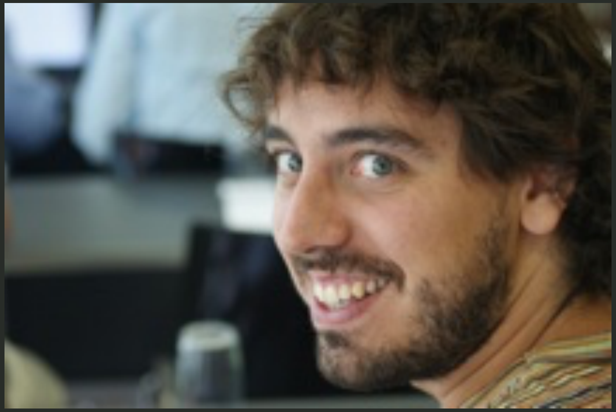
Extreme Statistics of Random Digraphs

Louigi Addario-Berry
McGill / Oxford



Borja Balle
McGill/Lancaster

Guillem Perarnau
McGill/Birmingham

Oxford, May 2015

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r-out digraph $D(n,r)$

Vertices $[n] := \{1, 2, \ldots, n\}$

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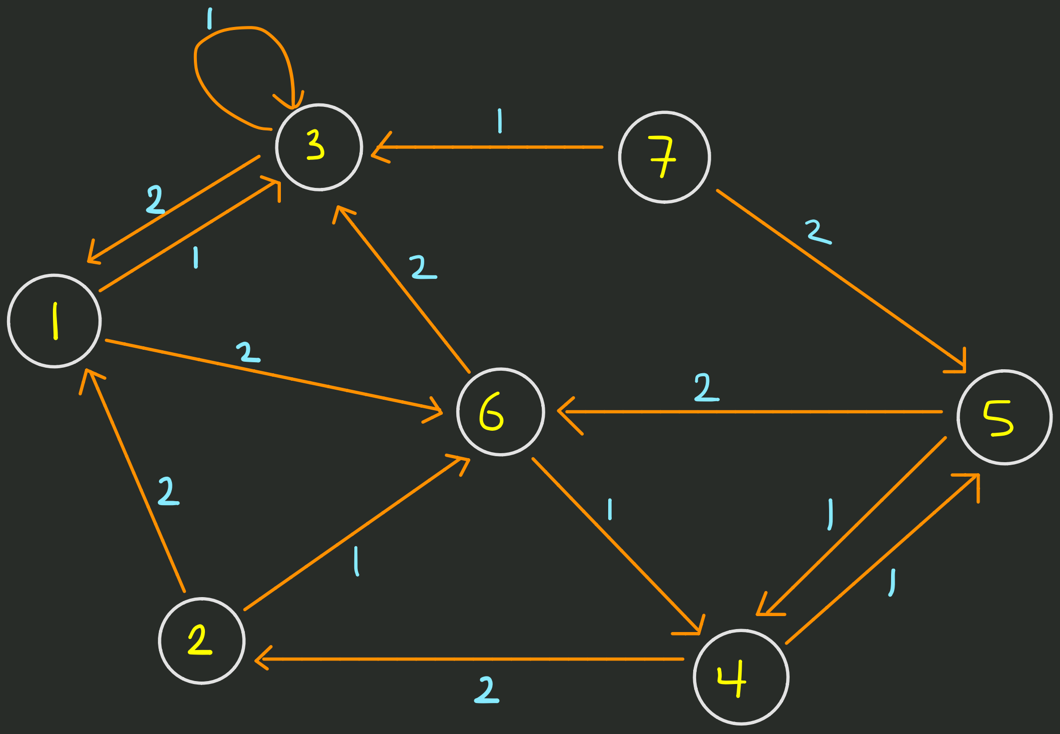

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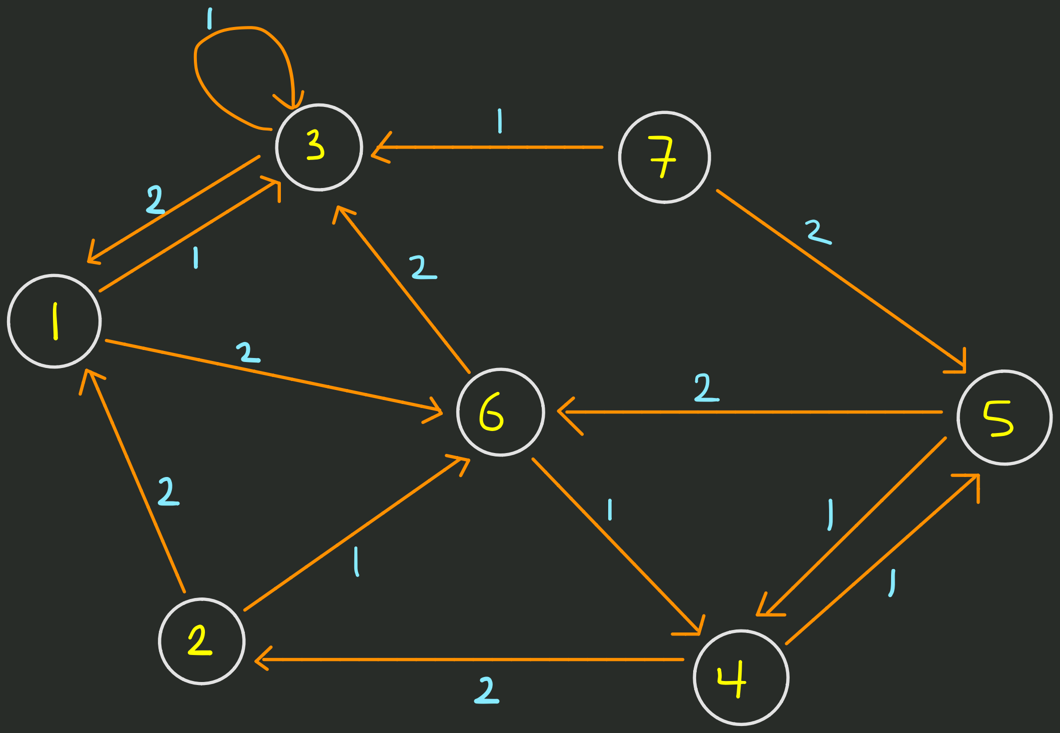

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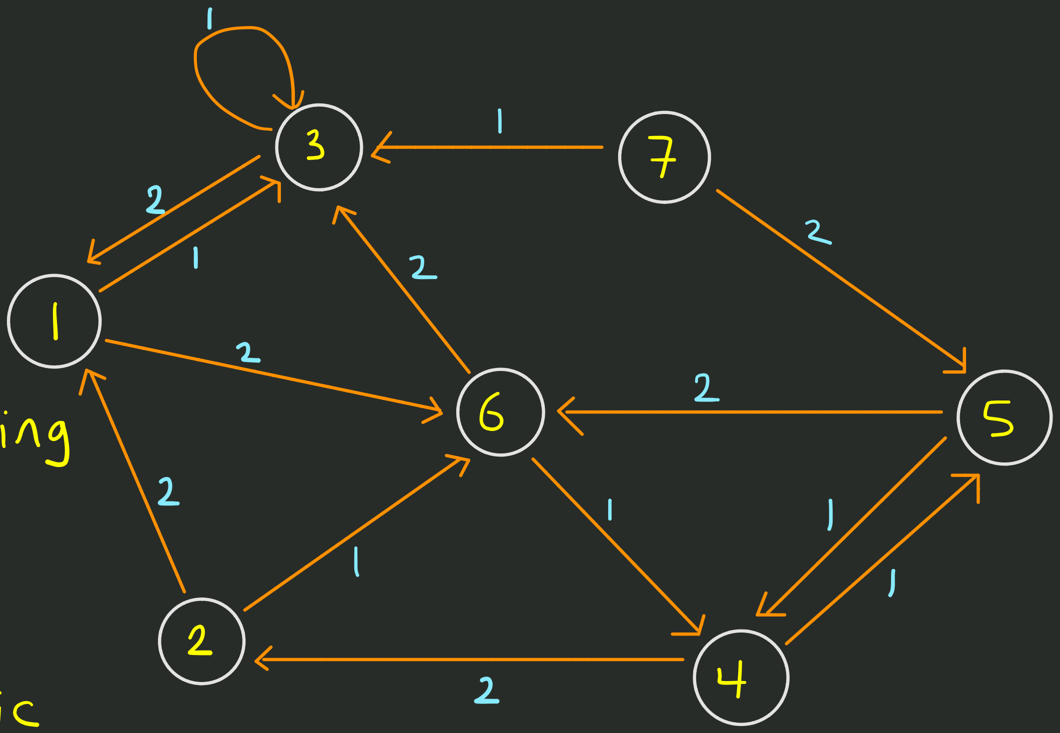
Motivation:

① Probabilistic Machine Learning of Deterministic Finite Automata/Regular Languages

② Directed graphs more realistic than undirected! We should be studying them.



Results 1: Diameter

Def: For a digraph D and $u,v \in D$,

$$\text{dist}(u,v) = \text{min \# edges on directed } u\text{-}v \text{ path}$$
$$= \infty \text{ if no } uv \text{ path.}$$

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Theorem (A-B, Balle, Perarnau 2015):

As $n \to \infty$,

$$\mathrm{diam}(D(n,r)) = (1 + c_r + o_p(1)) \log_r(n)$$

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where $c_r = (\log r)/(s r - \log r)$,

$s = \mathbb{P}(\text{Poisson}(r) \text{ Galton-Watson proc. survives})$

$1 - s = e^{-rs}$

Random walks on graphs

Simple random walk on an undirected graph

$G=(V,E) \longrightarrow \deg(v) = \#$ neighbours of $v = \#\{w : \{v,w\} \in E\}$

$X = (X_i, i \geq 0)$ Markov chain,

$$p_{vw} = \mathbb{P}(X_{i+1}=w \mid X_i = v) = \begin{cases} 1/\deg(v) & \text{if } \{v,w\} \in E \\ 0 & \text{otherwise} \end{cases}$$

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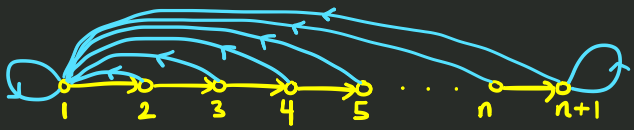
⚠ No simple description. Natural guess: $\pi(v) = \deg^-(v)/|E|$.

(Counter) Example:



Guess: $\pi(1) = \frac{1}{2}$, $\pi(i) = \frac{1}{2n}$, $i \geq 2$

Truth: $\pi(n+1) = 2^{-n}$, $\pi(i) = 2^{-i}$, $i \leq n$

Results 2: Stationary distribution

Proposition (Grusho, 1973): The largest component $D_0(n,r)$ of $D(n,r)$ is with high probability the unique recurrent component.

Corollary: Simple random walk has unique stat. dist π whp.

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i.e. $\forall \varepsilon > 0$, $\mathbb{P}(\pi_{max} > n^{-1+\varepsilon} \text{ or } \pi_{max} < n^{-1-\varepsilon}) \to 0$

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Remark: C_r same as in theorem on diameter

A People's History of the Diameter of Random Graphs

$D(n,r)$ (1973) Trakhtenbrot & Bardin: $\mathrm{diam}(D(n,r)) = O_p(\log n)$

$G(n,r)$ (1982) Bollobás & de la Vega: $\mathbb{P}(|\mathrm{diam}(G(n,r) - \log_{r-1}(\log n)| \in [-5,5]) \to 1.$

$G(n,\frac{\lambda}{n})$ (1974) Burtin: $\lambda(n)/\log n \to \infty$: Two-point concentration.

$$\mathbb{P}(\mathrm{diam}(G(n,\tfrac{\lambda}{n}) \in \{\lfloor \log_\lambda n \rfloor, \lfloor \log_\lambda n \rfloor + 1\}) \to 1$$

(1981) Bollobas: $\lambda(n) \geq \log^3 n$: "Explicit" two point concentration.

$$\mathbb{P}(\mathrm{diam}(G(n,\tfrac{\lambda}{n}) = \lfloor \log_\lambda n \rfloor) = p_1(n), \quad \mathbb{P}(\mathrm{diam}(G(n,\tfrac{\lambda}{n}) = \lfloor \log_\lambda n \rfloor + 1) = p_2(n),$$

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(2010) Riordan & Wormald: $G(n,\frac{\lambda}{n})$

$\lambda > 1$ fixed: $\mathrm{diam}(G(n,\lambda/n)) = (1 + 2c_\lambda)\log_\lambda n + O_p(1)$

$\lambda \to \infty$: 2-point concentration.

$\lambda(n) = 1 + o(1)$: $\mathrm{diam}(G(n,\lambda/n)) = d(n) + O_p(\frac{1}{\lambda - 1})$

$$d(n) = (1 + o(1)) \frac{3}{\lambda - 1} \log((\lambda - 1)^3 n).$$

(see also Ding-Kim-Lubetzky-Peres 2010)

Random edge weights

- Bhamidi & van der Hofstad → Conv. in dist., complete graph
- Amini & Lelarge → $G(n,\vec{r})$, $C \log n + o_p(\log n)$
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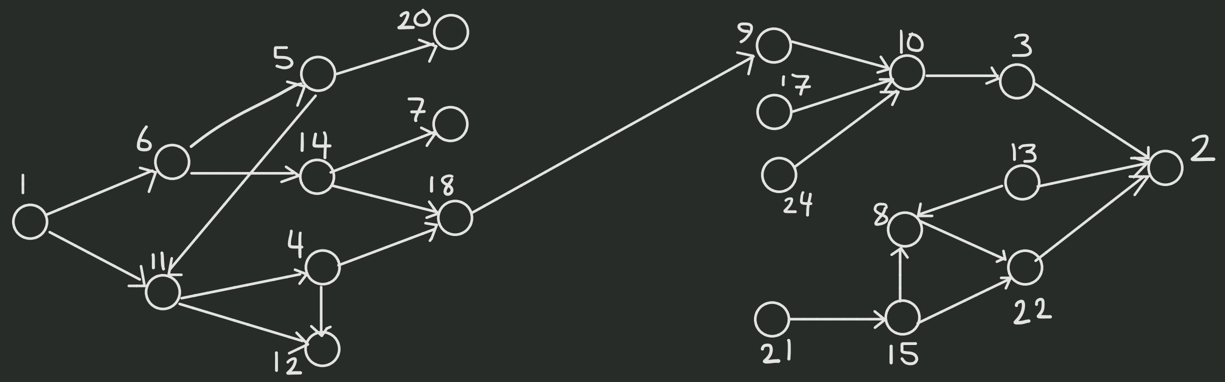

Aim: Find a shortest path from 1 to 2.

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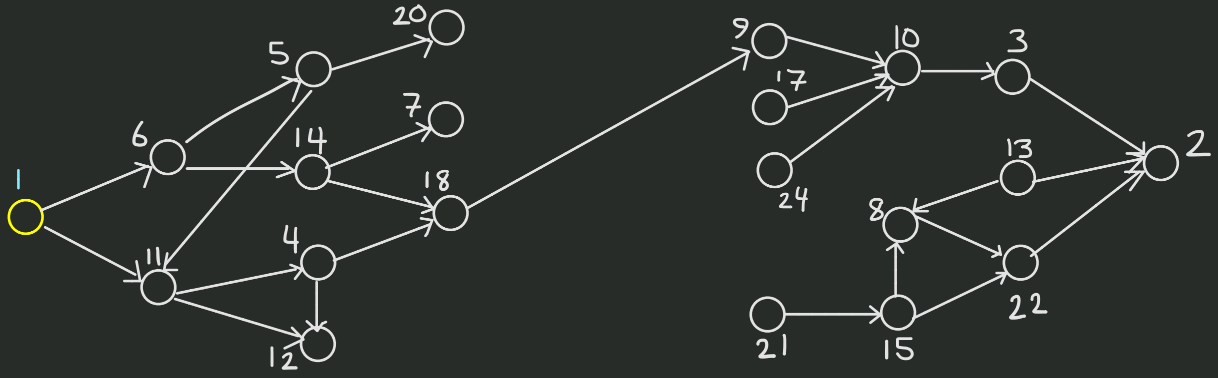
Outward



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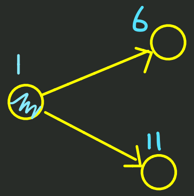

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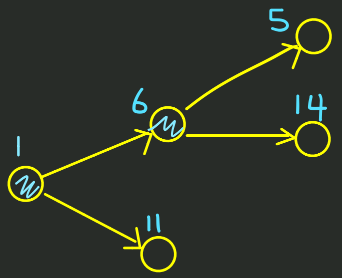

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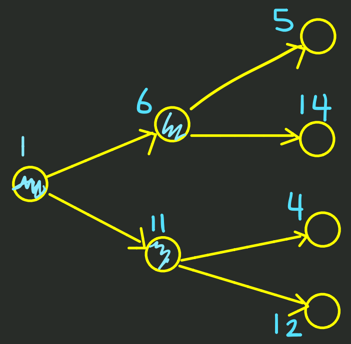

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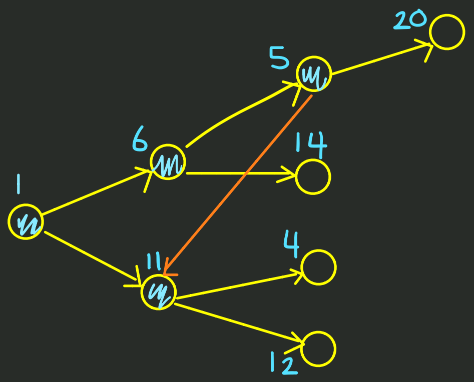

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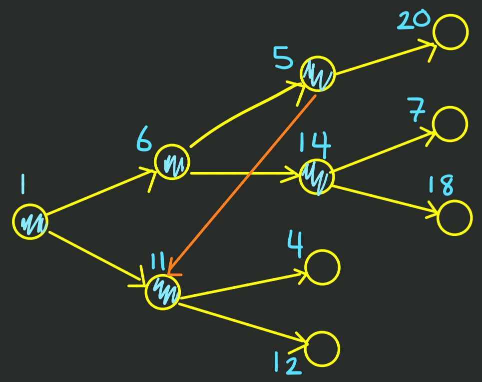

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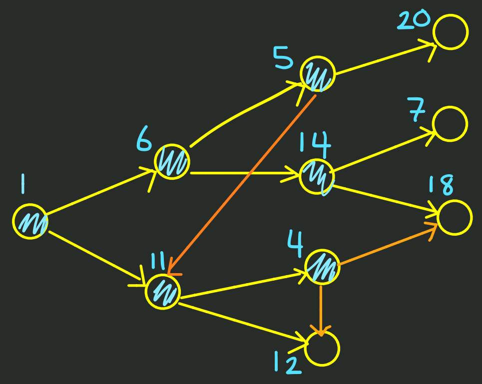

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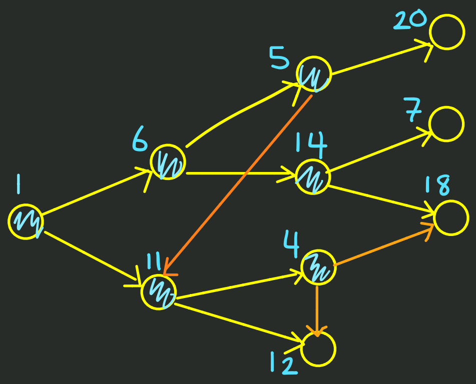

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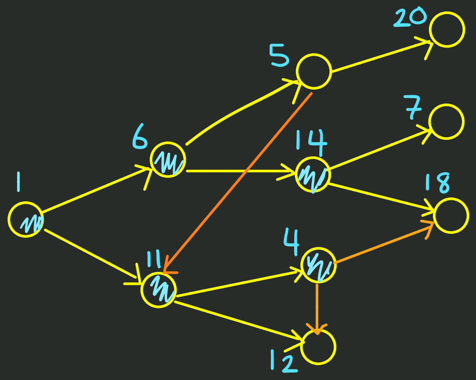

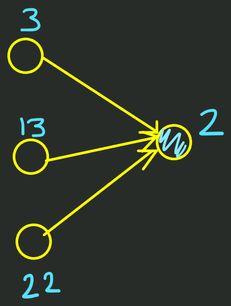

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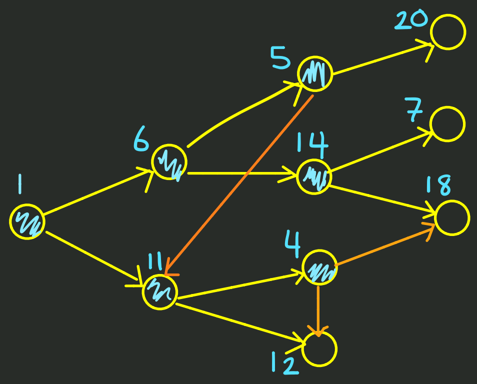

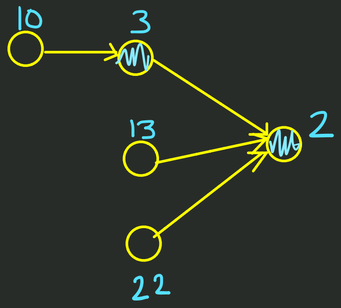

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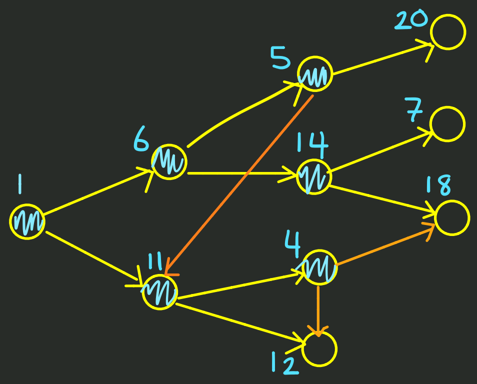

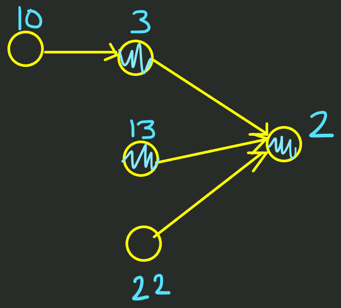

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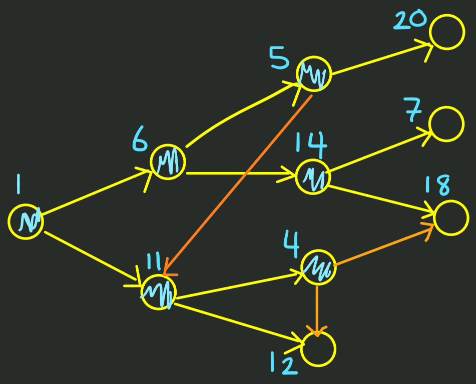

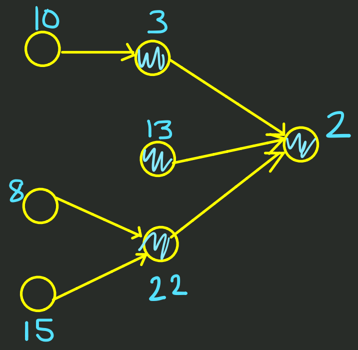

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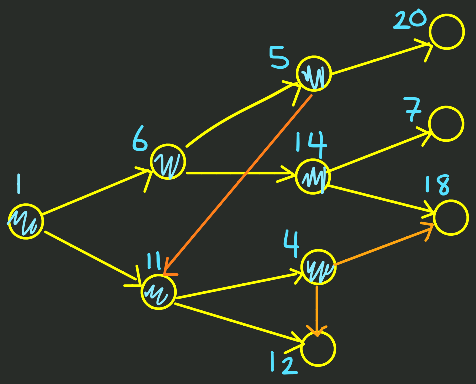

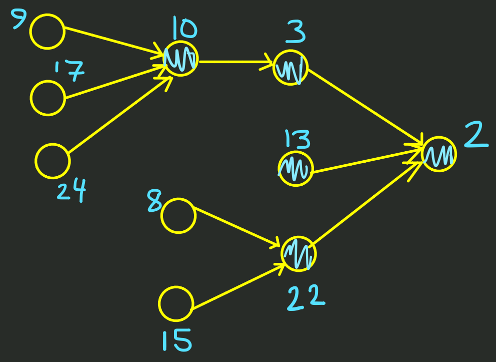

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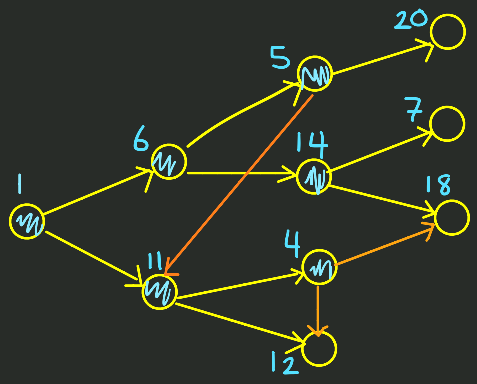
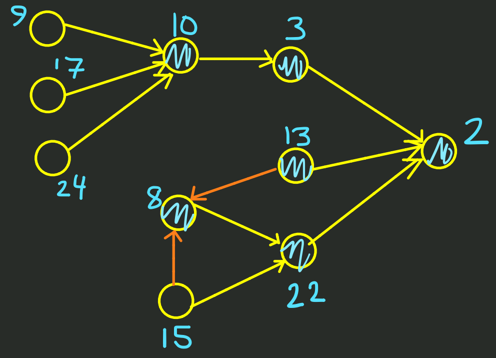
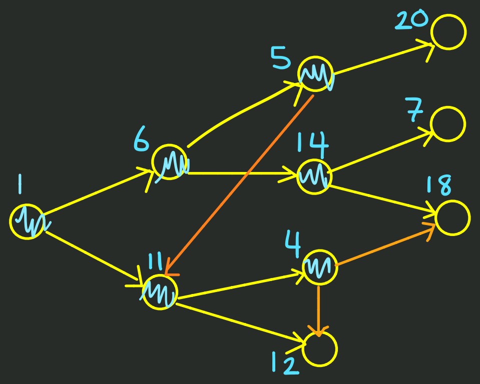


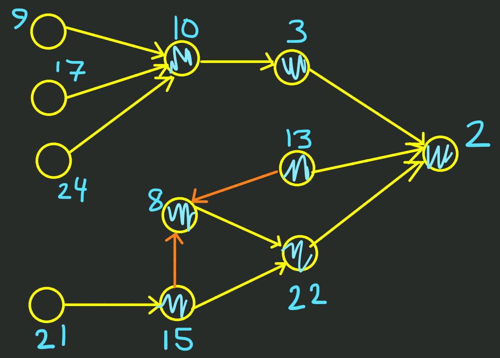


Inward

Breadth-first search

Outward





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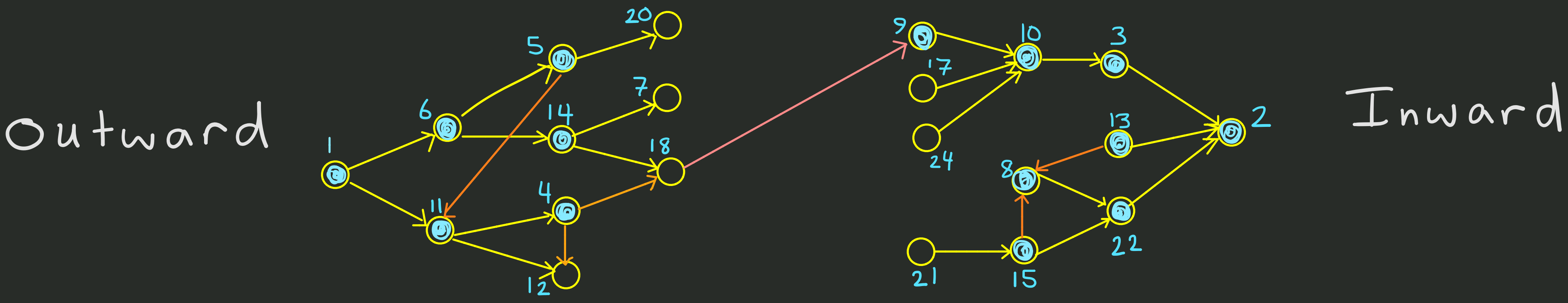
Breadth-first search



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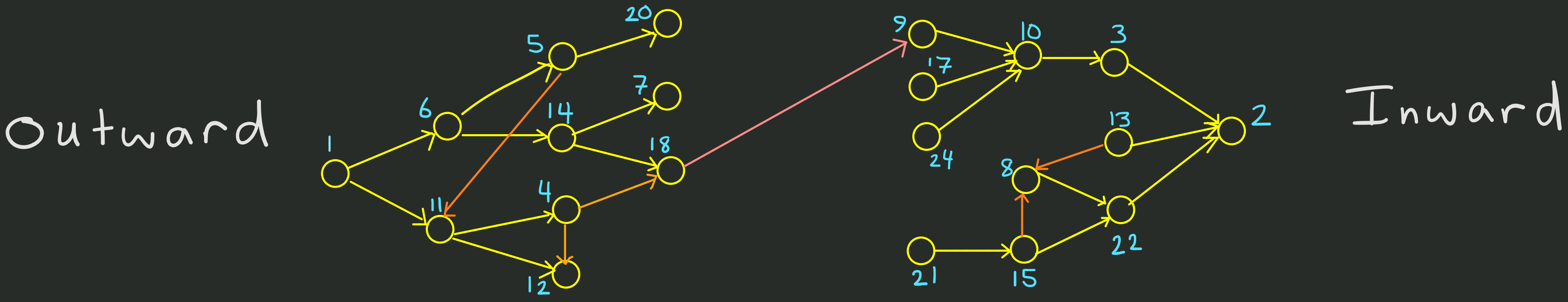
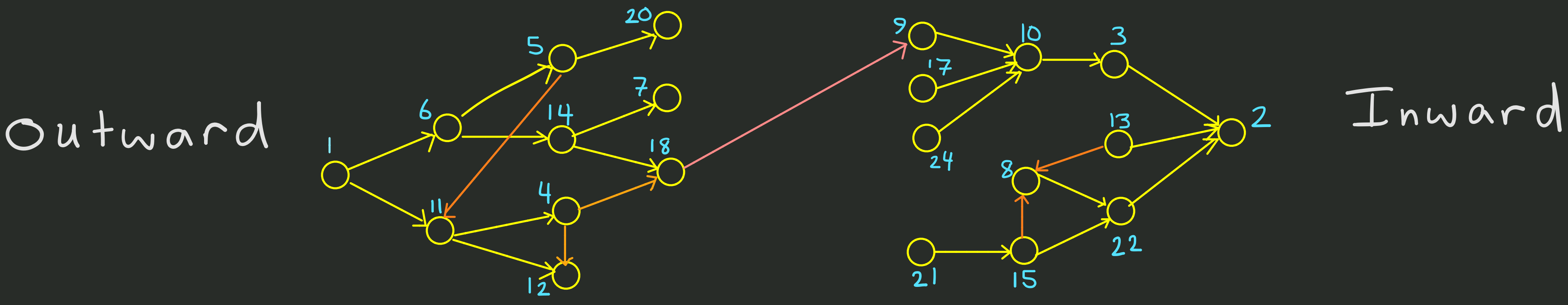


Observation : $\boxed{\mathrm{dist}(u,v) = \min\{\, j+k : N_j^+(u) \cap N_k^-(v) \neq \phi \,\}}$

Breadth-first search



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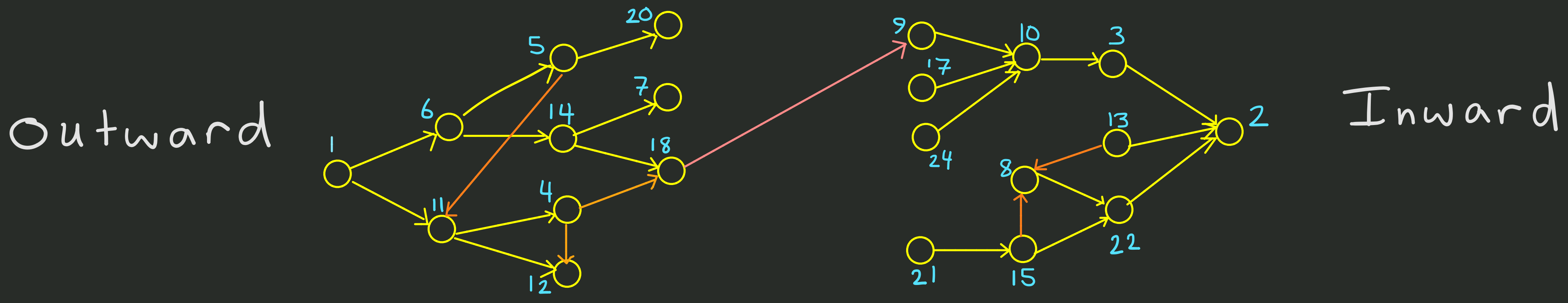
Heuristics:

1) oBFS looks like BFS on complete r-ary tree
2) iBFS looks like BFS on a Poisson(r) Galton-Watson tree.
3) out-nbhds much more regular than in-nbhds so in-nbhds cause any fluctuations in distances.

Breadth-first search



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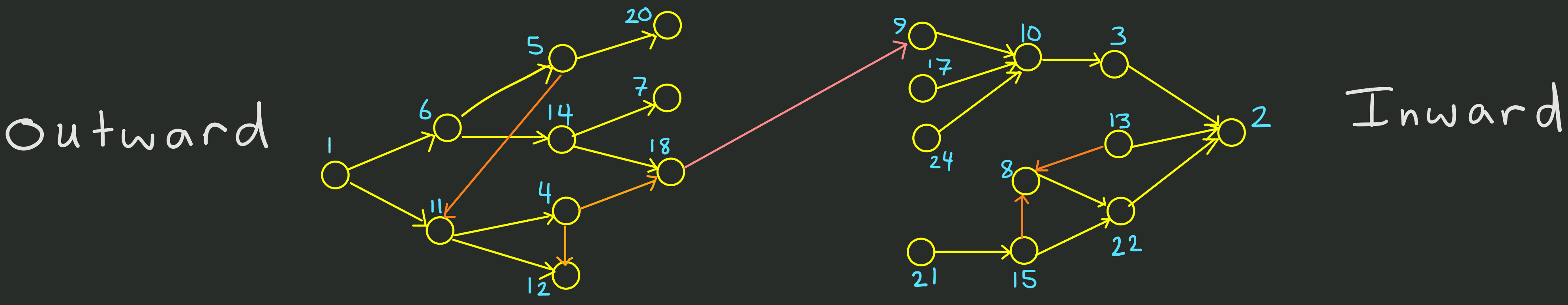
$\boxed{\text{For node } u, \text{ let } k(u) = \min\{ k : |N_k^-(u)| \geq \log^4 n \}}$

Prop: $\mathrm{diam}(D(n,r)) = \max_u k(u) + (1+o_p(1)) \log_r n$

Breadth-first search



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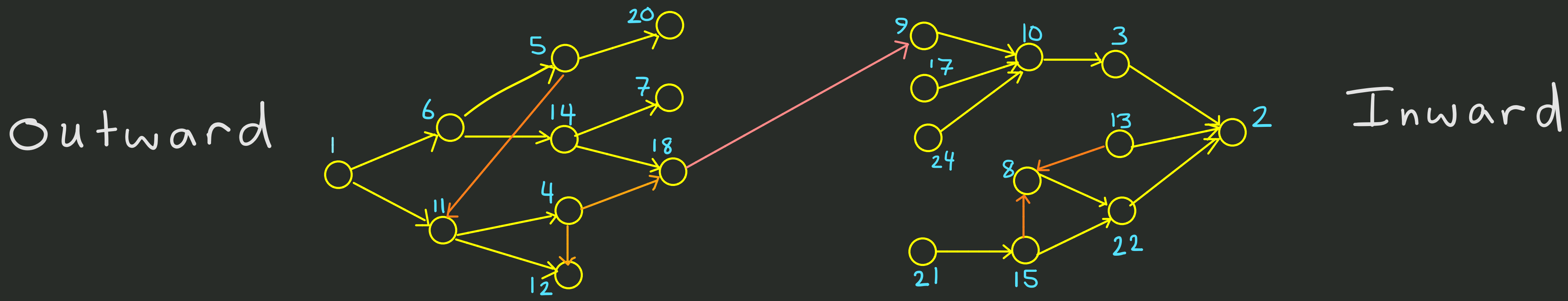
Proof idea:

If $|N_j^+(v)| \simeq r^j$ then $\mathbb{P}(N_j^+(v) \cap N_{k(u)}^-(u) \neq \phi) \simeq \left(1 - \frac{\log^4 n}{n}\right)^{r^j} \begin{cases} \simeq 1 \text{ when } r^j \ll \frac{n}{\log^4 n} \\ \simeq 0 \text{ when } r^j \gg \frac{n}{\log^4 n} \end{cases}$

Breadth-first search



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So first have $N_j^+(v) \cap N_{k(u)}^-(u) \neq \phi$ when $r^j \simeq \frac{n}{\log^4 n}$; $j = \log_r n - \log_r(\log^4 n)$

At first I was afraid...

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First try to understand a single vertex u.

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Idea (Athreya & Ney; Kingman; McDiarmid; Riordan & Wormald; Berestycki, Gantert, Mörters & Sidorova; ...)

Long-term growth is due to surviving lineages.

Let $T \sim$ Poisson(r) GW, root ρ

$$\boxed{\mathbb{P}(T = \infty) = s \qquad 1-s = e^{-rs}}$$

Say $v \in T$ survives if v has ∞ many descendants.

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$\mathbb{P}(\rho$ has a surviving children$) = \mathbb{P}(\text{Poisson}(rs) = a)$

$\mathbb{P}(\rho$ has a surviving children $| \rho$ survives$) = s^{-1} \cdot \mathbb{P}(\text{Poisson}(rs) = a)$

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$$\begin{aligned} \mathbb{P}(\rho \text{ has } 1 \text{ surviving child} \mid \rho \text{ survives}) &= s^{-1}\, \mathbb{P}(\text{Poisson}(rs) = 1) \\ &= s^{-1} \cdot (rs e^{-rs}) = r e^{-rs} \end{aligned}$$

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$k(u) = \min\{k : |N_k^-(u)| \geq \log^4 n\}$ $\mathrm{diam}(D(n,r)) = \max_u k(u) + (1+o_p(1)) \log_r n$

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$\mathbb{P}(\rho$ has 1 surviving child $| \rho$ survives$) = s^{-1} \mathbb{P}(\text{Poisson}(rs) = 1)$

$= s^{-1} \cdot (rse^{-rs}) = re^{-rs}$

So $\mathbb{P}($exactly 1 survivor in gen $g) = \mathbb{P}(1$ survivor in gen $1)^g = (re^{-rs})^g$

$k(u) = \min\{k : |N_k^-(u)| \geq \log^4 n\}$ $\qquad$ $\mathrm{diam}(D(n,r)) = \max_u k(u) + (1+o_p(1))\log_r n$

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$$\mathbb{P}(\text{exactly } 1 \text{ survivor in gen } g) = (r e^{-rs})^g \cong \mathbb{P}(\text{exactly } 1 \text{ in}\underline{\text{dividual}} \text{ in gen. } g)$$

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$\mathbb{P}(\text{exactly 1 survivor in gen } g) = (r e^{-rs})^g \overset{\smile}{=} \mathbb{P}(\text{exactly 1 individual in gen. } g)$

Back to in-neighbourhoods of u.

Idea: (Riordan-Wormald) For $k-t$ large, least $\underbrace{\text{costly}}_{\text{unlikely!}}$ way to have $0 < N_k^-(u) < r^t$ is to have $N_{k-t}^-(u) = 1$.

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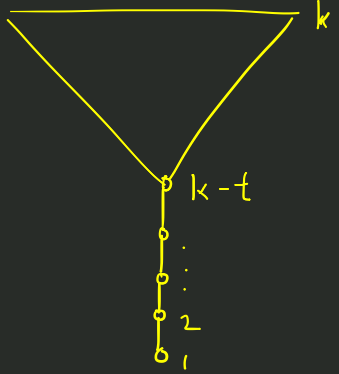

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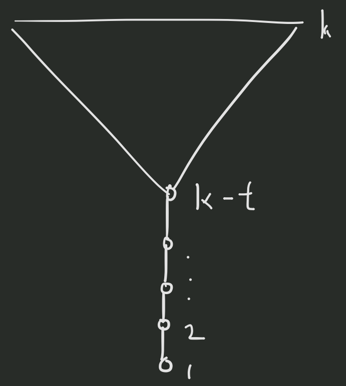
Cost: $\mathbb{P}(0 < |N_k^-(u)| < r^t) \simeq \mathbb{P}(|N_{k-t}^-(u)| = 1,\ 0 < N_k^-(u) < r^t)$

$$\simeq \mathbb{P}(|N_{k-t}^-(u)| = 1,\ N_k^-(u) > 0)$$

$$\simeq \mathbb{P}(|N_{k-t}^-(u)| = 1) \cdot \mathbb{P}(\mathrm{Po}(r)\ \mathrm{GW}\ \text{survives})$$

$$\simeq (r e^{-rs})^{k-t} \cdot s$$



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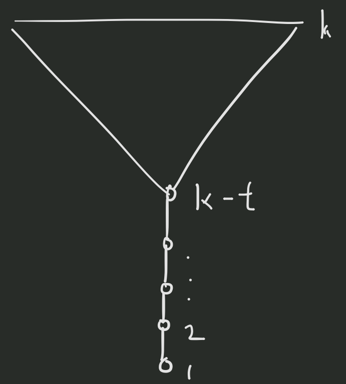
\mathbb{P}(0 < |N_k^-(u)| < r^t) &\cong \mathbb{P}(|N_{k-t}^-(u)| = 1,\ 0 < N_k^-(u) < r^t) \\
&\cong \mathbb{P}(|N_{k-t}^-(u)| = 1,\ N_k^-(u) > 0) \\
&\cong \mathbb{P}(|N_{k-t}^-(u)| = 1) \cdot \mathbb{P}(\mathrm{Po}(r) \text{ GW survives}) \\
&\simeq (r e^{-rs})^{k-t} \cdot s
\end{aligned}
$$



<u>Prop</u> For $k-t$ large and k, t small $\leftarrow (k \cdot 2^t = o(n^{1/2}))$ enough that GW approximation is good,

$$\mathbb{P}(0 < |N_k^-(u)| < r^t) = \Theta(1) \cdot (r e^{-rs})^{k-t}$$

$k(u) = \min\{k : |N_k^-(u)| \geq \log^4 n\}$ $\qquad$ $\mathrm{diam}(D(n,r)) = \max_u k(u) + (1+o_p(1)) \log_r n$

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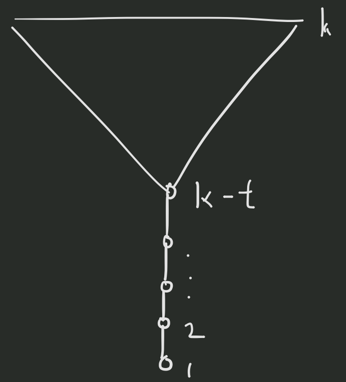

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Here: $r^t = \log^4 n$. Then $(r e^{-rs})^{k-t} = \frac{1}{n}$ when $k = c_r \log_r n + O(\log\log n)$

Bounding the diameter: wrap-up.

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So if $k > c_r \log_r n + C\log\log n$ then $\mathbb{E}[\#\{u : k(u) \geq k\}] \longrightarrow 0$

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First moment method $\Rightarrow$ $k(u) \leq c_r \log_r n + O(\log\log n)$ whp

Second moment method $\Rightarrow$ $k(n) \geq c_r \log_r n - O(\log\log n)$ whp

$\downarrow$

Count $\mathbb{E}[\#$ pairs u,v s.t. $k(u) \geq k,\ k(v) \geq k]$

Bounding Π_{max} and Π_{min}

View $N^-_{\leq k}(u)$ as a maze.

(Treasure at u.)

Maze is h-hard if every path from $N^-_k(u)$ to u has $\geq h$ traps.

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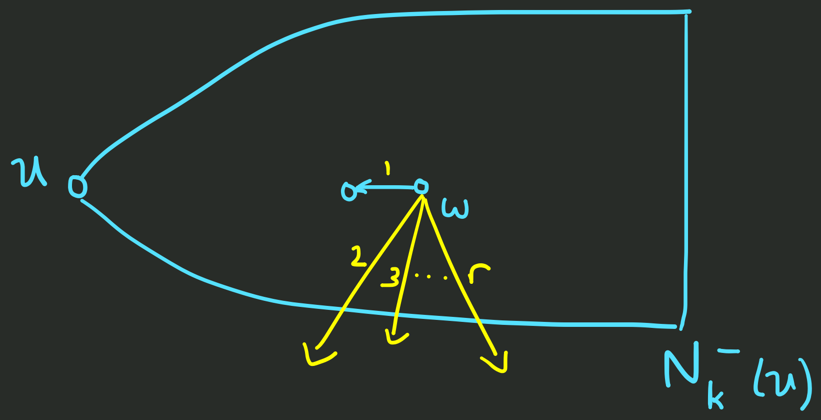

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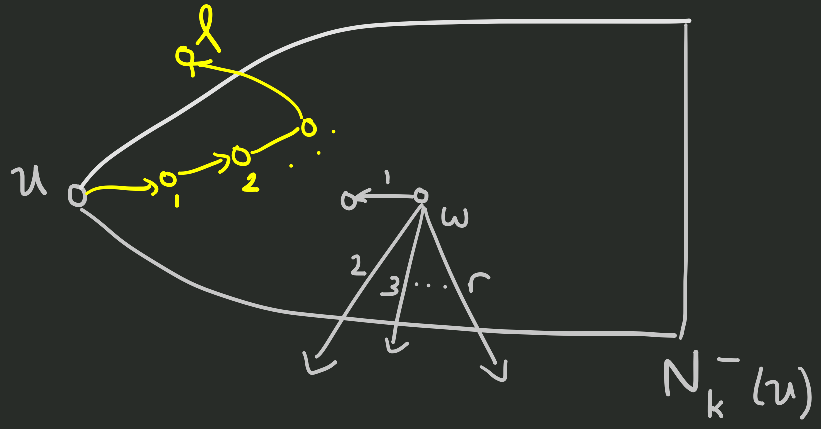

Prop: If $N^-_{\leq k}(u)$ is h-hard and $\exists$ at least one path of length ℓ from u to $N^-_{\leq k}(u)^c$ then

$\pi(u) \leq r^{\ell-h}$

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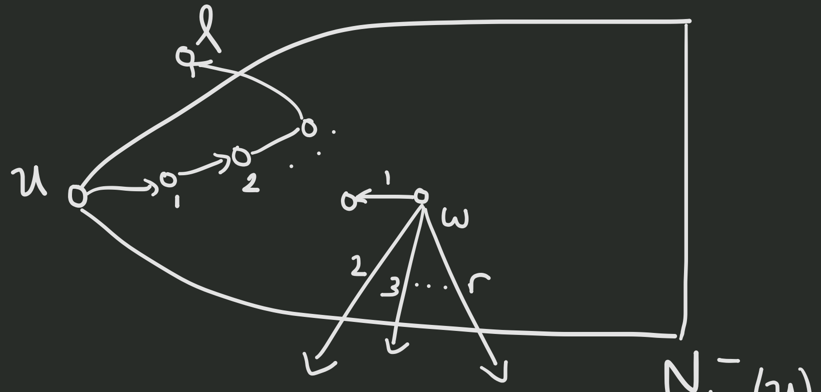
Trap:

A vertex w with $|N^+(w) \cap N^-_{\leq k}(u)| = 1$.



Prop: If $N^-_{\leq k}(u)$ is h-hard and $\exists$ at least one path of length ℓ from u to $N^-_{\leq k}(u)^c$ then

$$\pi(u) \leq r^{\ell-h}$$

Proof: In this case

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Bounding π_{max} and π_{min}

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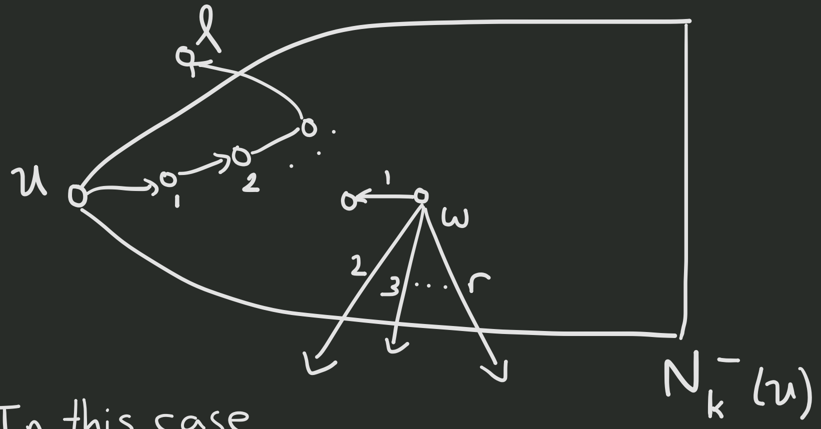

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Fact: Whp, for all u

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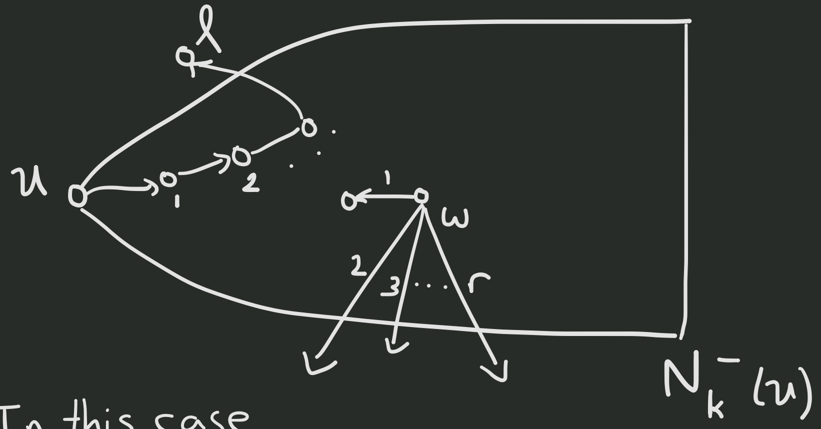

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So $\pi_{\max} = \max_u \pi(u) \le r^{\ell - k} < n^{-(1-3\varepsilon)}$ whp.

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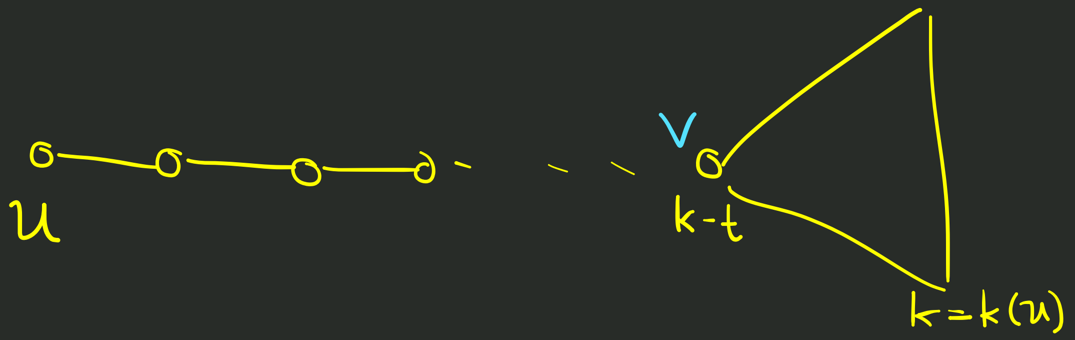

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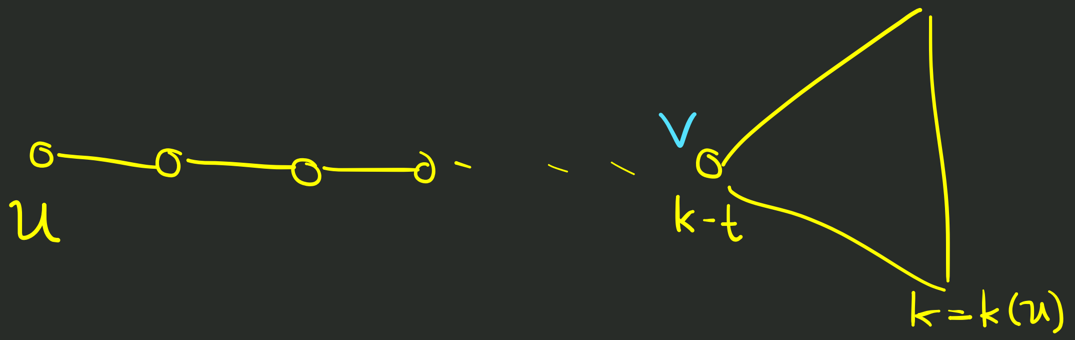

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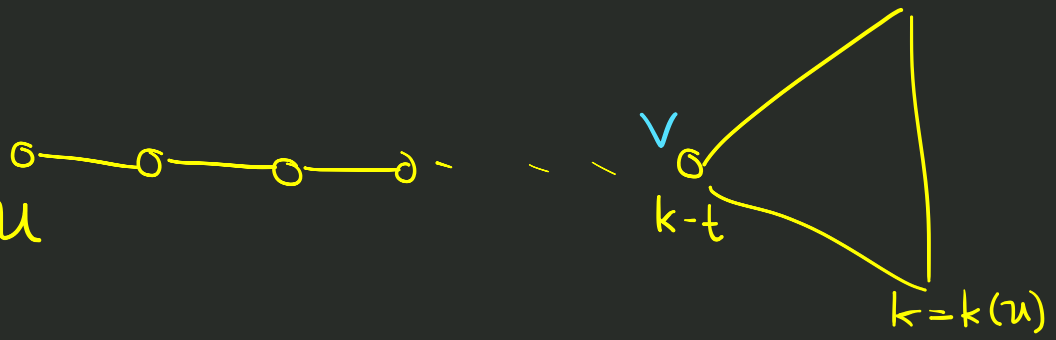

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So $\mathbb{E}_u[\tau_u^+] \gtrsim r^{k-t}\,\mathbb{E}_v[\tau_v^+] \gtrsim r^{k-t}\cdot 1/n$

$= n^{-(1+c_r+o_p(1))}$.

Thank you!

$k(u) = \min\{k : |N_k^-(u)| \geq \log^4 n\}$ $\qquad$ $\operatorname{diam}(D(n,r)) = \max_u k(u) + (1+o_p(1)) \log_r n$

Prop: $\max_u k(u) = (1+o(1))\, c_r \log n$

First try to understand a single vertex u.

Idea (Athreya & Ney; Kingman; McDiarmid; Riordan & Wormald):

Study BP of survivors.

Let $T \sim$ Poisson(r) GW, $\mathbb{P}(T=\infty)=s$. $\qquad 1-s=e^{-rs}$.

Say $v \in T$ survives if v has ∞ many descendants.

Let ρ be root of T, $Z =$ # surviving children of ρ

$\mathbb{P}(\rho \text{ has } a \text{ surviving children}) = \mathbb{P}(\text{Poisson}(rs) = a)$

$\mathbb{P}(\rho \text{ has } a \text{ surviving children} \mid \rho \text{ survives}) = s^{-1} \cdot \mathbb{P}(\text{Poisson}(rs)=a)$

$$\begin{aligned}\mathbb{P}(\rho \text{ has } 1 \text{ surviving child} \mid \rho \text{ survives}) &= s^{-1}\, \mathbb{P}(\text{Poisson}(rs)=1)\\ &= s^{-1} \cdot (rs\, e^{-rs}) = r\, e^{-rs}\\ &= r(1-s)\end{aligned}$$

Idea: For $k-t$ large, least $\underbrace{\text{costly}}_{\text{unlikely!}}$ way to have $0 < N_k^-(u) < r^t$

is to have $N_{k-t}^-(u) = 1$.

Cost: $$\begin{aligned}\mathbb{P}(|N_k^-(u)| \leq 2^t) &\cong \mathbb{P}(|N_{k-t}^-(u)=1,\ 0 < N_k^-(u) < r^t)\\ &\cong \mathbb{P}(|N_{k-t}^-(u)=1,\ N_k^-(u) > 0)\\ &\cong \mathbb{P}(|N_{k-t}^-(u)|=1) \cdot \mathbb{P}(\text{Po}(r) \text{ GW survives})\end{aligned}$$

survivors in gen k is increasing in k