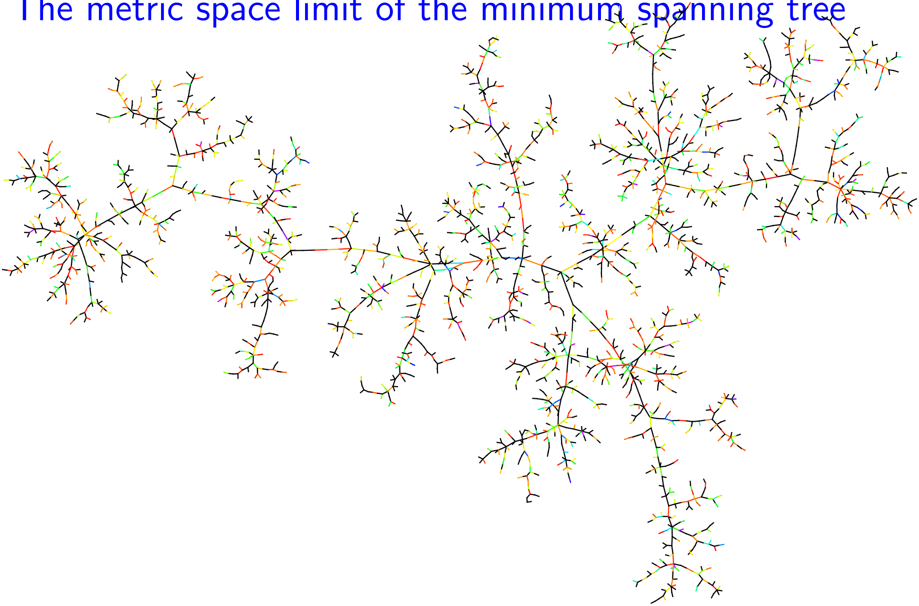

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Theorem (Addario-Berry, Broutin, Goldschmidt, Miermont 2013)

As $n \to \infty$, $M_n \longrightarrow \mathcal{M}$ in distribution, for a limiting random measured metric space $\mathcal{M}$.

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- Work started in 2011; published in 2017

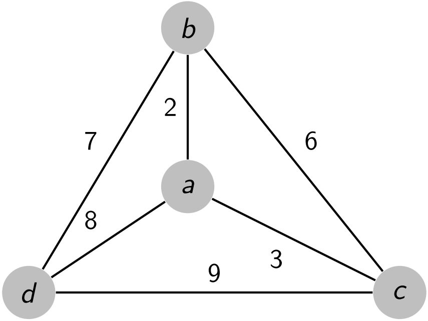
Grégory Miermont

2011 - Nicolas Broutin's wedding

Photo by Luc Devroye

Kruskal's minimum spanning tree algorithm

- Order edges in increasing order of weight as $e_1, \ldots, e_m$.



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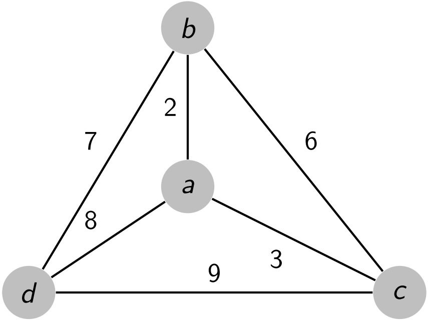

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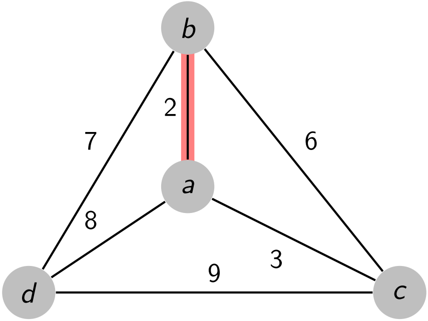

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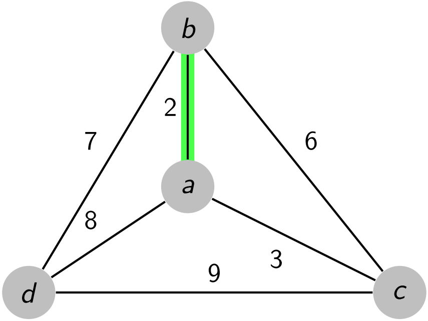

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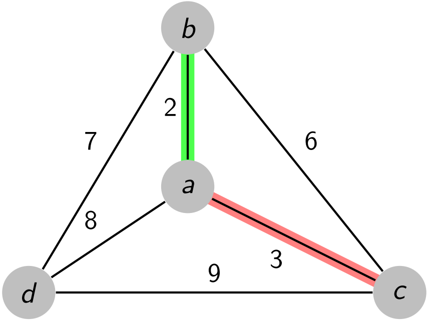

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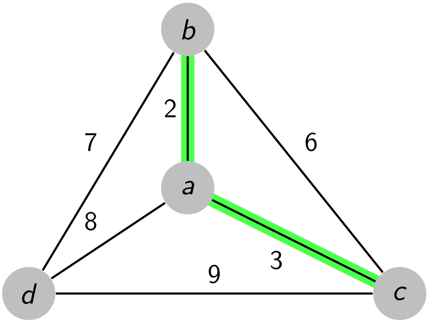

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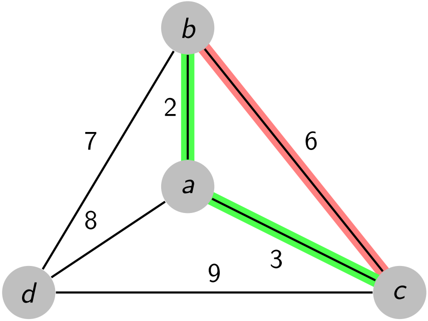

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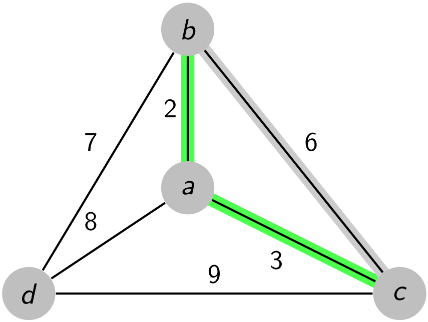

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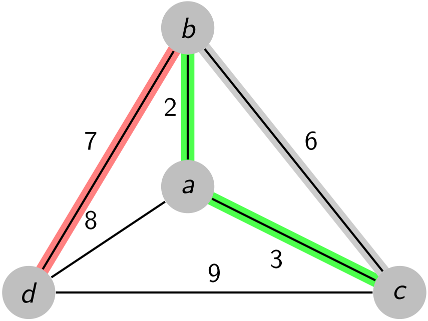

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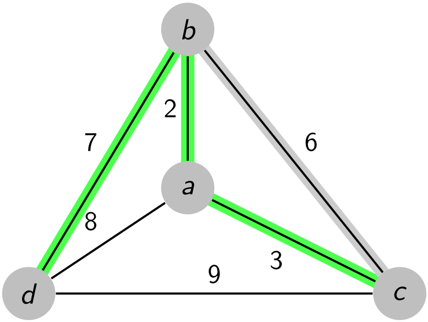

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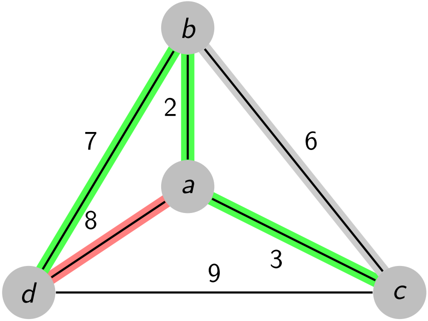

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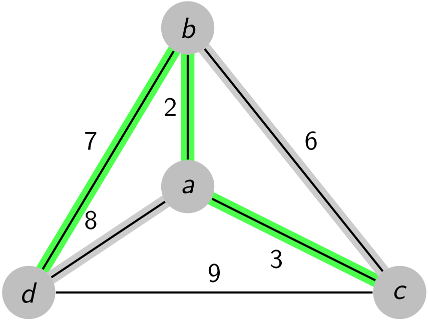

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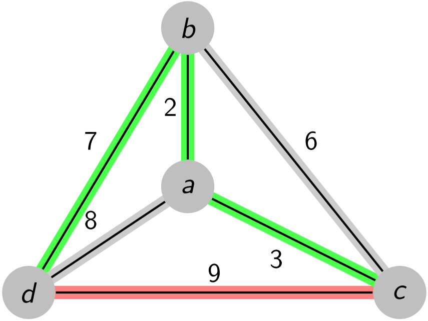

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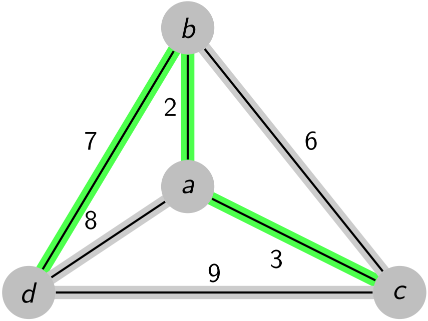

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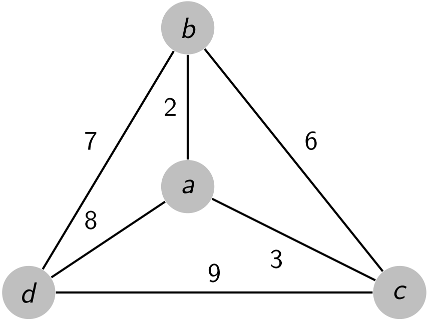

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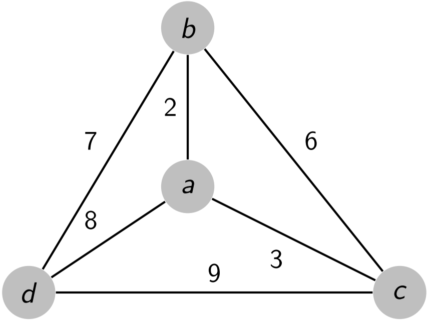

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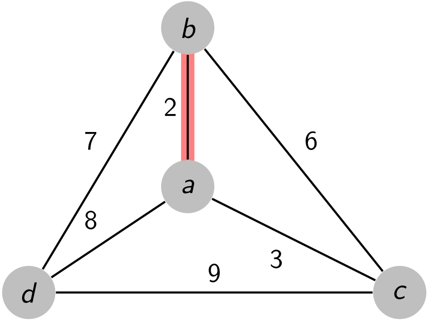

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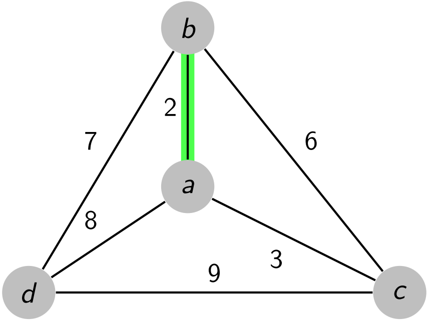

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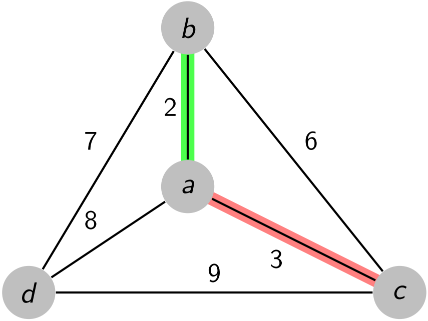

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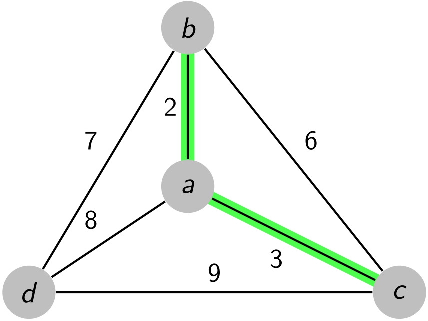

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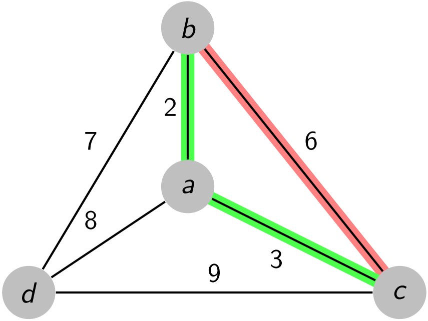

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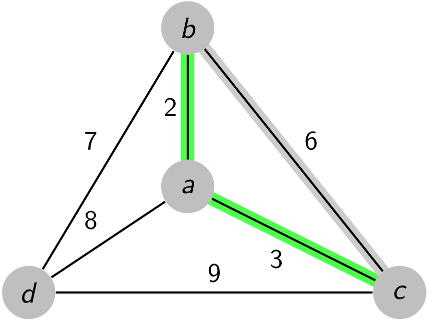

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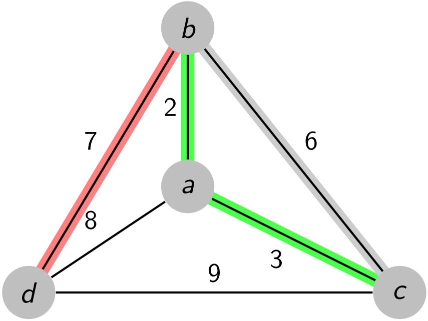

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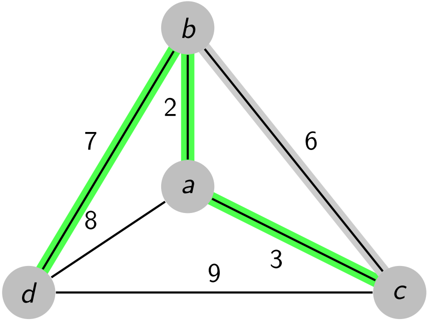

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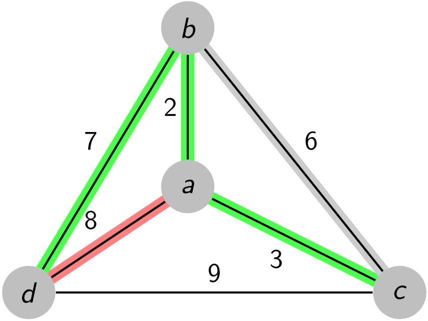

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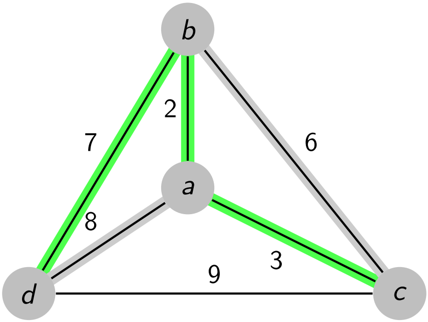

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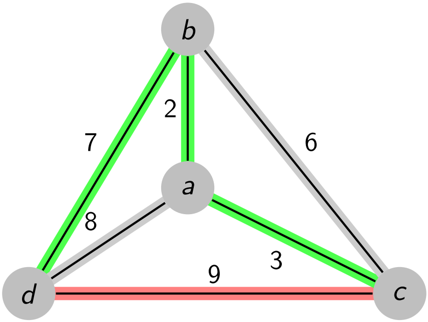

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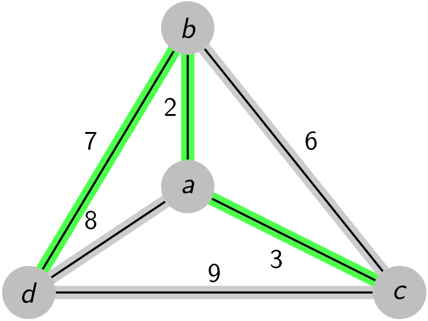

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- $G(n, 1/n)$: the breakpoint. MST structure is built here.

Proof idea

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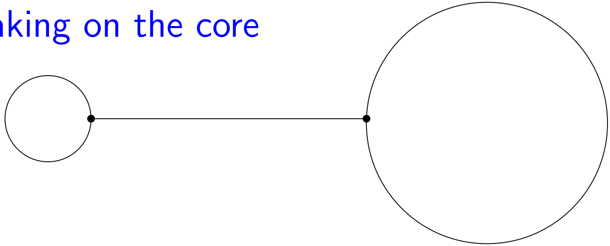
- Cycle breaking is *dual* to Kruskal: instead of adding edges (unless they would create cycles), remove edges (unless they would cause disconnections).

Cycle breaking on the core



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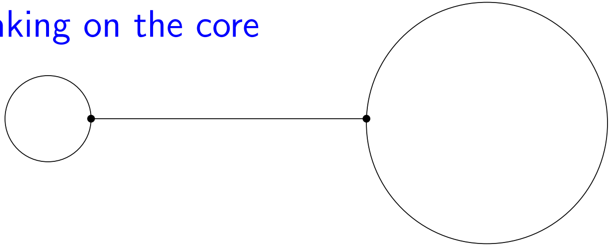

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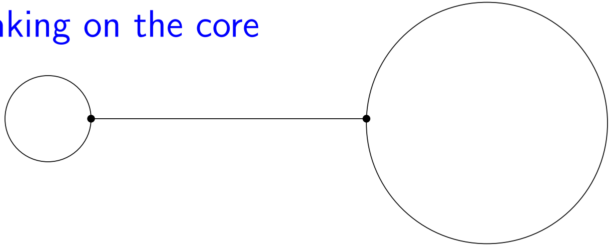

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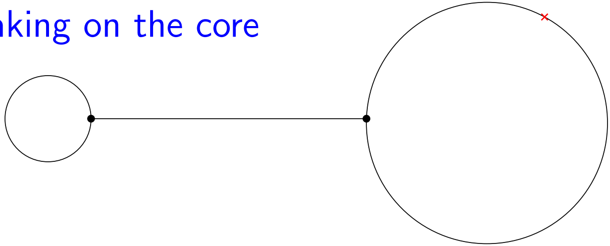

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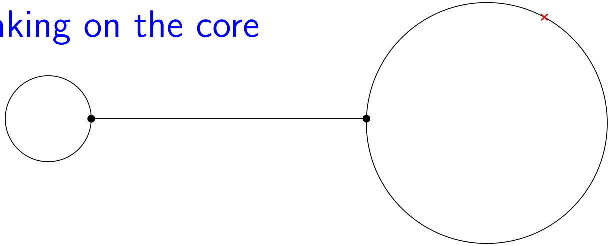

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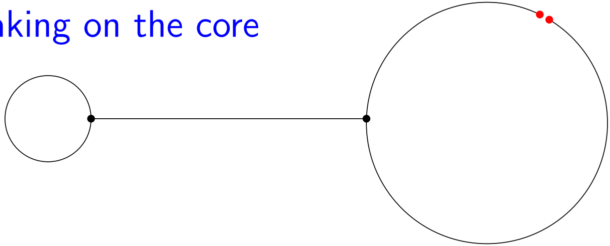

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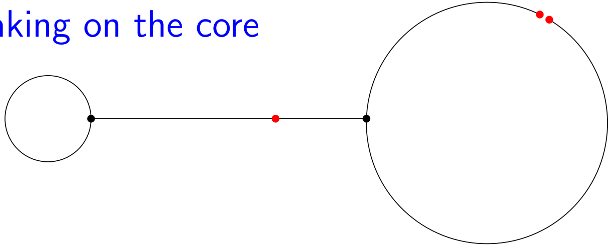

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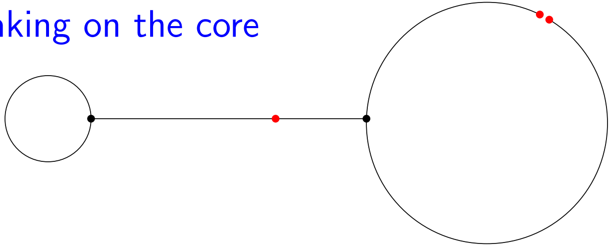

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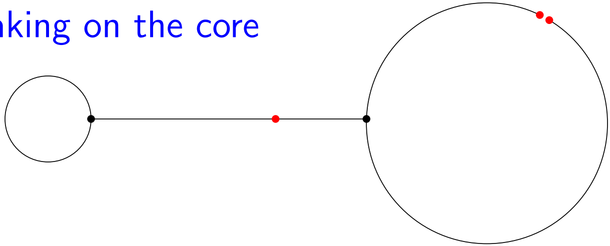

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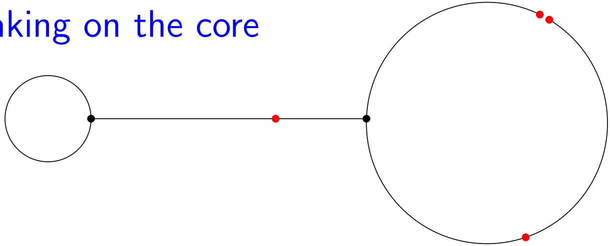

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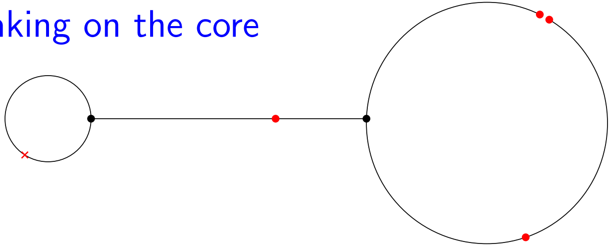

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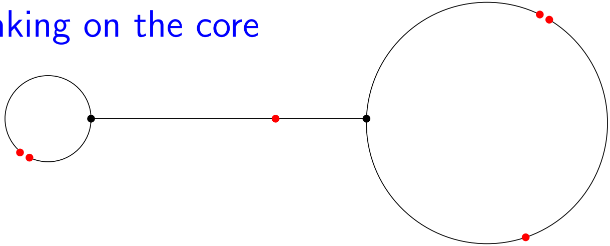

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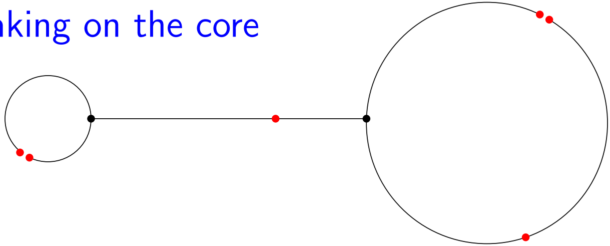

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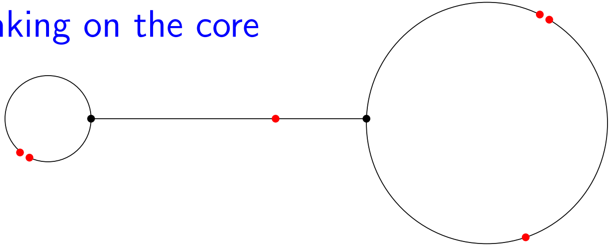

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Result: MST of a component of $G(n, 1/n)$ can be built (in the limit) by gluing randomly rescaled Brownian continuum random trees along the edges of a random discrete tree.

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Because $p = 1/n$ is the critical point for the Erdős-Rényi random graph, and due to the coupling between the random graph (percolation) process and Kruskal's algorithm, proving convergence for the minimum spanning trees of components of $G(n, 1/n)$, as sketched on the previous slide, is the main step in proving convergence of the full minimum spanning tree M_n to a limit $\mathcal{M}$.

The Annals of Probability
2017, Vol. 45, No. 5, 3075–3144
DOI: 10.1214/16-AOP1132

THE SCALING LIMIT OF THE MINIMUM SPANNING TREE OF THE COMPLETE GRAPH

BY LOUIGI ADDARIO-BERRY[1,2,*], NICOLAS BROUTIN[3,†], CHRISTINA GOLDSCHMIDT[2,4,‡] AND GRÉGORY MIERMONT[5,§]

McGill University, Inria Rocquencourt-Paris†, University of Oxford‡ and École Normale Supérieure de Lyon/Institut universitaire de France§*

Consider the minimum spanning tree (MST) of the complete graph with n vertices, when edges are assigned independent random weights. Endow this tree with the graph distance renormalized by $n^{1/3}$ and with the uniform measure on its vertices. We show that the resulting space converges in distribution as $n \to \infty$ to a random compact measured metric space in the Gromov–Hausdorff–Prokhorov topology. We additionally show that the limit is a random binary $\mathbb{R}$-tree and has Minkowski dimension 3 almost surely. In particular, its law is mutually singular with that of the Brownian continuum random tree or any rescaled version thereof. Our approach relies on a coupling between the MST problem and the Erdős–Rényi random graph. We exploit the explicit description of the scaling limit of the Erdős–Rényi random graph in the so-called critical window, established in [*Probab. Theory Related Fields* **152** (2012) 367–406], and provide a similar description of the scaling limit for a "critical minimum spanning forest" contained within the MST. In order to accomplish this, we introduce the notion of $\mathbb{R}$-graphs, which generalise $\mathbb{R}$-trees, and are of independent interest.

Universality of the minimum spanning tree limit. . .

. . . still an open problem in general!

That's not the whole history of our collaboration and friendship. . .

Oxford, 2014

INVERTING THE CUT-TREE TRANSFORM

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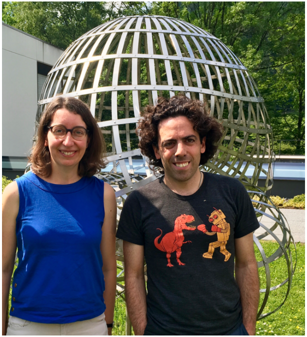
Abstract. We consider fragmentations of an $\mathbb{R}$-tree T driven by cuts arriving according to a Poisson process on $T \times [0, \infty)$, where the first co-ordinate specifies the location of the cut and the second the time at which it occurs. The genealogy of such a fragmentation is encoded by the so-called *cut-tree*, which was introduced by Bertoin and Miermont [17] for a fragmentation of the Brownian continuum random tree. The cut-tree was generalised by Dieuleveut [25] to a fragmentation of the α-stable trees, $\alpha \in (1, 2)$, and by Broutin and Wang [19] to the inhomogeneous continuum random trees of Aldous and Pitman [11]. In the first two cases, the projections of the forest-valued fragmentation processes onto the sequence of masses of their constituent subtrees yield an important family of examples of Bertoin's self-similar fragmentations [14]; in the first case the time-reversal of the fragmentation gives an additive coalescent. Remarkably, in all of these cases, the law of the cut-tree is the same as that of the original $\mathbb{R}$-tree.

In this paper, we develop a clean general framework for the study of cut-trees of $\mathbb{R}$-trees. We then focus particularly on the problem of *reconstruction*: how to recover the original $\mathbb{R}$-tree from its cut-tree. This has been studied in the setting of the Brownian CRT by Broutin and Wang [20], where they prove that it is possible to reconstruct the original tree in distribution. We describe an enrichment of the cut-tree transformation, which endows the cut-tree with information we call a *consistent collection of routings*. We show this procedure is well-defined under minimal conditions on the $\mathbb{R}$-trees. We then show that, for the case of the Brownian CRT and the α-stable trees with $\alpha \in (1, 2)$, the original tree and the Poisson process of cuts thereon can both be almost surely reconstructed from the enriched cut-trees. For the latter results, our methods make essential use of the self-similarity and re-rooting invariance of these trees.

Oberwolfach, 2017



Voronoi tessellations in the CRT and continuum random maps of finite excess

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Christina Goldschmidt¶

Abstract
Given a large graph G and k agents on this graph, we consider the Voronoi tessellation induced by the graph distance. Each agent gets control of the portion of the graph that is closer to itself than to any other agent. We study the limit law of the vector Vor $:= (V_1/n, V_2/n, ..., V_k/n)$, whose i'th coordinate records the fraction of vertices of G controlled by the i'th agent, as n tends to infinity. We show that if G is a uniform random tree, and the agents are placed uniformly at random, the limit law of Vor is uniform on the $(k-1)$-dimensional simplex. In particular, when $k = 2$, the two agents each get a uniform random fraction of the territory. In fact, we prove the result directly on the Brownian continuum random tree (CRT), and we also prove the same result for a "higher genus" analogue of the CRT that we call the continuum random unicellular map, indexed by a genus parameter $g \geq 0$. As a key step of independent interest, we study the case when G is a random planar embedded graph with a finite number of faces. The main idea of the proof is to show that Vor has the same distribution as another partition of mass Int $:= (I_1/n, I_2/n, ..., I_k/n)$ where I_j is the contour length separating the i-th agent from the next one in clockwise order around the graph.

closer to itself than to any other agent. We study the k-dimensional vector whose i'th coordinate records the fraction of vertices of G controlled by the i'th agent. More precisely, we will try to understand the law of this vector when the agents are placed uniformly at random on the graph G, and when the number of vertices of the graph becomes very large.

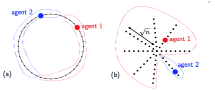
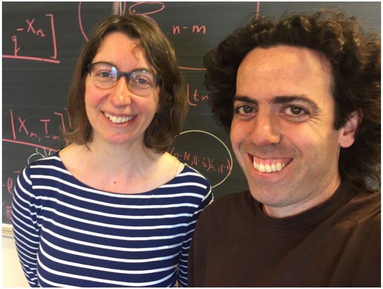


Figure 1: Examples of Voronoi competition between $k = 2$ agents on a deterministic graph. (a) On an n-cycle, each agent gets about half of the graph. (b) On a star made of $\sqrt{n}$ spikes, the winner takes almost all the graph.

Let us start with two instructive examples (Figure 1). If $k = 2$ and if G is an n-cycle, then, regardless

Montreal, 2018



MINIMIZING THE TIME TO A DECISION

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Suppose we have three independent copies of a regular diffusion on [0, 1] with absorbing boundaries. Of these diffusions, either at least two are absorbed at the upper boundary or at least two at the lower boundary. In this way, they determine a majority decision between 0 and 1. We show that the strategy that always runs the diffusion whose value is currently between the other two reveals the majority decision whilst minimizing the total time spent running the processes.

Some thoughts

- Talk to one another! You have something to offer that the other person doesn't.
- Don't be scared. (More realistically, recognize that it's normal to be intimidated by other mathematicians, and that many others in your immediate surroundings are probably feeling the same way.)
- Being exposed to math is good even if you don't understand it all. You can't know in advance which math will come in handy later. Knowing what's out there will help you recognize what tools you can use in your own work.
- Recognise the role of chance if you succeed!