# The front location for branching Brownian motion with decay of mass

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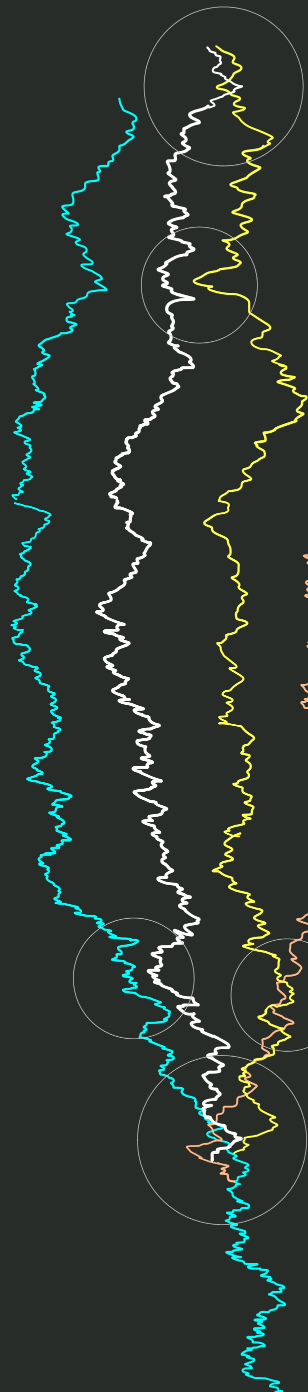

Duke, Nov 2015

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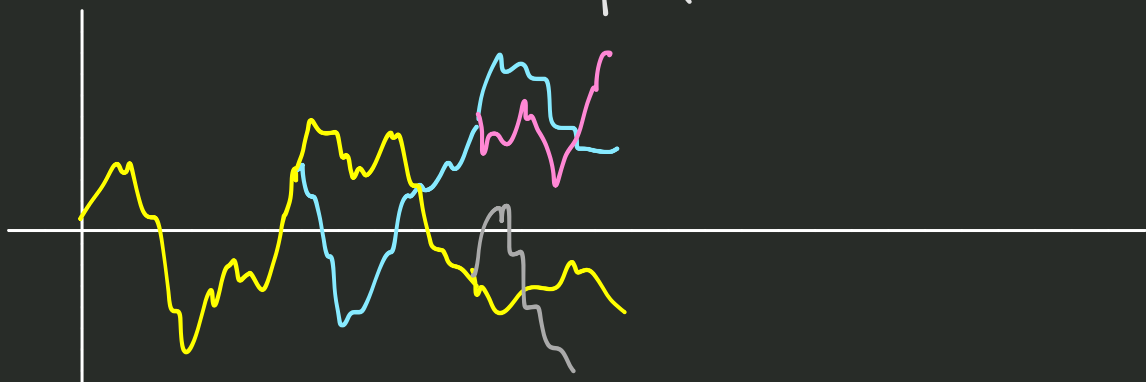
- Particles move as Brownian Motion
- Particles move and branch indep. from one-another.



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- Write $\mu(t,x) =$ expected density of particles at $x$

$$= \mathbb{E}[\#\{i : X_i(t) \in dx\}]$$

$$= e^t \, \mathbb{P}(X_1(t) \in dx)$$

$$= e^t \, \mathbb{P}(N(0,t) \in dx)$$

$$= e^t \cdot (2\pi t)^{-1/2} \cdot \exp(-x^2/2t)$$

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| |
|---|
| <u>Intuition</u>: (For some purposes): $e^t$ indep. Brownian Motions. |

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- Competition $\Rightarrow$ insufficient resources $\Rightarrow$ Loss of mass.
- Branching $\Rightarrow$ Mass doubles.

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- Parts of analysis work for:
  - $M_i(t+dt) = M_i(t) \cdot \left(2 - dt \cdot \sum_{\{j : |X_j(t) - x| \in [0,1)\}} M_j(s)\right)$
  - Branching $\longrightarrow$ Mass splits

# Main Result

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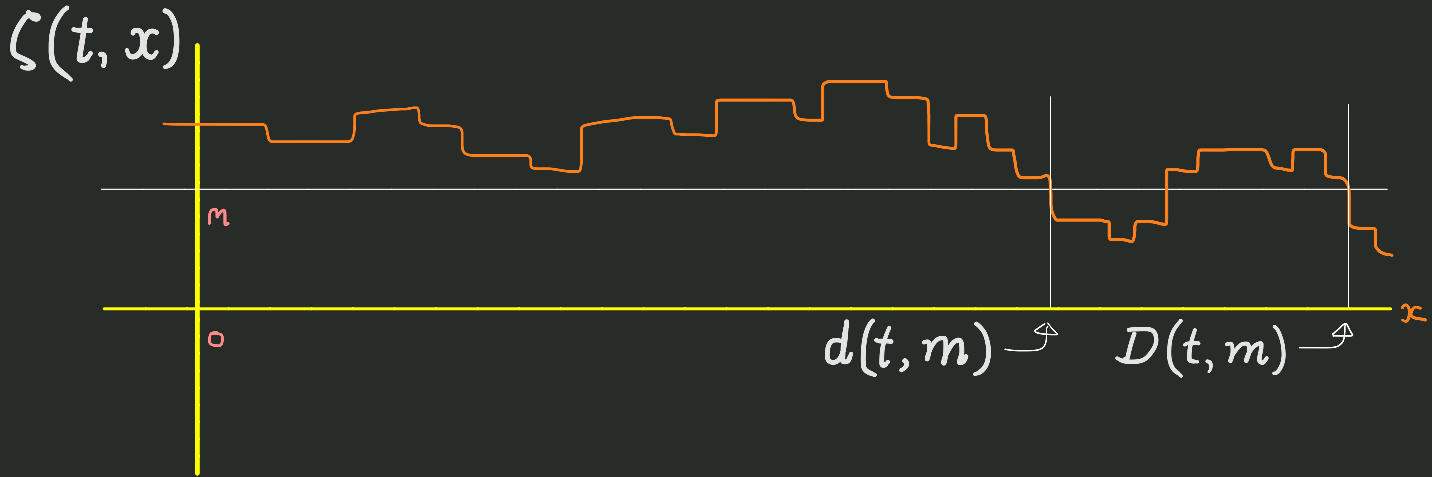

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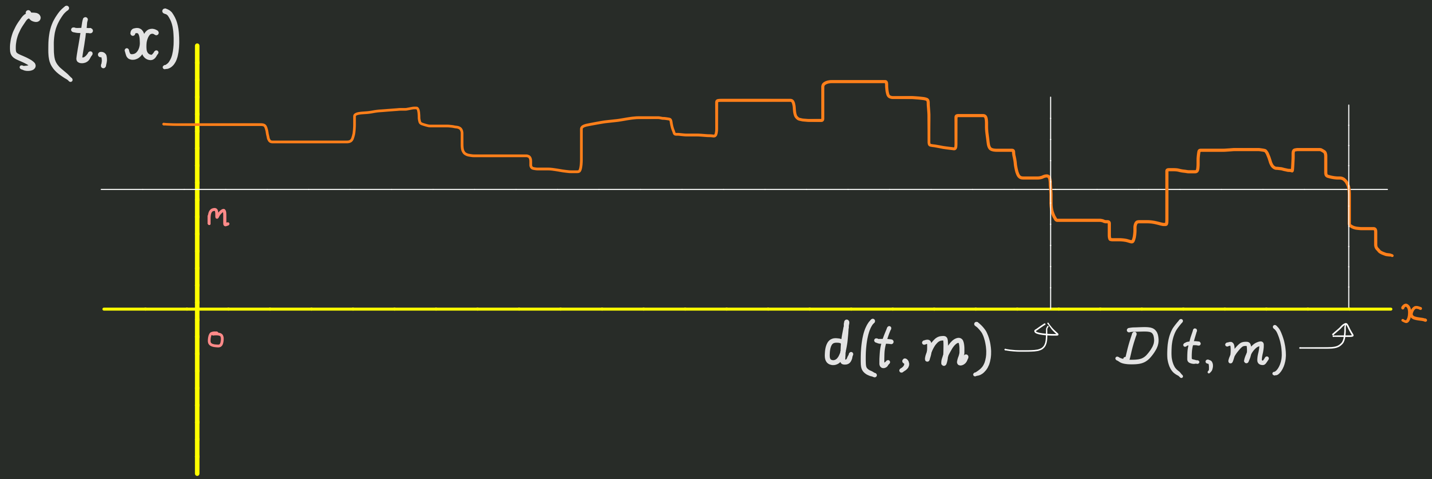
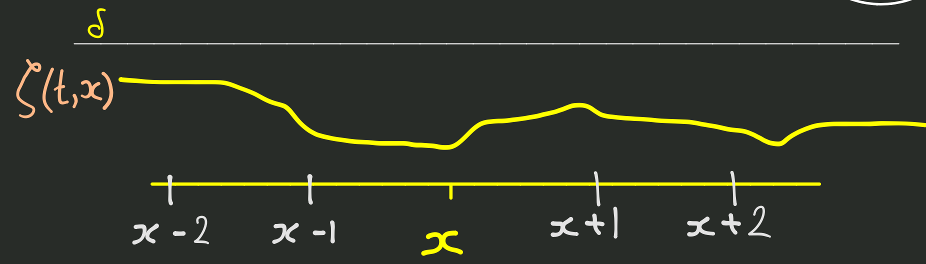



**Theorem** *There exists* $c \in (0,\infty)$ *such that a.s.* $\forall\, m < 1$,

$$\limsup_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} \geq c \qquad \liminf_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c$$

# Proof idea I. Density stabilizes quickly

$$\frac{d}{dt}\zeta(t,x) \cong \zeta(t,x) - \sum_{\{i : |X_i(t)-x| \in (0,1)\}} M_i(t)\cdot\zeta(t,X_i(t))$$



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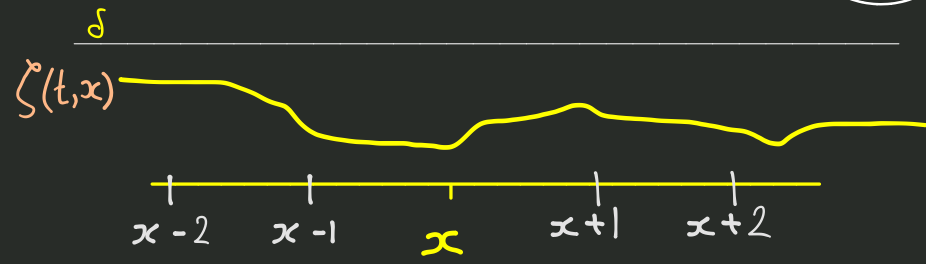


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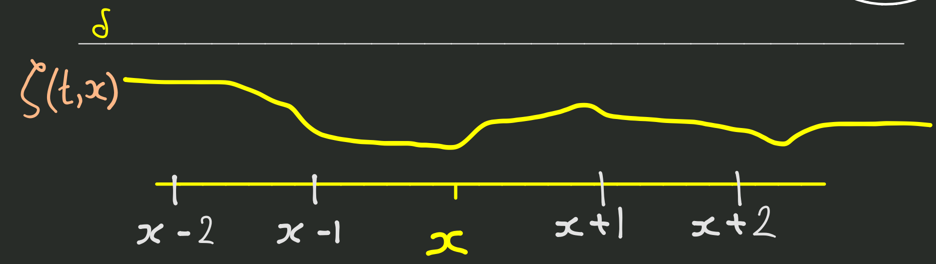


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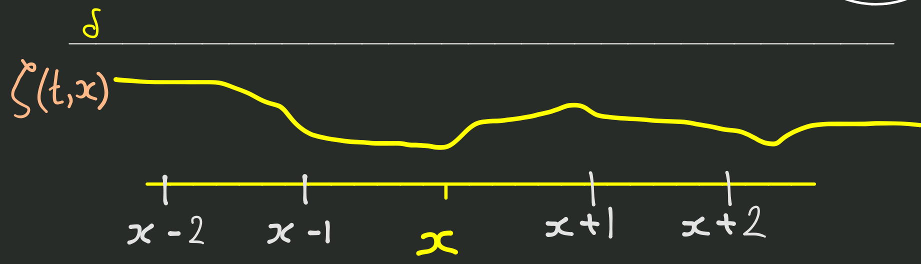


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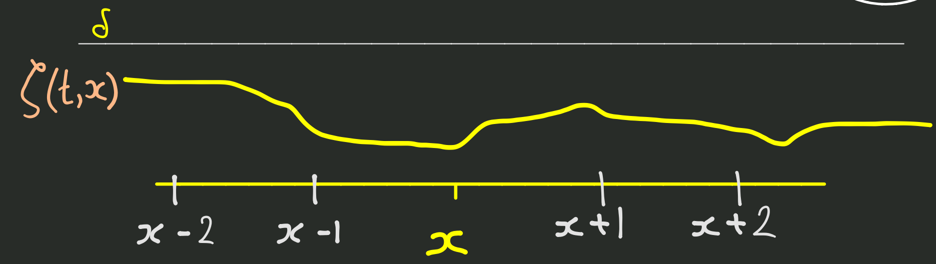


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Density $\delta < 1$ $\rightarrow$ Density $\sim 1$ in time $O(\log\frac{1}{\delta})$

Density $\Delta > 1$ $\rightarrow$ Density $\sim 1$ in time $O(1)$

# Proof idea

II. Proof by contradiction.

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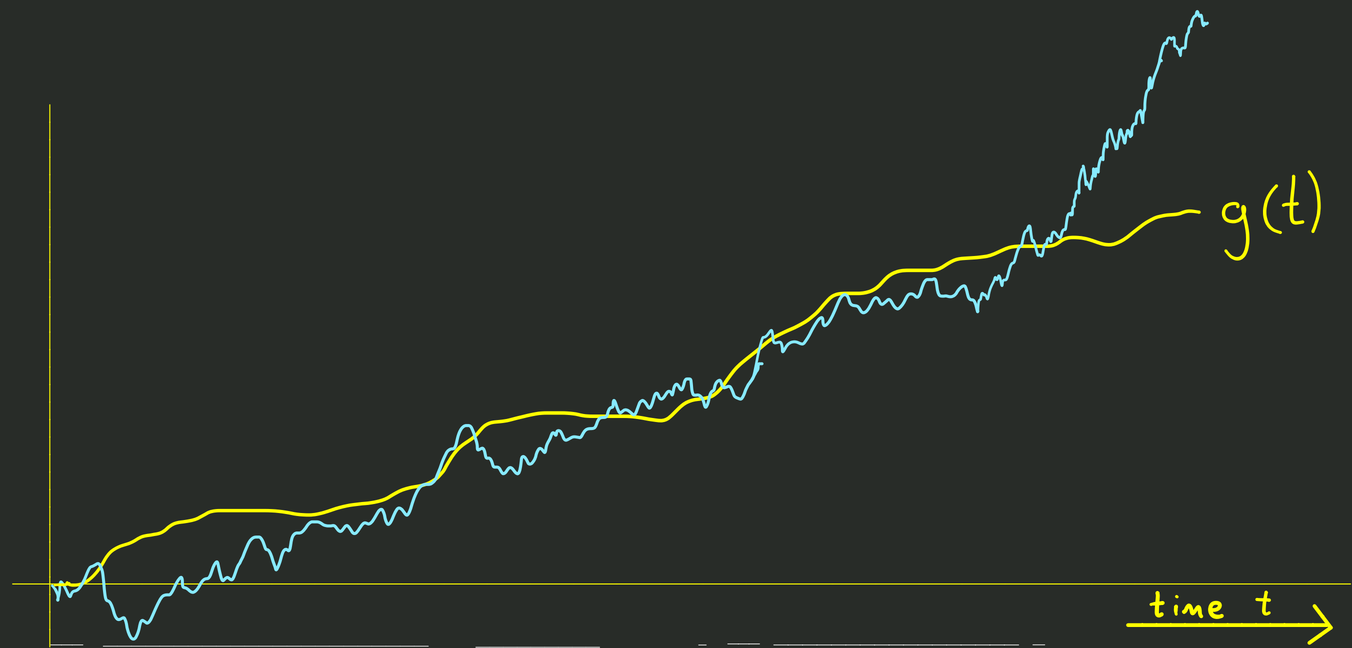
A particle $X_i(t)$ whose ancestral path $(X_{i,t}(s), s \leq t)$ has $\mathrm{Leb}(\{s : |X_{i,t}(s)| \leq d(s)\}) \geq t/2$ (a slowpoke) then has $M_t(i) \leq e^{-t/4}$.

# Proof idea  II.b). Self-correction

Write $g(t)$ for the "slowpoke threshold":

$g(t)$ deterministic, chosen s.t. w.h.p. $\forall\, i$ s.t. $X_i(t) \geq g(t)$

$$\mathrm{Leb}(\{s : |X_{i,t}(s)| \leq g(s)\}) \geq t/2$$



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If $d(s) \geq g(s)$ $\forall s \in (t/4, t)$ then

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In this case at time t

NEED $\exp(t/8)/2$ slowpokes to make $\sum_{\{i : X_i(t) \geq g(t)\}} M_i(t) > 1/2$.

## Proof idea III. Upper Bound.

Lemma: If can choose $g(t) = \sqrt{2}\,t - o(t)$ then

$\forall\, r > 0$ large $\exists\, s \in [r/4, r]$ s.t. $d(s) \leq g(s) + 1$

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**Proof:**

Suppose $g(t)=\sqrt{2}\,t-o(t)=v_t-o(t)$.

Then $\#\{i : X_i(t)\geq g(t)\}=\exp\left((1+o(1))\sqrt{2}\cdot(v_t-g(t))\right)=e^{o(t)}$.

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If $d(s) \geq g(s) + 1 \;\; \forall s \in (r/4, r)$ then $\forall |x| \geq g(r) + 1$

$$\mathcal{J}(r, x) \leq \sum_{\{i :\, X_i(r) \geq g(r)\}} M_i(r)$$

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$$f(r,x) \leq \sum_{\{i : X_i(r) \geq g(r)\}} M_i(r) \leq e^{-r/8} \cdot \#\{i : X_i(r) \geq g(r)\}$$

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$$\approx e^{-r/8} \cdot e^{o(r)} = o(1).$$

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$\forall\, r > 0$ large $\exists\, s \in [r/4, r]$ s.t. $d(s) \leq g(s) + 1$

**Proof:**

Suppose $g(t) = \sqrt{2}\,t - o(t) = v_t - o(t)$.

Then $\#\{i : X_i(t) \geq g(t)\} = \exp\big((1+o(1))\sqrt{2}\cdot(v_t - g(t))\big) = e^{o(t)}$.

If $d(s) \geq g(s) + 1 \;\; \forall s \in (r/4, r)$ then $\forall |x| \geq g(r) + 1$

$$\mathcal{J}(r,x) \leq \sum_{\{i : X_i(r) \geq g(r)\}} M_i(r) \leq e^{-r/8} \cdot \#\{i : X_i(r) \geq g(r)\}$$

$$\approx e^{-r/8} \cdot e^{o(r)} = o(1).$$

So $d(r) \leq g(r) + 1$ $\blacksquare$

## Proof idea III. Upper Bound.

Lemma: If $g(t) = \sqrt{2}\,t - o(t)$ then $\forall\, r > r_0\ \exists\, s \in [r/4, r] : d(s) \leqslant g(s) + 1$

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**Proof**: Let $B = (B(t), t \geq 0)$ be Brownian motion.

**Easy**: $\mathbb{P}\left(|B_t| < 1,\ \mathrm{Leb}\{s \in [0,t] : |B_s| < 1\} > t/2\right) = e^{-\Theta(t)}$.

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**Conclusion:**
$$\mathbb{E}\left(\#\{i : |X_i(t) - \sqrt{2}\,t| < \ell,\ X_{i,t}(s) \geqslant \sqrt{2}\,s - \ell \text{ for time} \geqslant t/2\}\right) \approx e^{\sqrt{2}\,\ell} \cdot e^{-\Theta(t/\ell^2)}$$

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$$= \Theta(1) \text{ when } \ell \approx t/\ell^2 \ ; \ \ell \approx t^{1/3}$$

Proof idea IV: Lower bound.

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- $\exists\, s < t$ s.t. $D(s, \frac{1}{2t}) \geq \sqrt{2}\, s - O(s^{1/3})$
- or: $D(t, \frac{1}{2}) \geq \sqrt{2}\, t - O(t^{1/3})$

# Proof idea IV: Lower bound.

**Strict fastpoke threshold:**

$h(t)$ s.t. whp $\forall t$ $\exists i$ s.t. $\forall s \leq t$ $X_{i,t}(s) \geq h(s)$.

If $D(s) < h(s)$ $\forall s < t$ then any fastpoke has mass $\geq e^{-t/2}$.

If $D(s, \frac{1}{2t}) < h(s)$ $\forall s < t$ then any fastpoke has mass $\geq 1/2$

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Now use that density stabilizes quickly to go from $D(s, \frac{1}{2t})$ to $D(s + O(\log t), \frac{1}{2})$

Thank you!