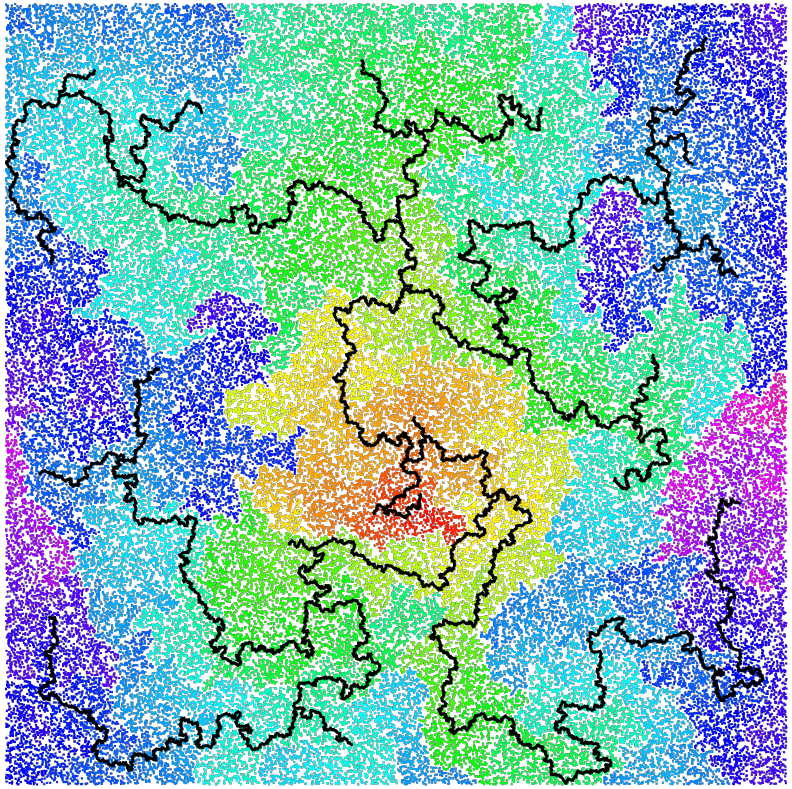
The scaling limit of the minimum spanning tree, and the critical Erdős-Rényi random graph



L. Addario-Berry

Summer school on random graphs and probabilistic methods

Fields Institute

June 9, 2017

Parts are joint with Nicolas Broutin, Christina Goldschmidt, Grégory Miermont

Percolation on the complete graph

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- In $G_{n,p}$, each edge is independently present with probability p.
- Write T_n for the MST of K_n with these weights, $T_{n,p}$ for the graph obtained by only keeping edges of T_n of weight less than p.

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$\star$ Set-up: K_n, edge weights $\{W_e, e \in E(K_n)\}$ are iid, Uniform$[0,1]$ (though the precise distribution is unimportant).

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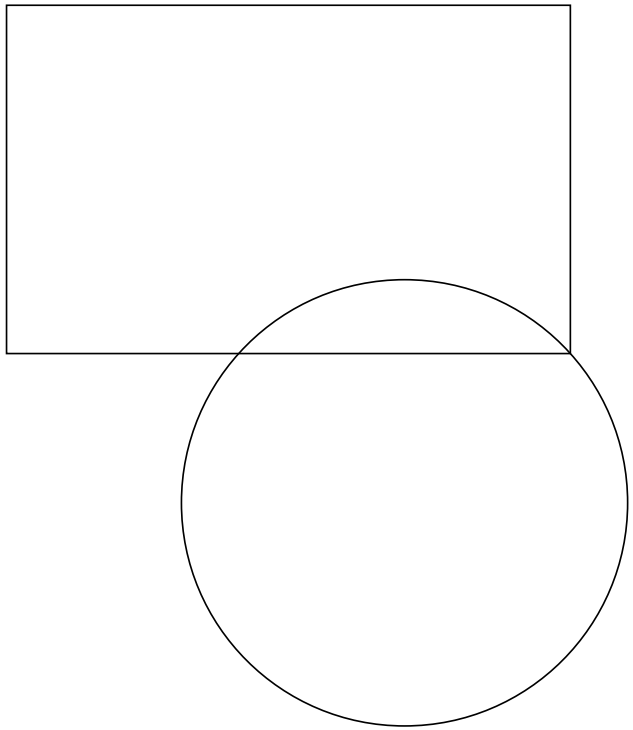
We can also say some things about the limit $\mathcal{M}$; but first things first – a brief explanation of the notion of convergence.

Metric space convergence.

Given sets S, T in a metric space (M, d), let

$$b(S, T) = \inf\{r \ : \ T \subset b_r(S)\}.$$



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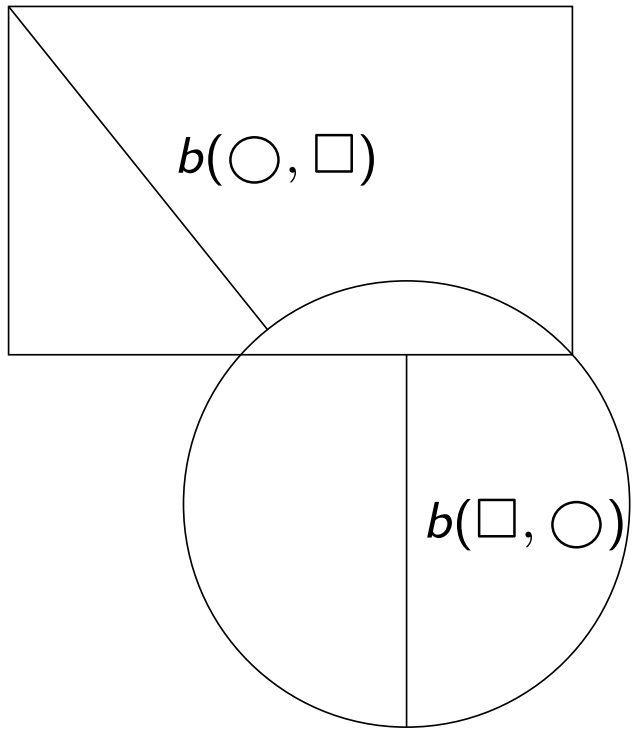

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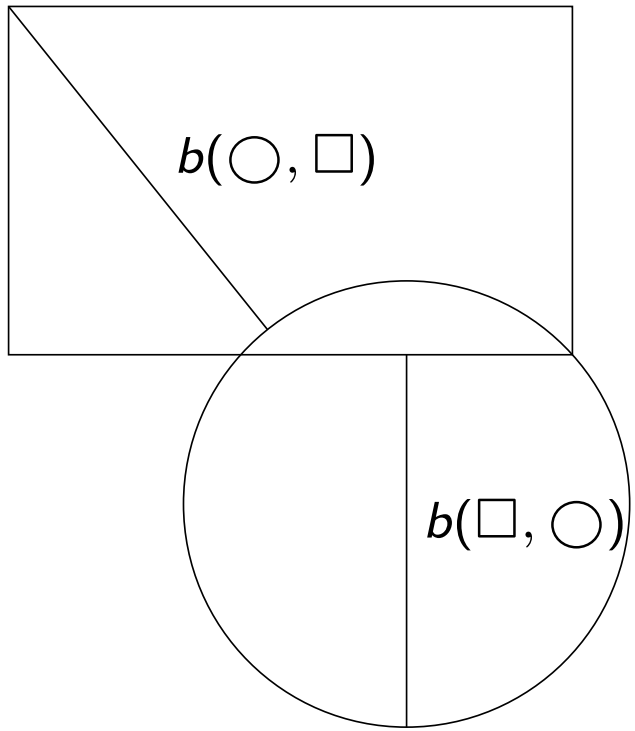

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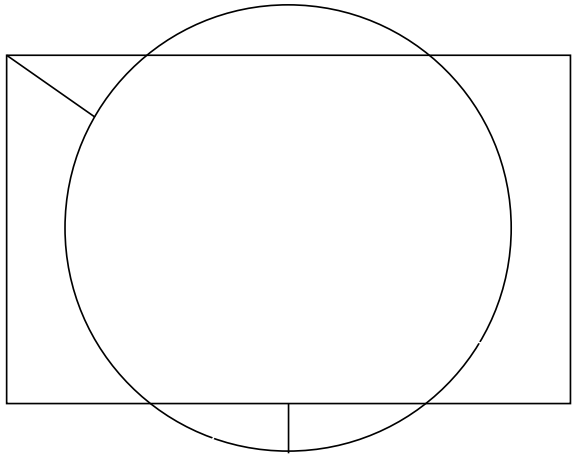

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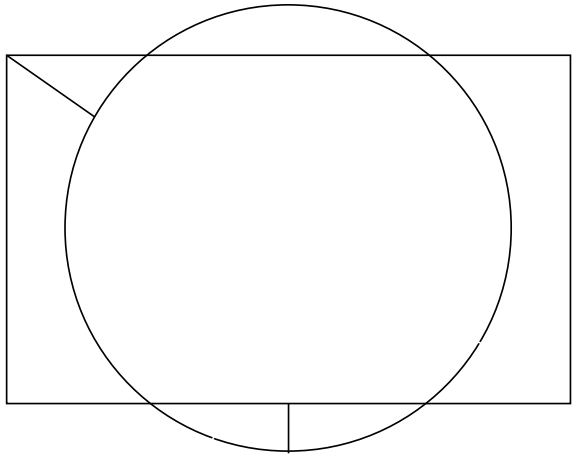

The *Gromov-Hausdorff* distance between metric spaces (S, d_1) and (T, d_2) is

$$\inf\{d_H(S, T)\},$$

where the infimum is over all metric spaces (M, d) containing both $(S, d_1$ and (T, d_2) as subspaces.

Similarity of metric spaces

Two metric spaces (X, d_X) and (Y, d_Y) are similar in the Gromov-Hausdorff Sense if we can map both into a common space (Z, d_Z) via maps f_X, f_Y so that:

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This yields *Gromov–Hausdorff–Prokhorov* convergence.

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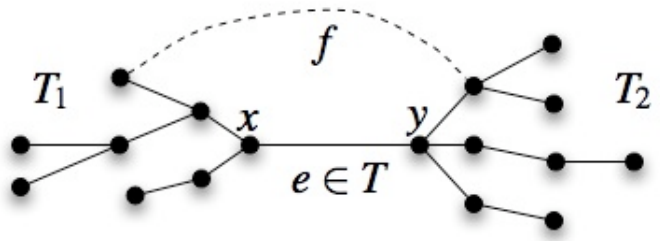
For finite G, an edge $e = xy$ is in the MST $\Leftrightarrow \nexists$ a path from x to y all of whose edges have weights less than w_e

$\forall f$ from T_1 to T_2, $w_f > w_e$.



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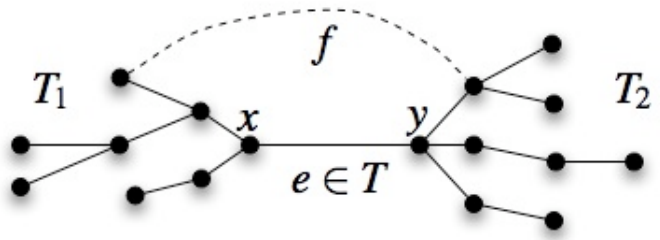

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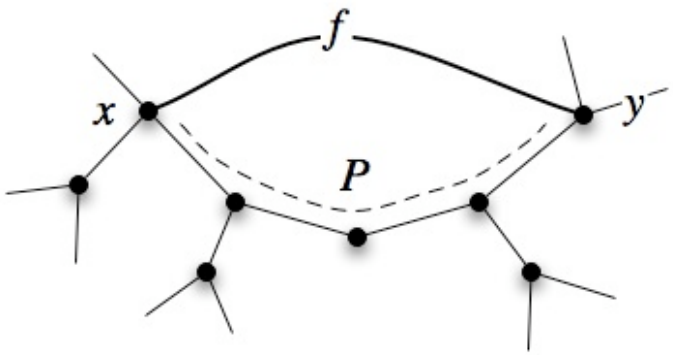

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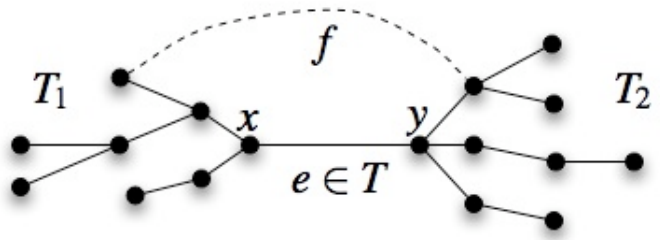

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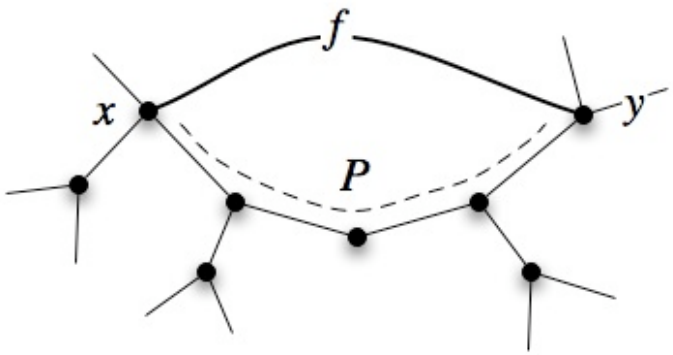

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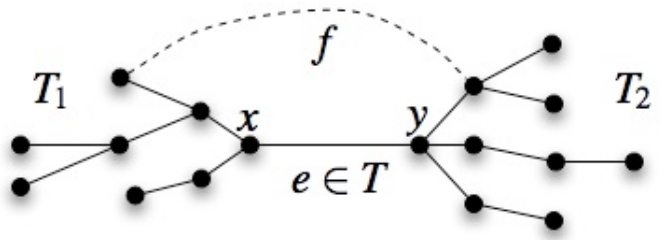

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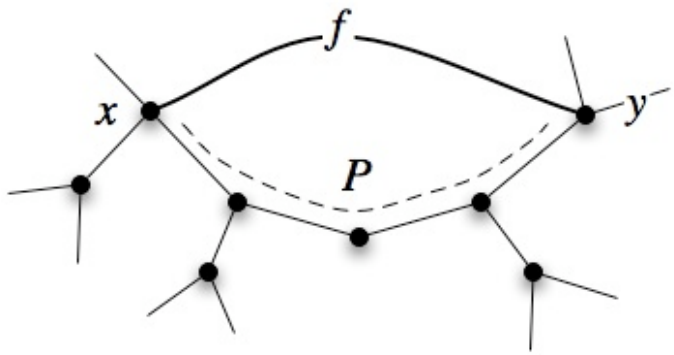

Kruskal's algorithm adds an edge $e = x, y$ of weight p if and only if x and y are in distinct components of $G_{n,p}$.

For all weights p, $G_{n,p}$ has the same set of components as $T_{n,p}$.

Erdős & Renyi, 1960

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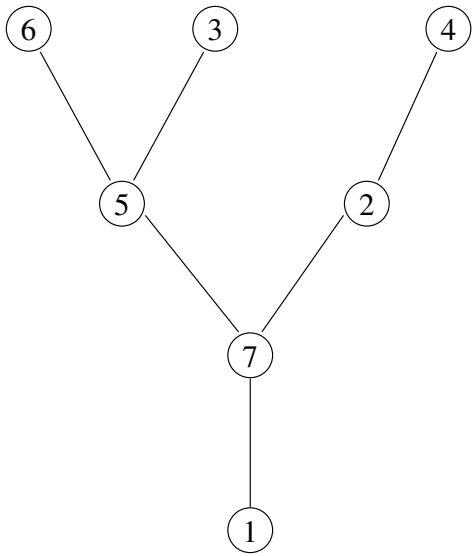
So to understand the tree, we really need to understand the graph process at time around $1/n$.

A reminder about trees.

A tree is a connected graph with no cycles. A tree always has one more vertex than it has edges.



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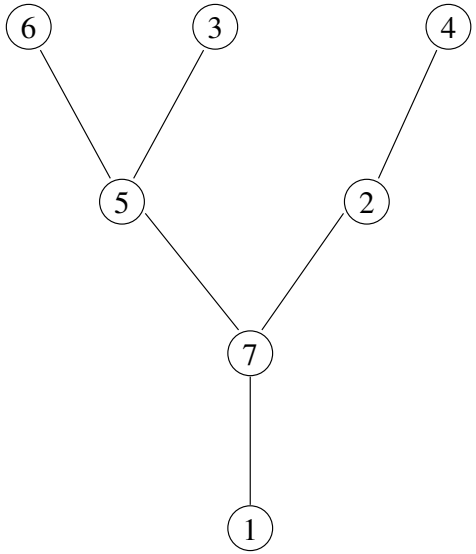

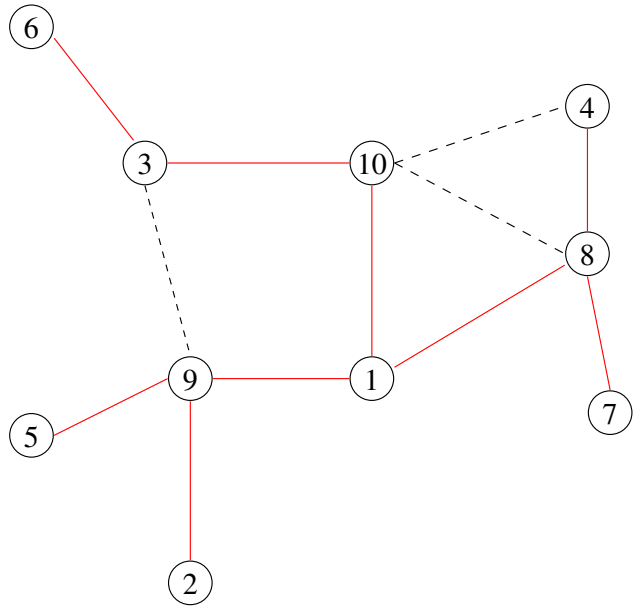

A connected graph which is not a tree has surplus equal to the number of edges more than a tree that it has. The graph on the right has surplus 3.

Convergence of component sizes of $G_{n,1/n}$.

- Let $C_1^{(n)}, C_2^{(n)}, \ldots$ be the sizes of the connected components of $G_{n,1/n}$, listed in decreasing order, let $S_1^{(n)}, S_2^{(n)}, \ldots$ be their surplusses.

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Theorem (Aldous, 1997)

$$(C_1^{(n)}/n^{2/3}, C_2^{(n)}/n^{2/3}, \ldots) \to (L_1, L_2, \ldots), \quad \textit{and}$$
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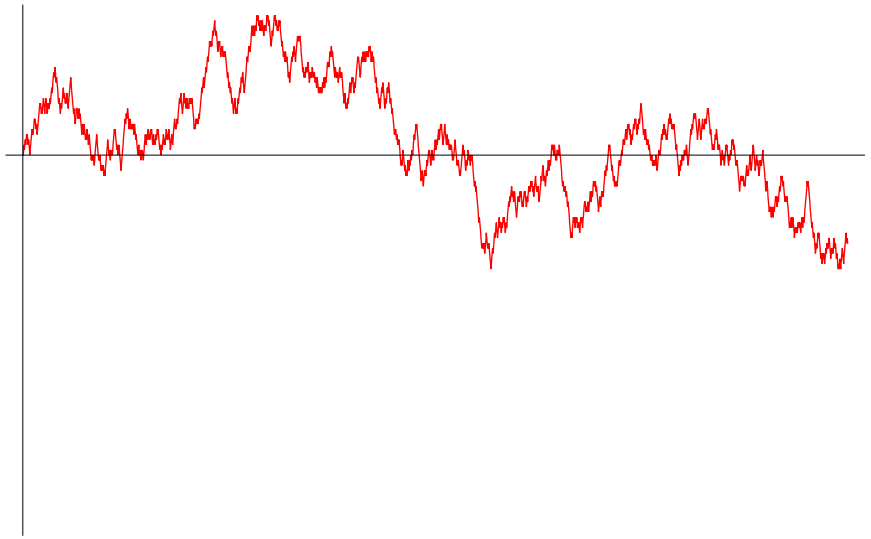
- The second is of finite-dimensional distributions.

Convergence of component sizes of $G_{n,1/n}$.

What are the limit sequences?

Convergence of component sizes of $G_{n,1/n}$.

- Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion



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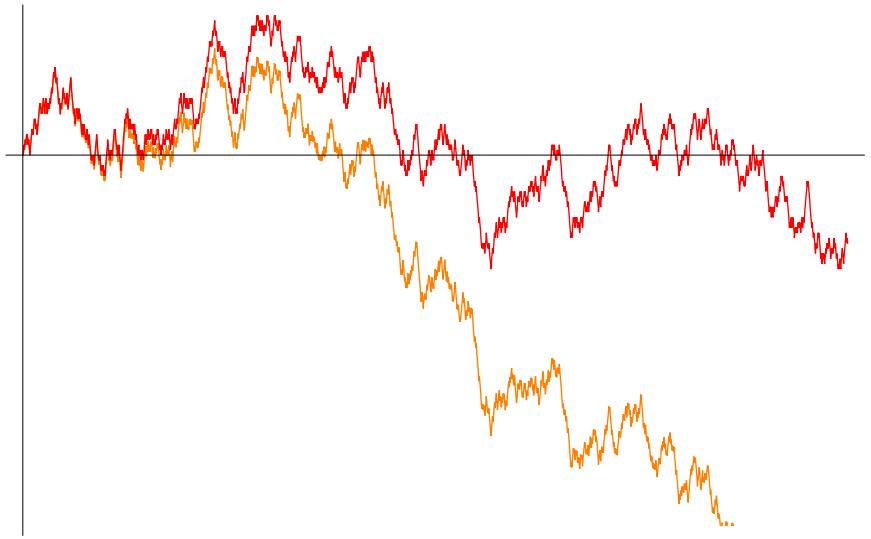

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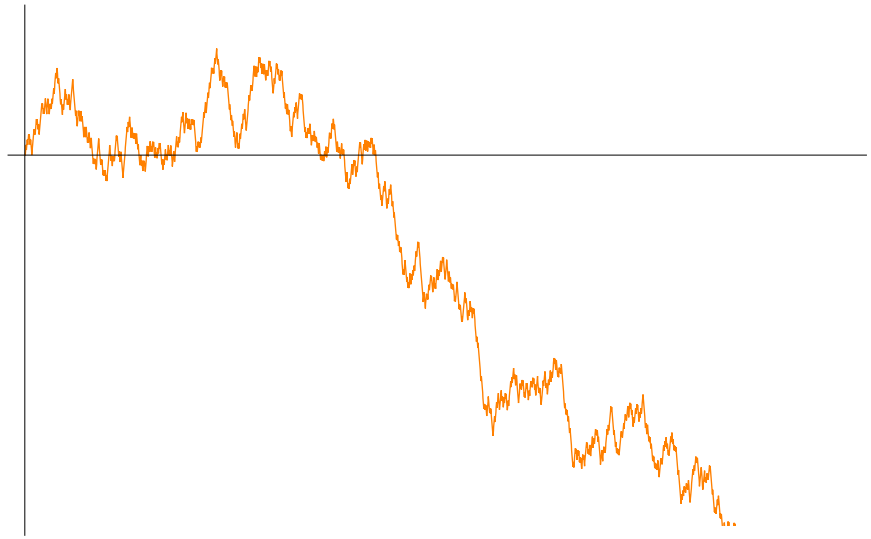

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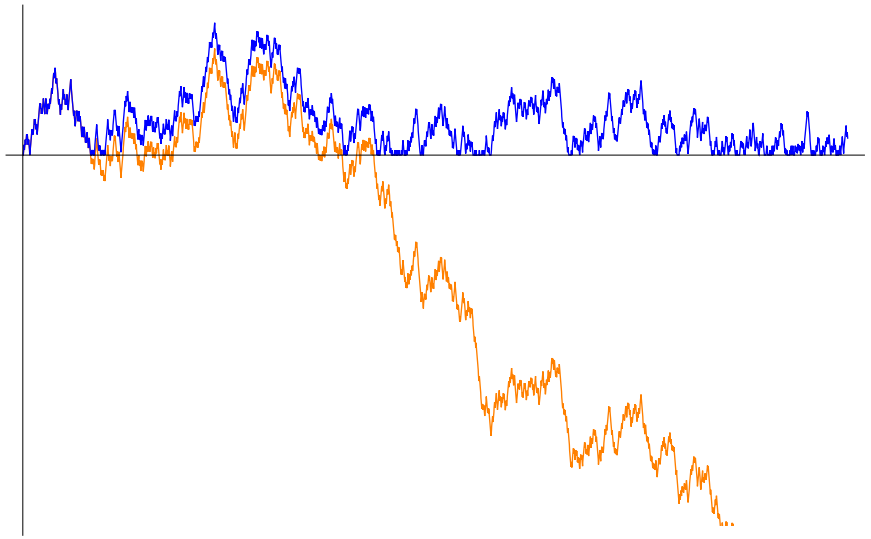

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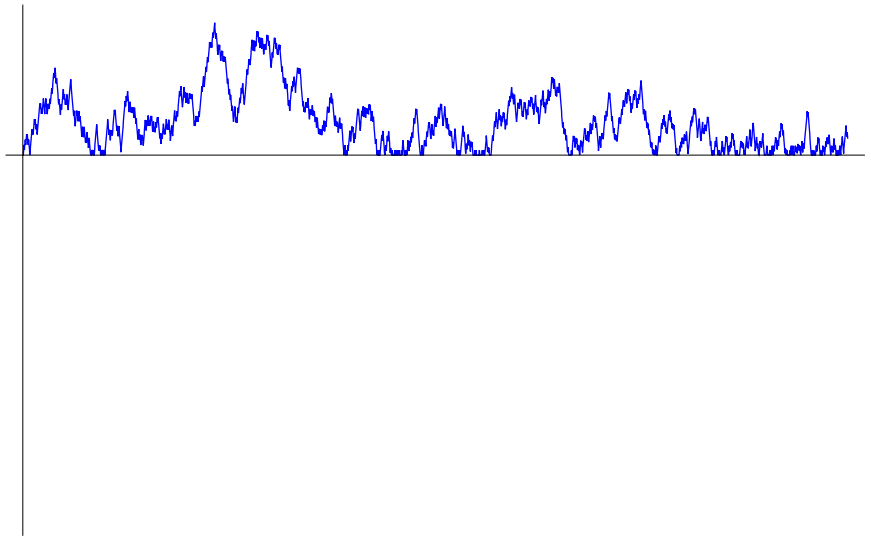

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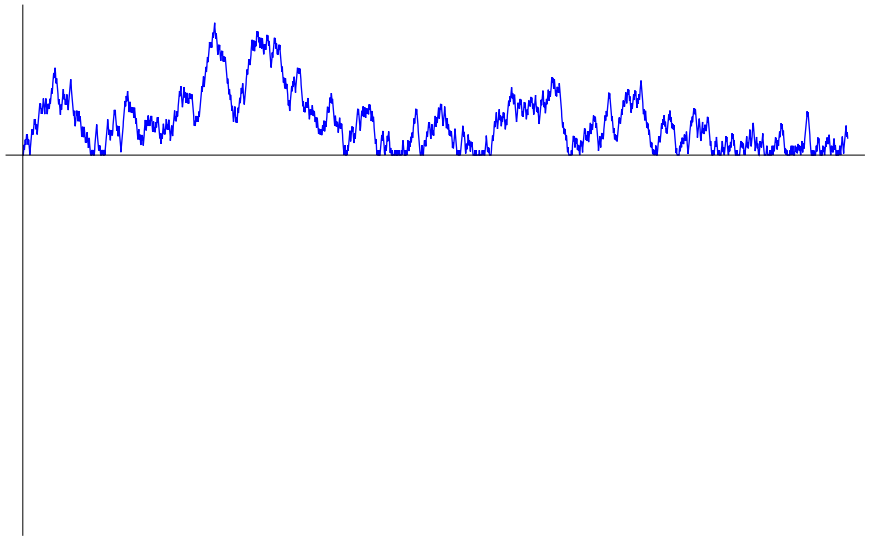

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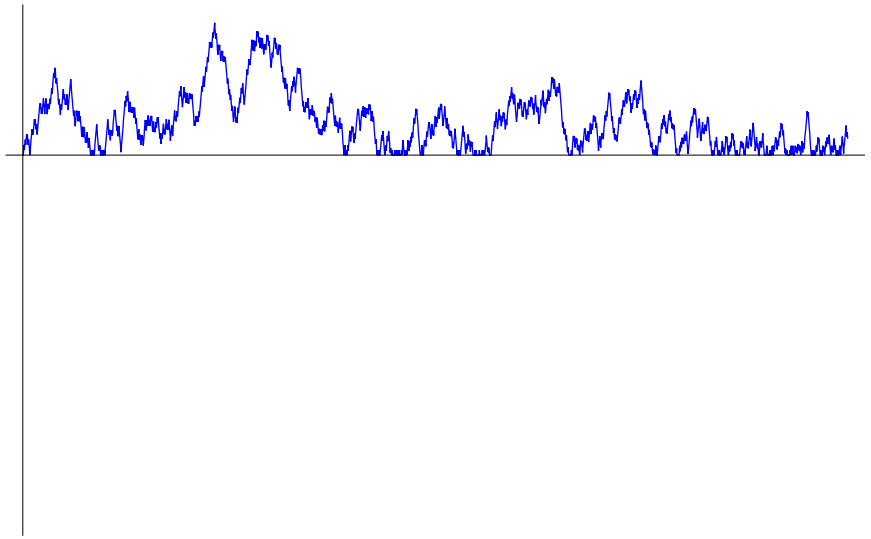
- The first limit object $(L_1, L_2, \ldots)$ is the sequence of the lengths of the excursions above 0 of W_t^* (in decreasing order).
- Let $\mathcal{P}$ be a homogeneous Poisson process of "marks" under the excursions of W_t^*.



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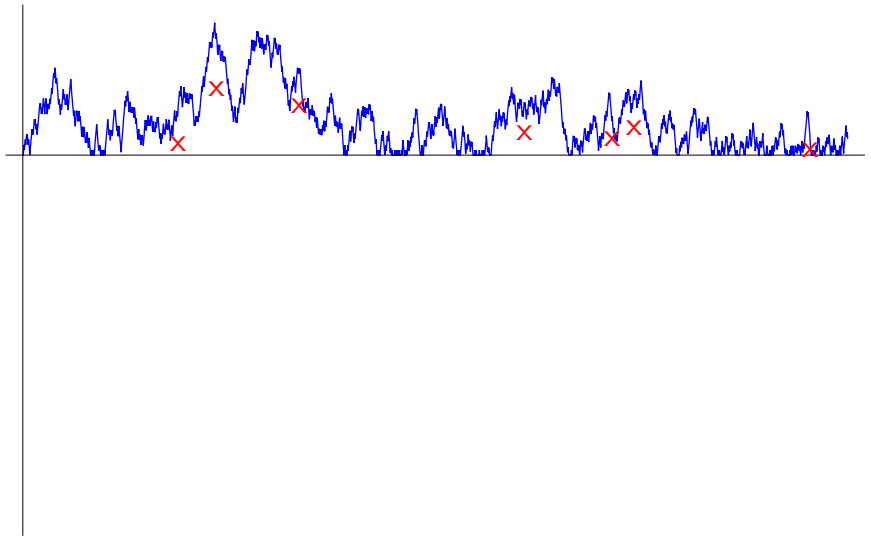
- The first limit object $(L_1, L_2, \ldots)$ is the sequence of the lengths of the excursions above 0 of W_t^* (in decreasing order).
- Let $\mathcal{P}$ be a homogeneous Poisson process of "marks" under the excursions of W_t^*.



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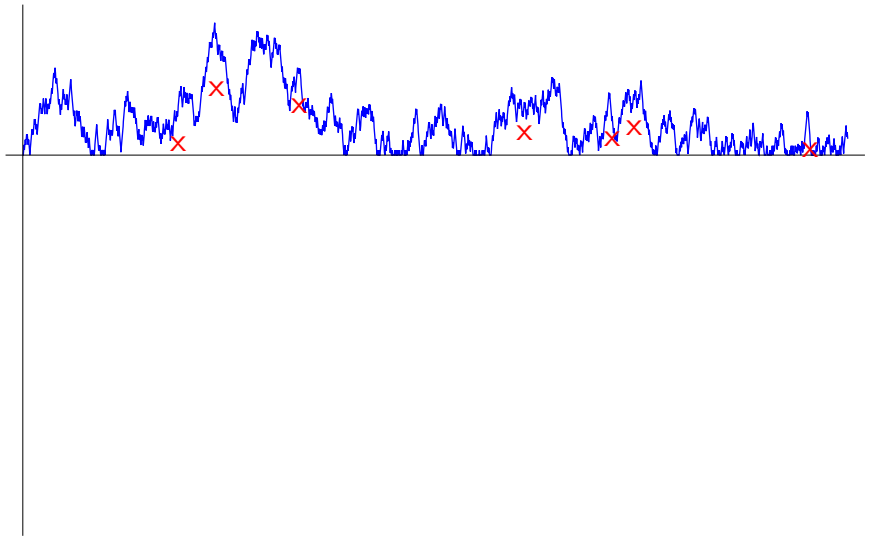

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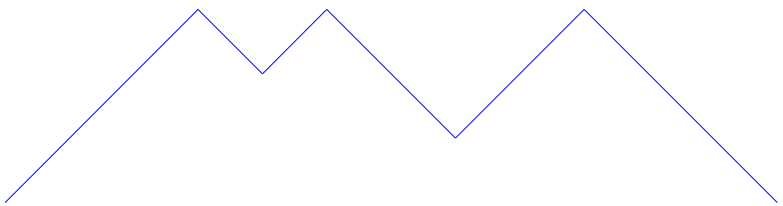

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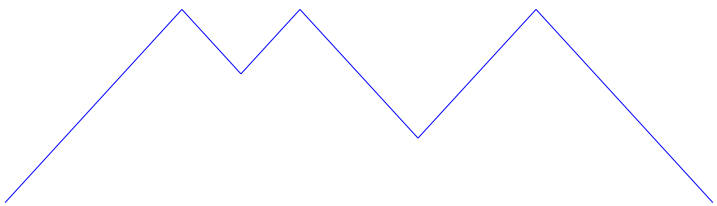

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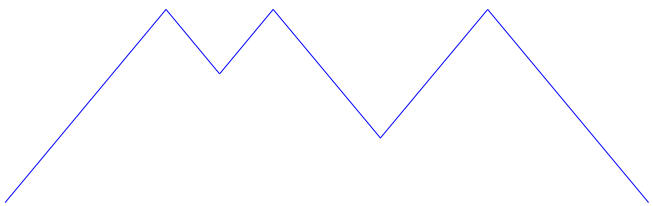

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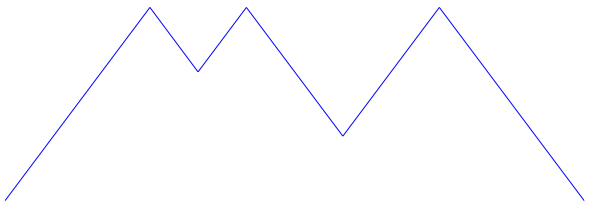

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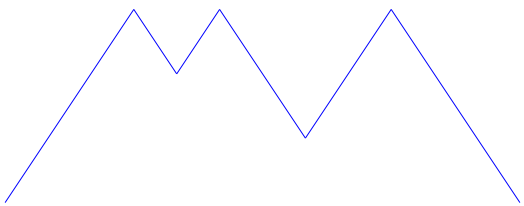

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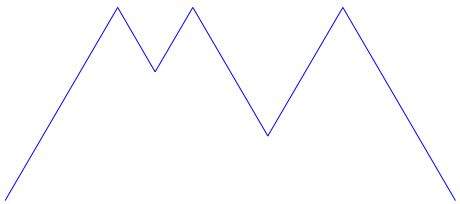

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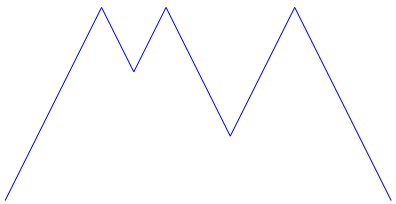

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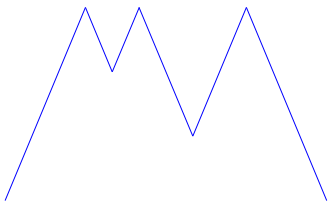

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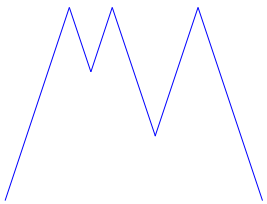

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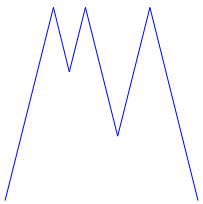

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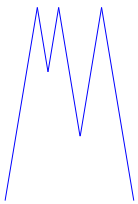

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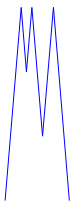

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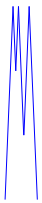

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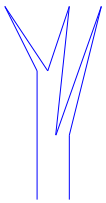

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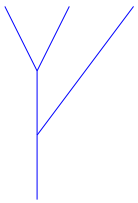

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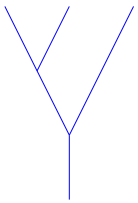

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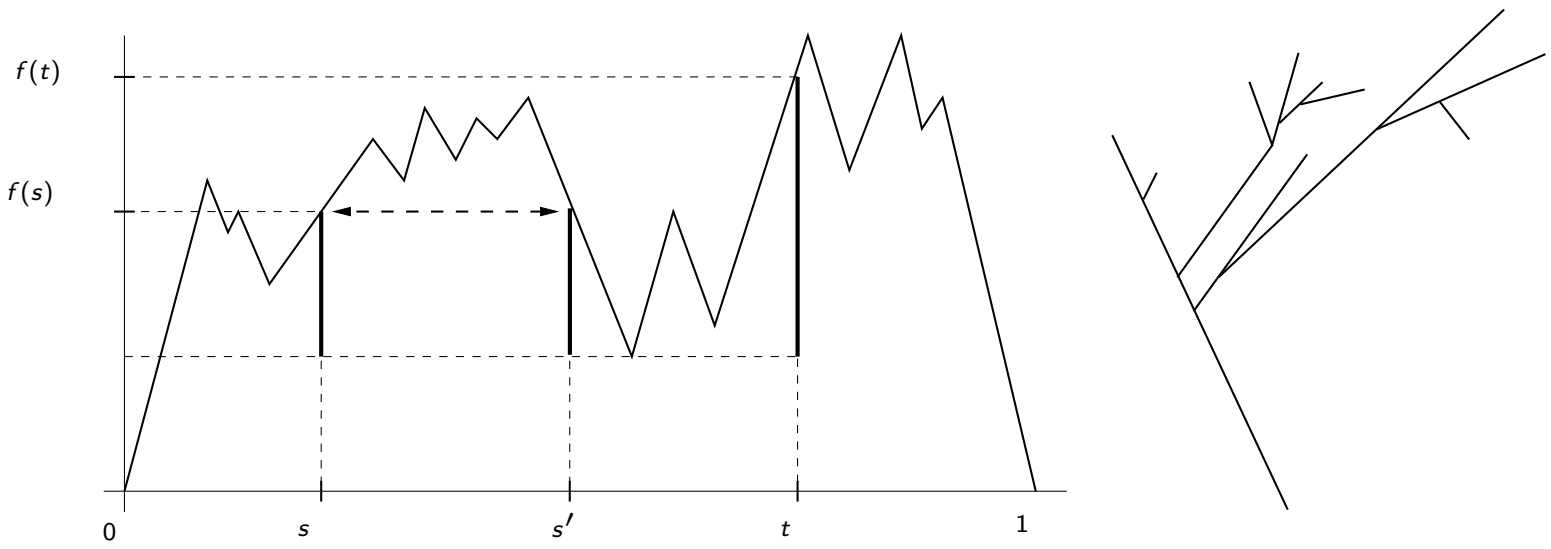

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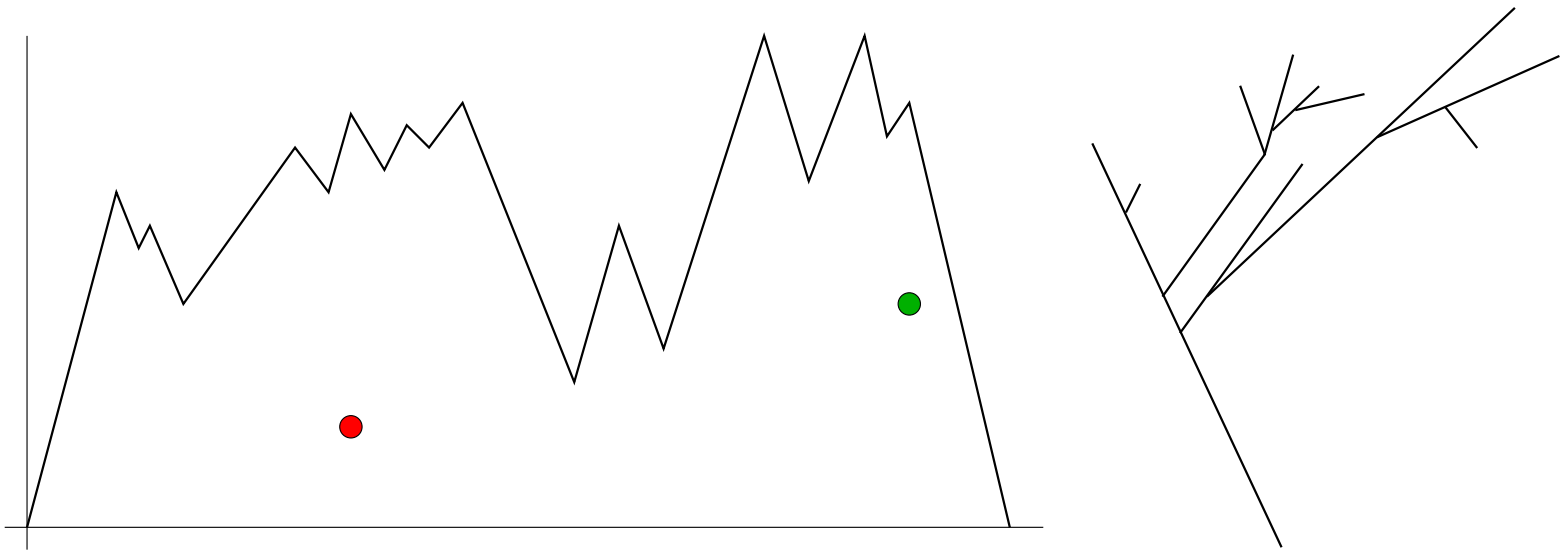

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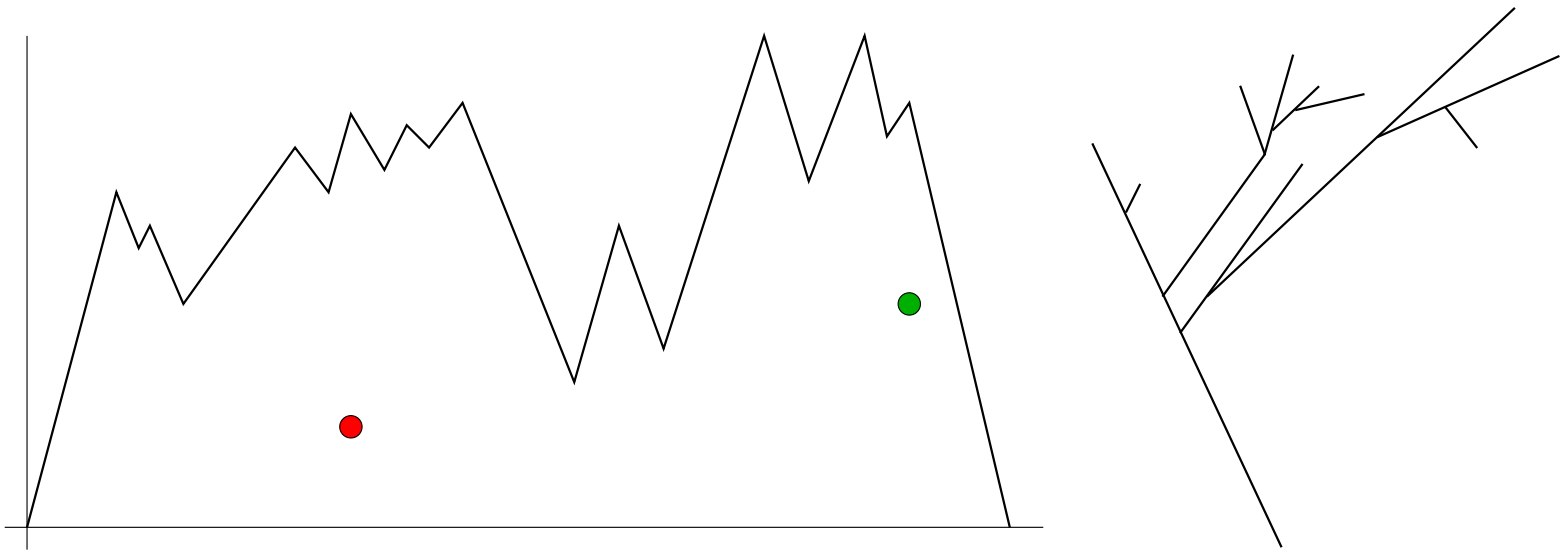

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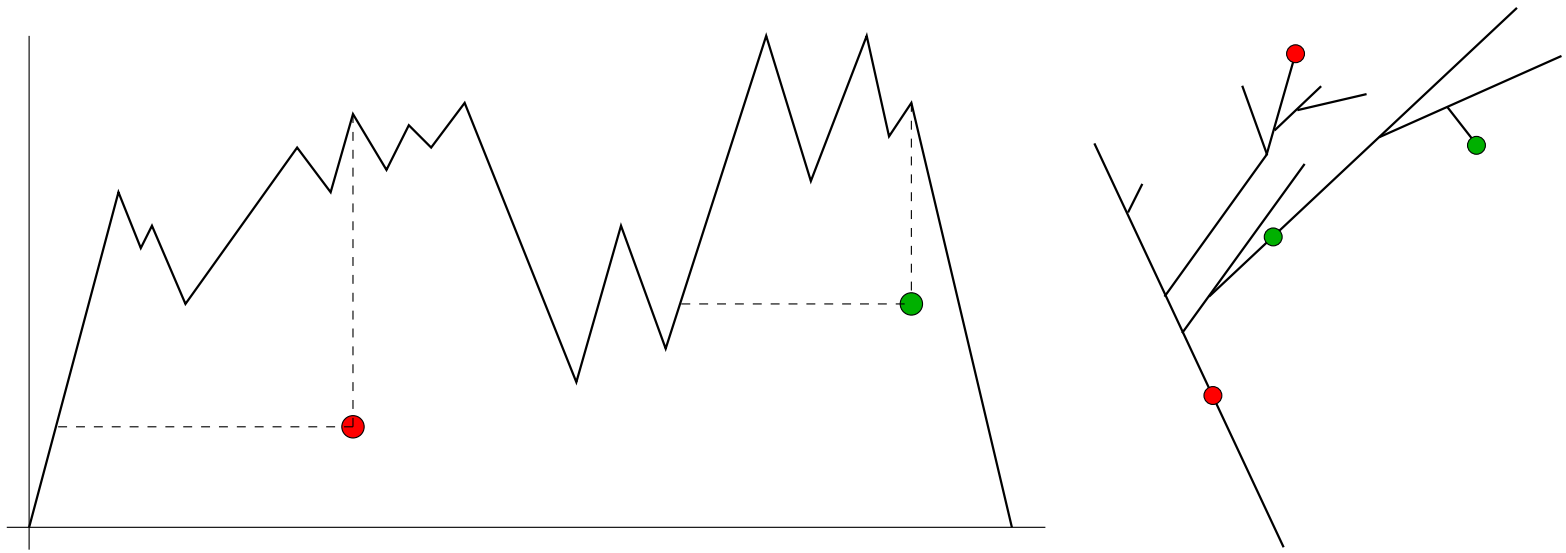

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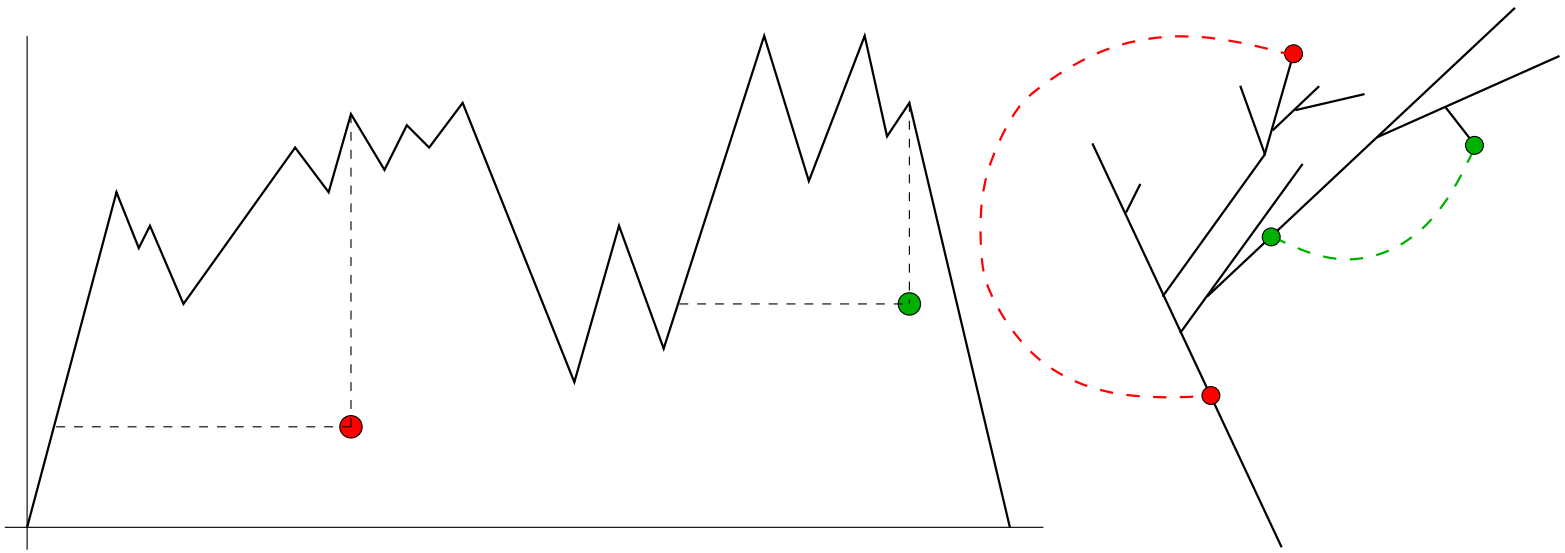

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We obtain a new metric space by identifying these points.

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- Let $\mathcal{C}$ be a component of $G_{n,1/n}$ with m vertices, and suppose we condition $\mathcal{C}$ to be a tree.
- Then $\mathcal{C}$ is distributed as a *uniform random* tree on m labelled vertices; any spanning tree of the vertex set is equally likely. (There are m^{m-2} such possible trees.)

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So fix an integer m and generate a uniform random tree on labels $1, 2, \ldots, m$.

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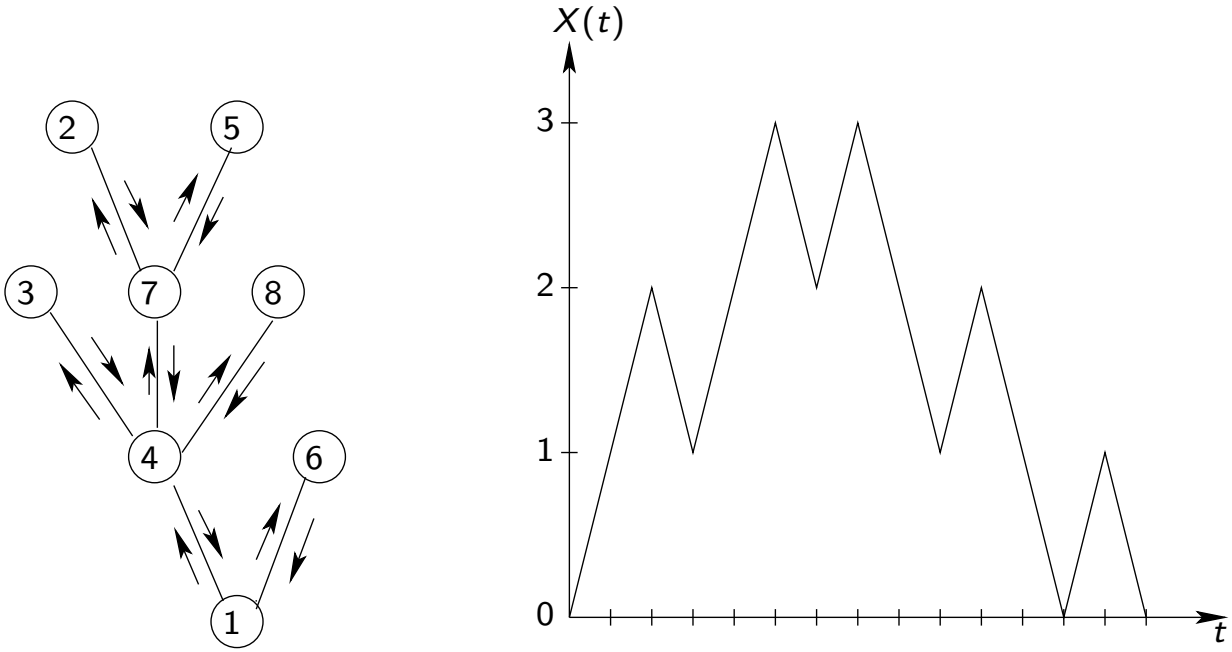
These trees are well understood; they are distributed as Poisson Galton-Watson trees conditional on their size. In particular, they have height $\Theta(\sqrt{m})$.

The Harris walk.

The Harris walk is an excursion encoding heights in a finite tree.



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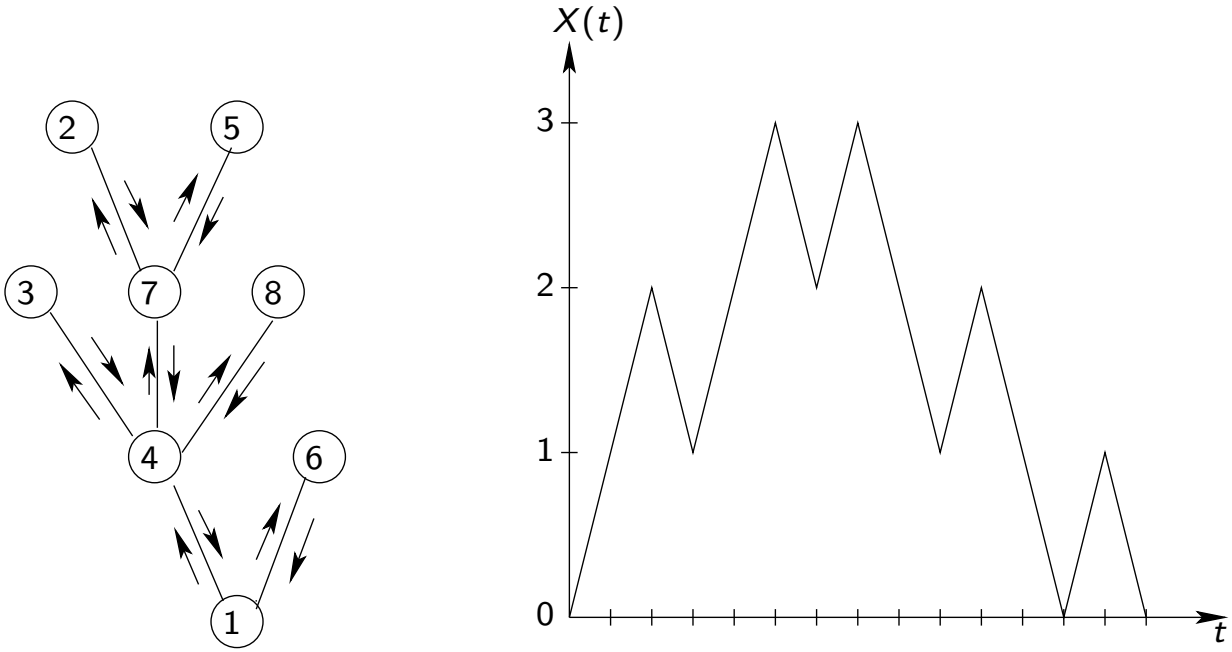

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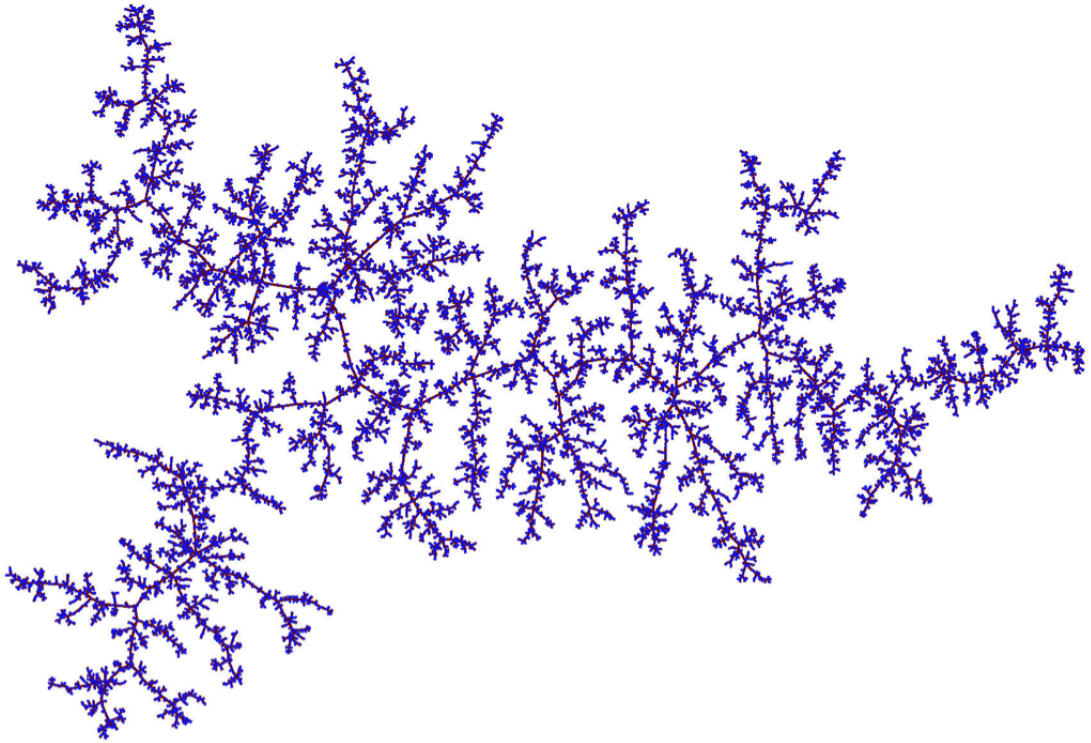
When the tree is uniformly random on $1, 2, \ldots, m$, this essentially looks like a random walk with $2m$ steps, conditioned to stay positive and return to zero at time $2m$.

Rescaled, its limit is a Brownian excursion.

The continuum random tree.

If the *tree itself* is rescaled (edge lengths $1/\sqrt{m}$ instead of 1), then as $m \to \infty$, there is a limiting object, called the Brownian continuum random tree.



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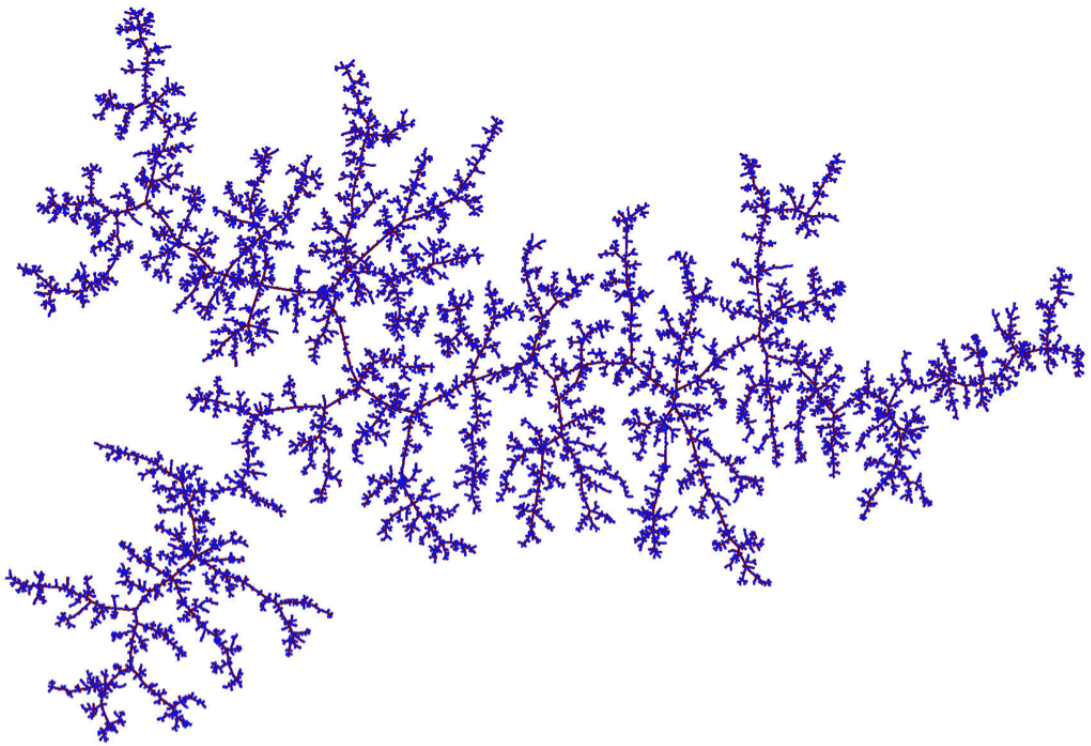

This is just the tree coded by a standard Brownian excursion.

Metric convergence of uniform trees.

The uniform random tree on $1, \ldots, m$, with distances rescaled by $1/\sqrt{m}$, and mass $1/m$ for each vertex, converges in distribution to the Brownian continuum random tree, in the Gromov-Hausdorff-Prokhorov sense.

Limiting random graph: what kind of limit.

Let $\mathcal{C}^{(n)} = (\mathcal{C}_1^{(n)}, \mathcal{C}_2^{(n)}, \ldots)$ be the size-ordered sequence of components of $G_{n,1/n}$, with distances rescaled by $n^{-1/3}$ and masses rescaled by $n^{-2/3}$.

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Here convergence is with respect to the metric

$$d(\mathcal{C}, \mathcal{M}) = \left(\sum_{i=1}^{\infty} d_{GHP}(\mathcal{C}_i, \mathcal{M}_i)^4 \right)^{1/4}.$$

Proof idea.

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- Warning: the excursion is *not* the Harris walk of this spanning tree.

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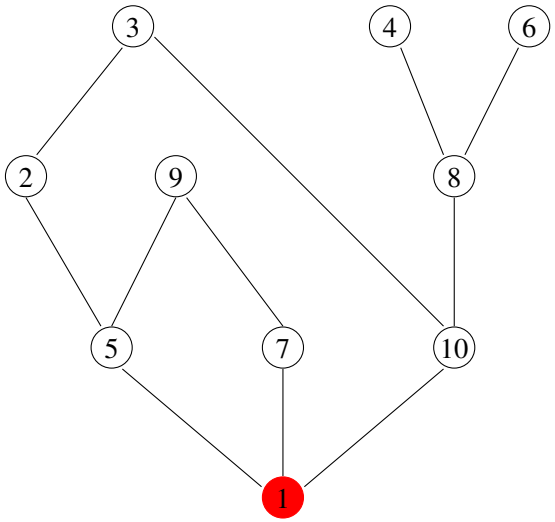
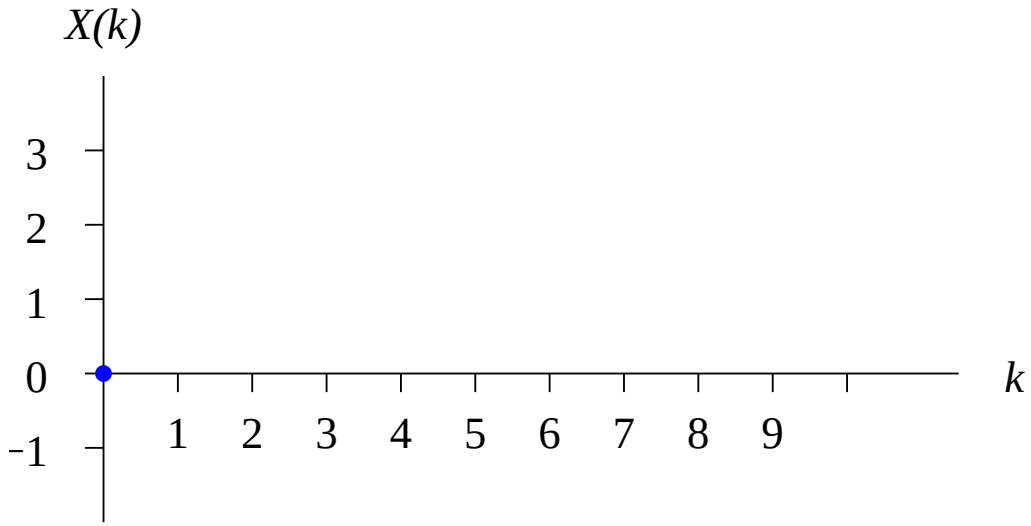
We will also want to keep track of some information. At step k, let $X(k)$ be the number of vertices which are alive.

Depth-first walk

Step 0: initialization



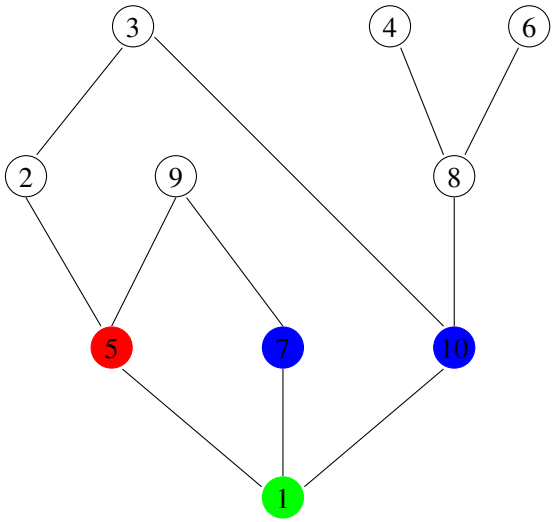


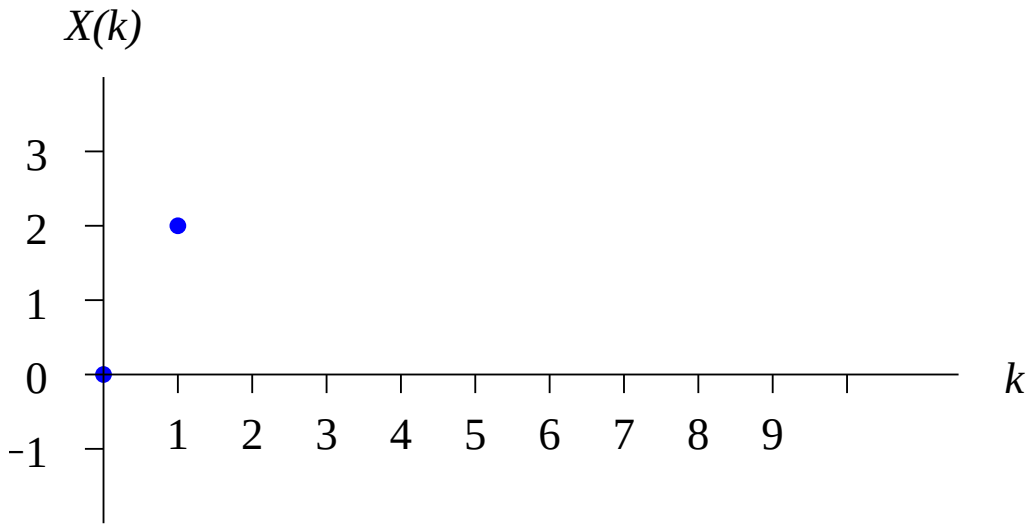
[Picture by Christina Goldschmidt.]

Current: 1 Alive: none Dead: none

Depth-first walk

Step 1



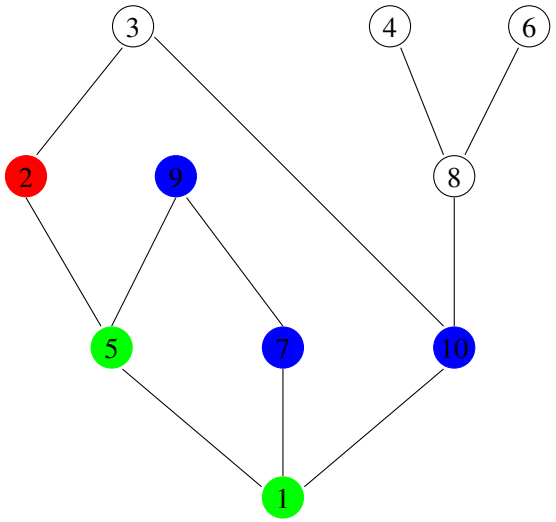


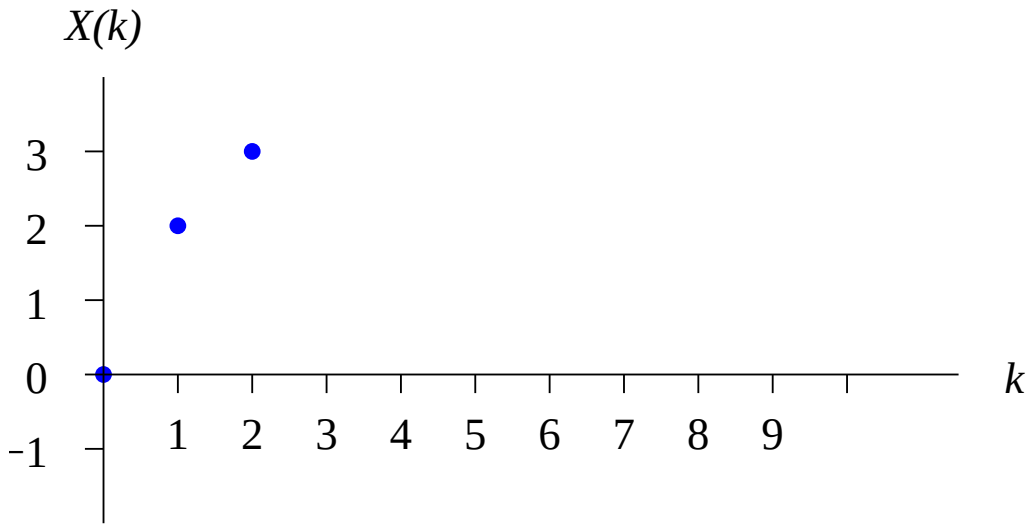
[Picture by Christina Goldschmidt.]

Current: 5 Alive: 7,10 Dead: 1

Depth-first walk

Step 2

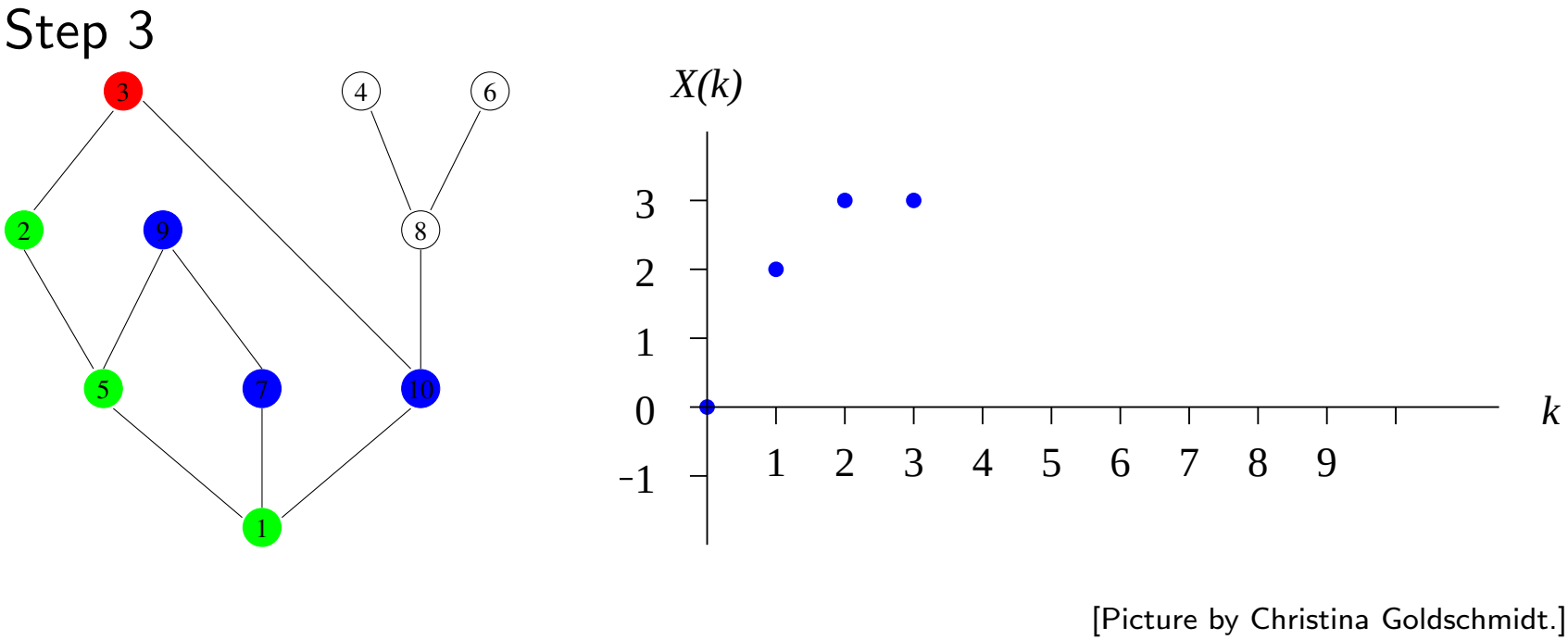




[Picture by Christina Goldschmidt.]

Current: 2 Alive: 9,7,10 Dead: 1,5

Depth-first walk

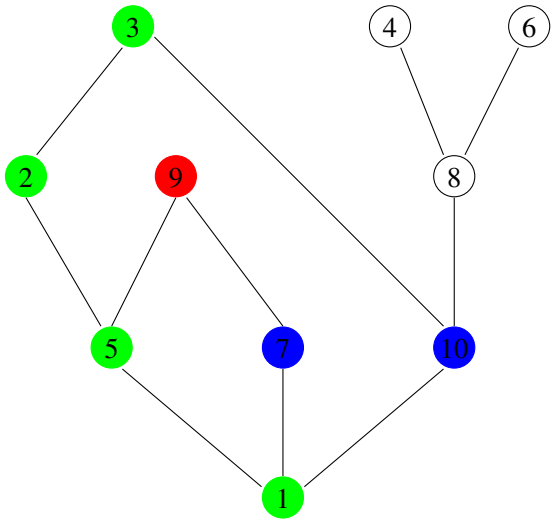


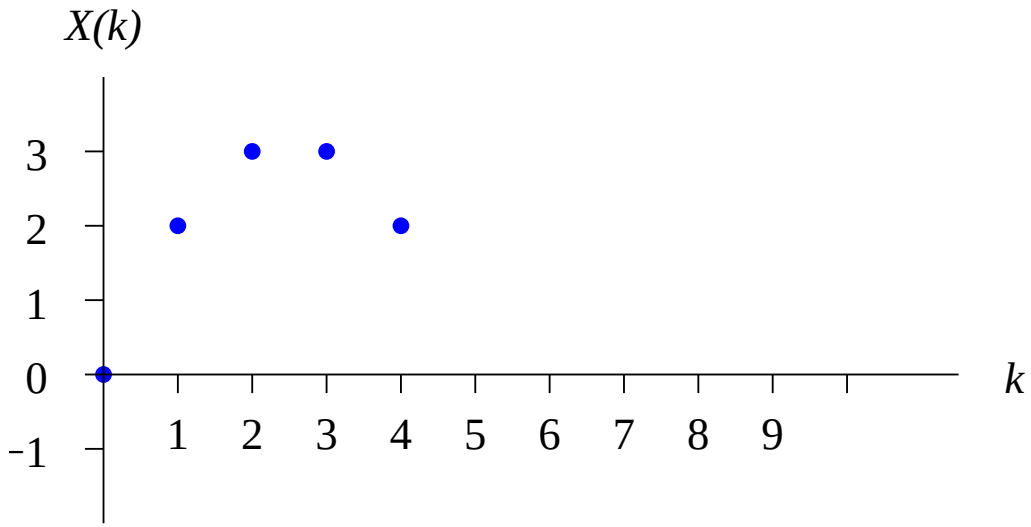
[Picture by Christina Goldschmidt.]

Current: 3 Alive: 9,7,10 Dead: 1,5,2

Depth-first walk

Step 4

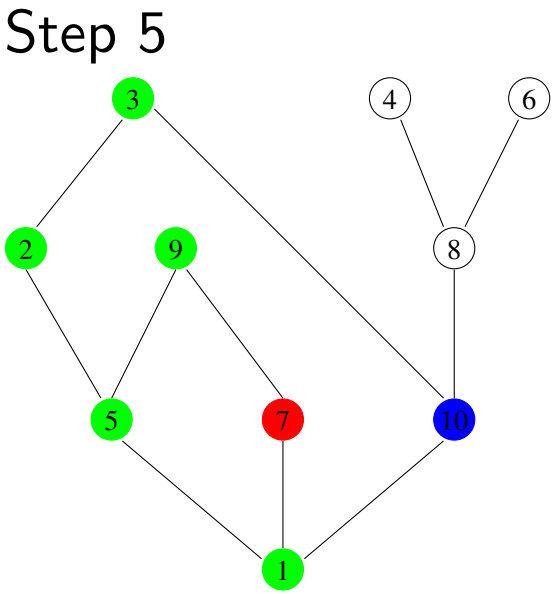


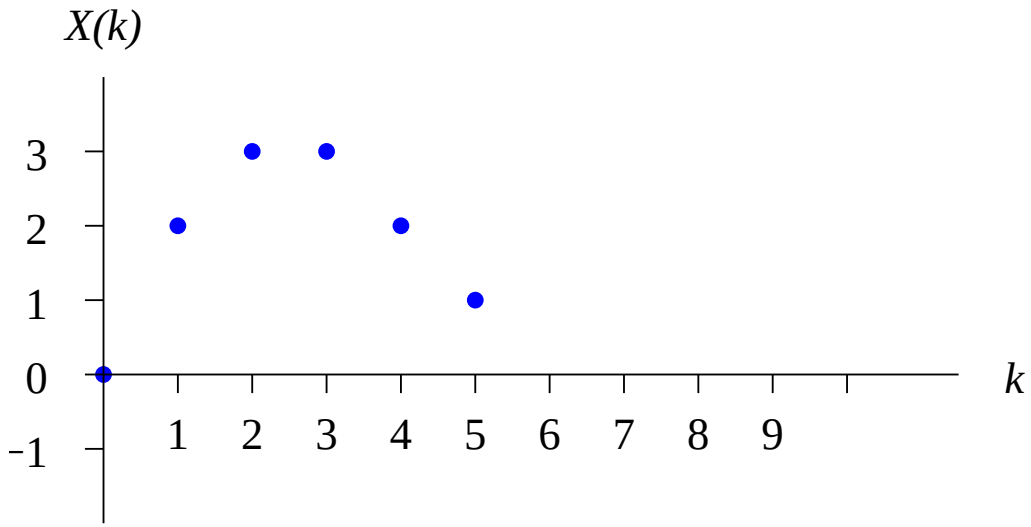


[Picture by Christina Goldschmidt.]

Current: 9 Alive: 7,10 Dead: 1,5,2,3

Depth-first walk

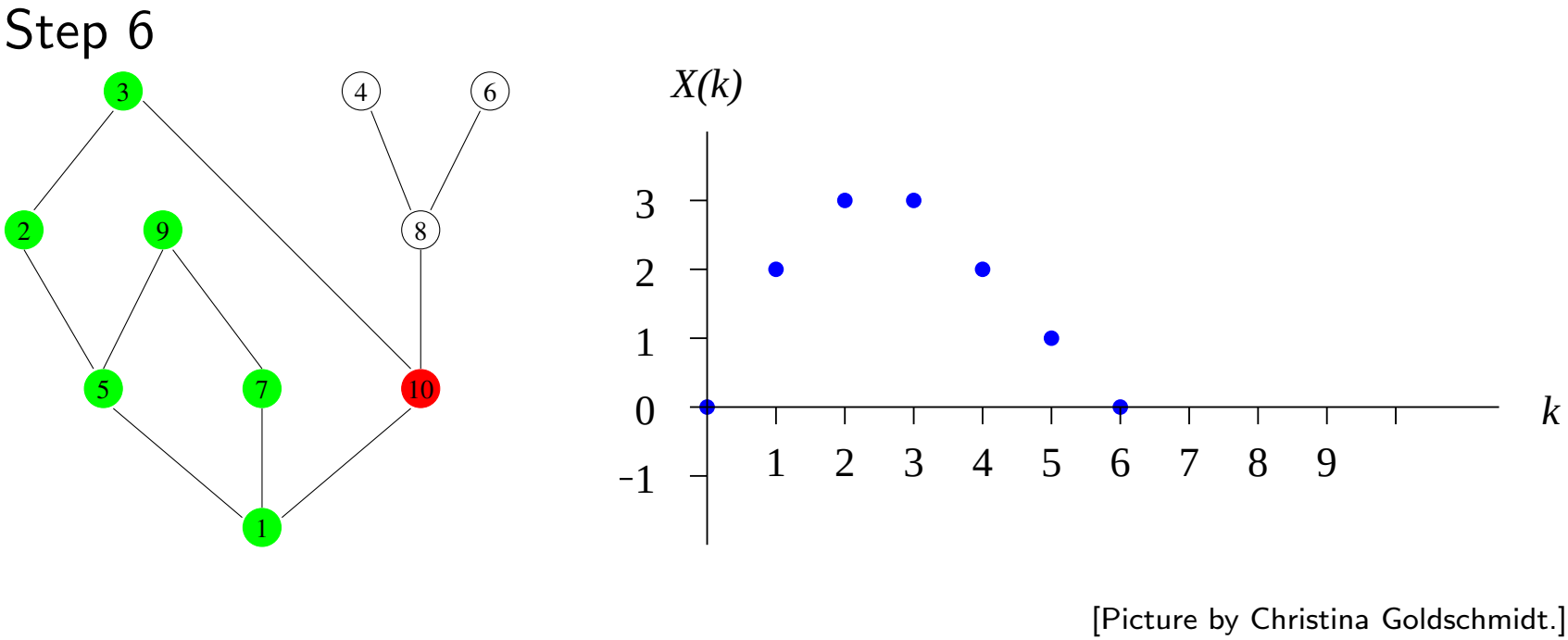




[Picture by Christina Goldschmidt.]

Current: 7 Alive: 10 Dead: 1,5,2,3,9

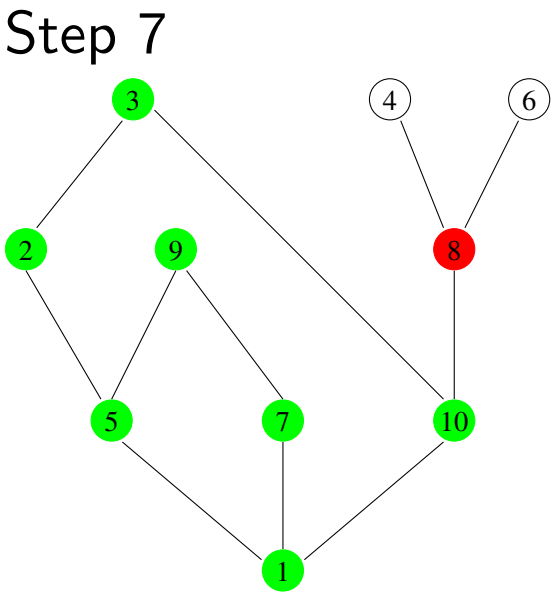
Depth-first walk

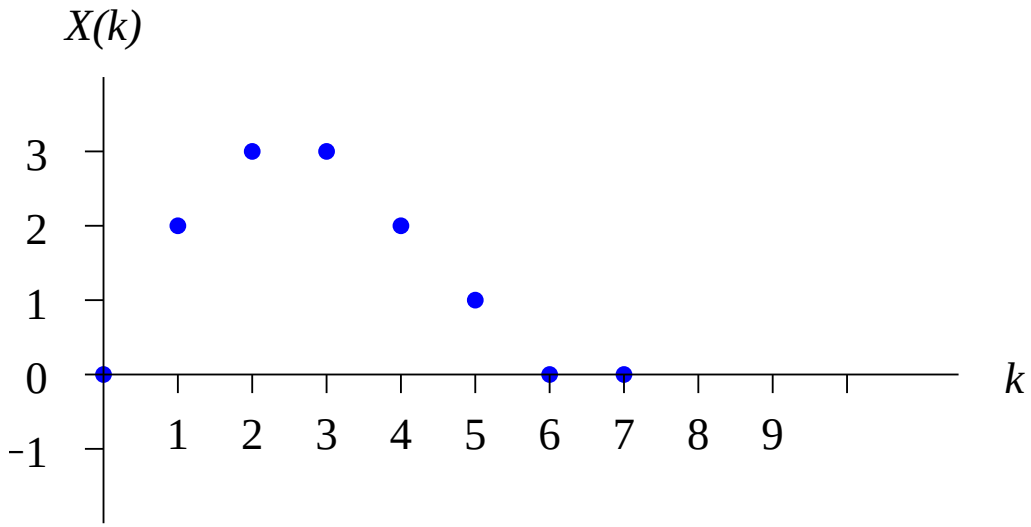


[Picture by Christina Goldschmidt.]

Current: 10 Alive: none Dead: 1,5,2,3,9,7

Depth-first walk

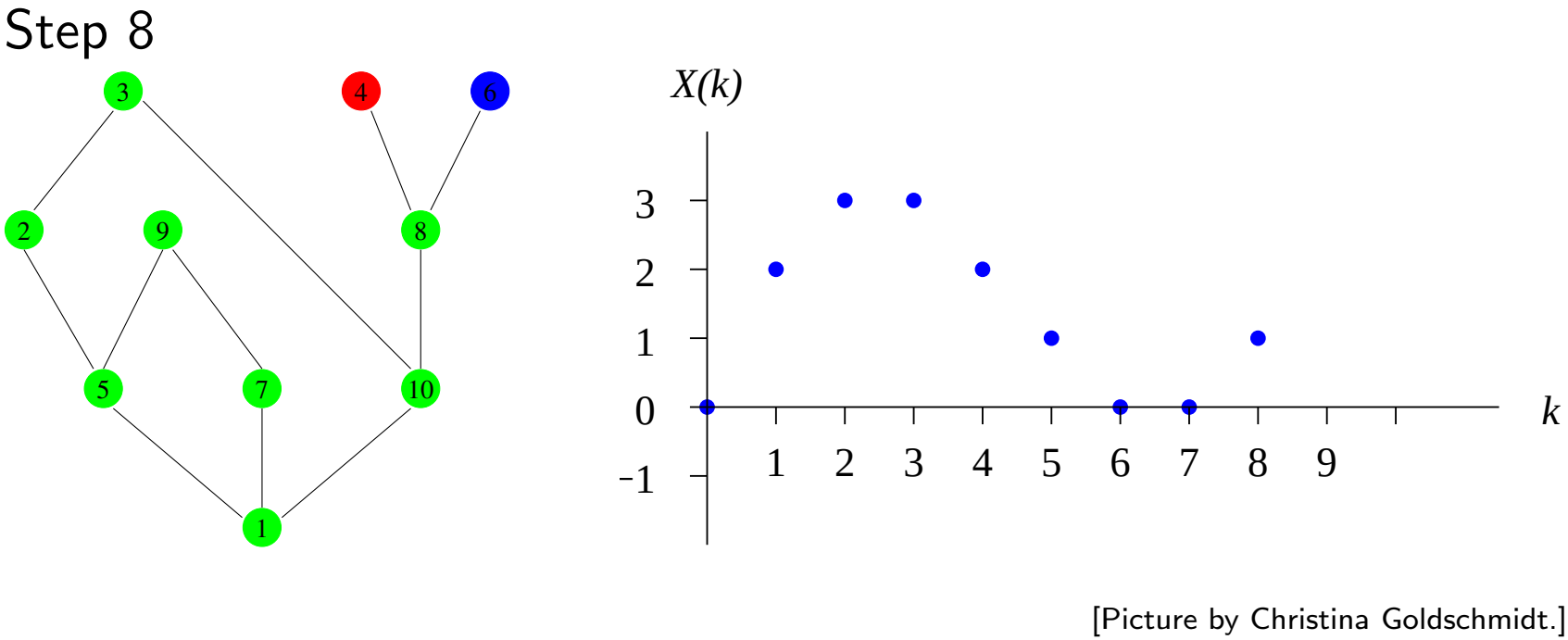




[Picture by Christina Goldschmidt.]

Current: 8 Alive: none Dead: 1,5,2,3,9,7,10

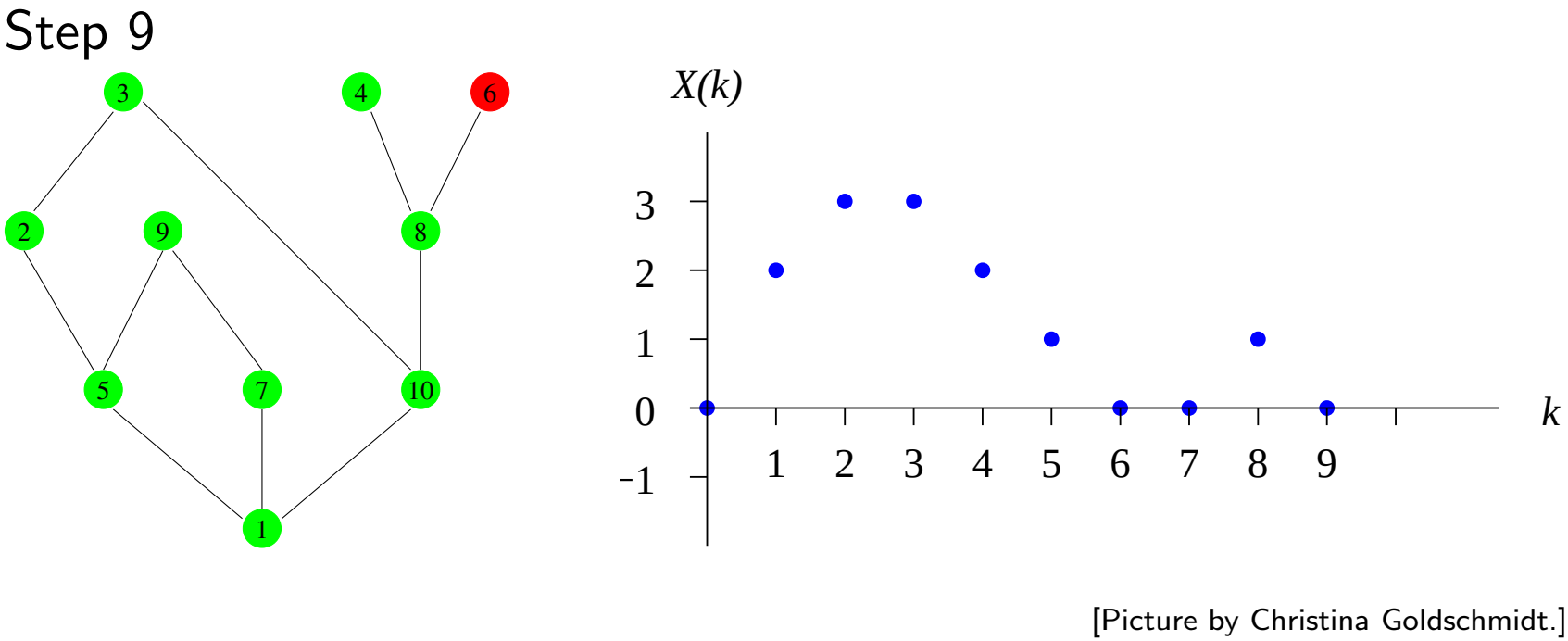
Depth-first walk



[Picture by Christina Goldschmidt.]

Current: 4 Alive: 6 Dead: 1,5,2,3,9,7,10,8

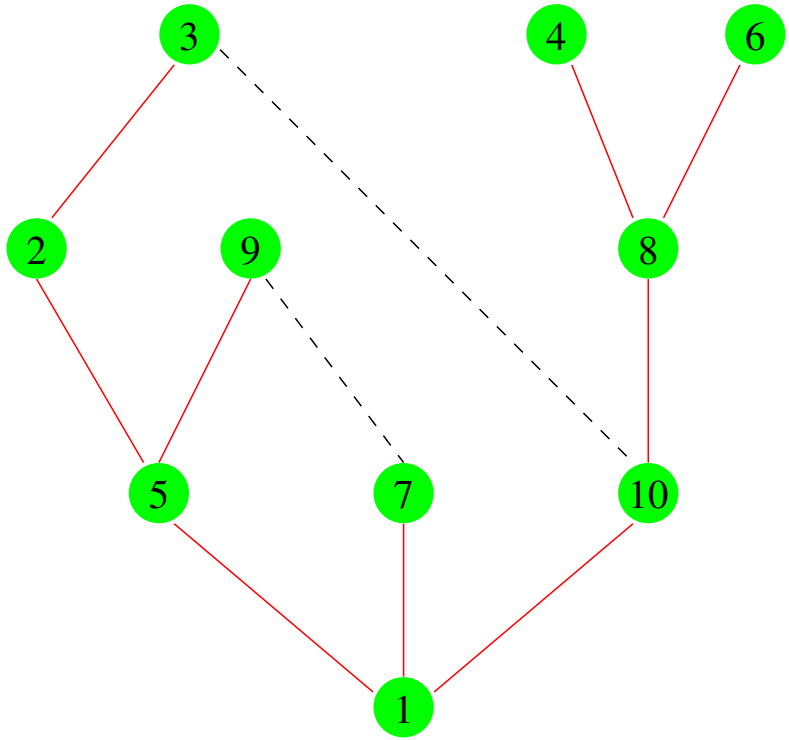
Depth-first walk



[Picture by Christina Goldschmidt.]

Current: 6 Alive: none Dead: 1,5,2,3,9,7,10,8,4

Depth-first tree



We essentially explored this underlying tree (the dashed edges made no difference to the depth-first walk). This tree is the depth-first tree associated with the graph.

Permitted edges

What are the set of graphs with a given depth-first tree/canonical discrete excursion?

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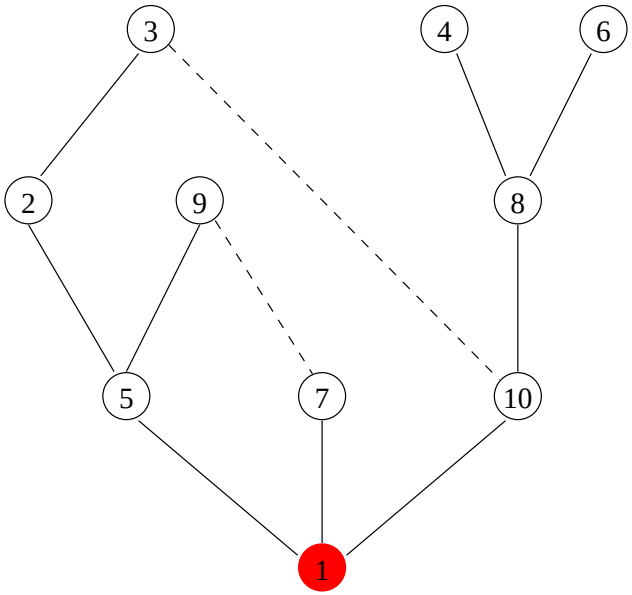
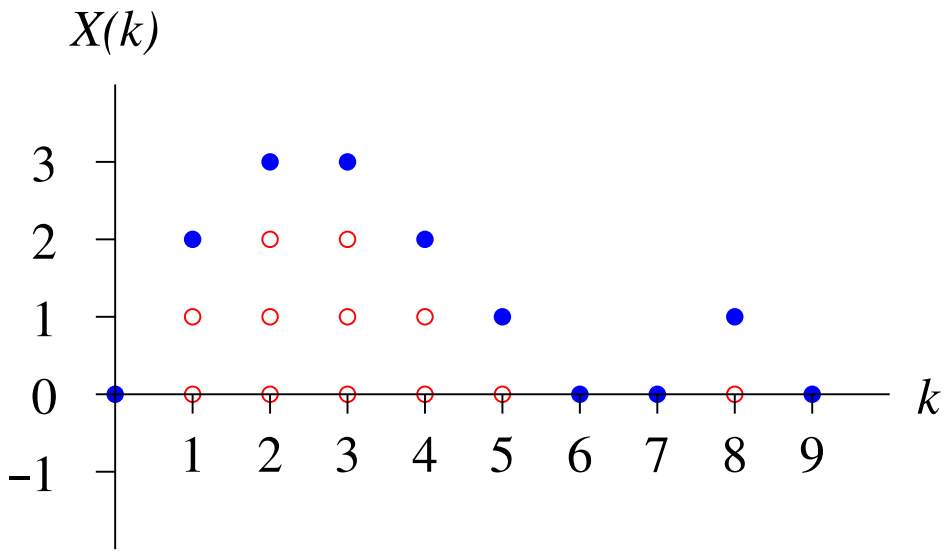
Call such edges permitted.

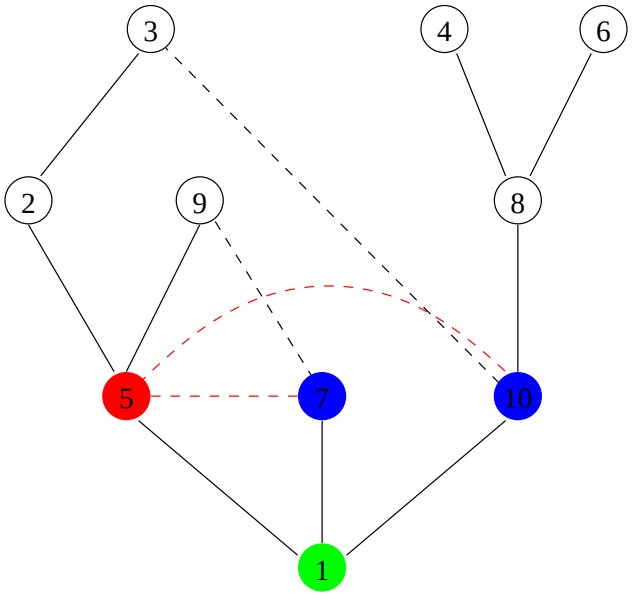
Depth-first walk and permitted edges

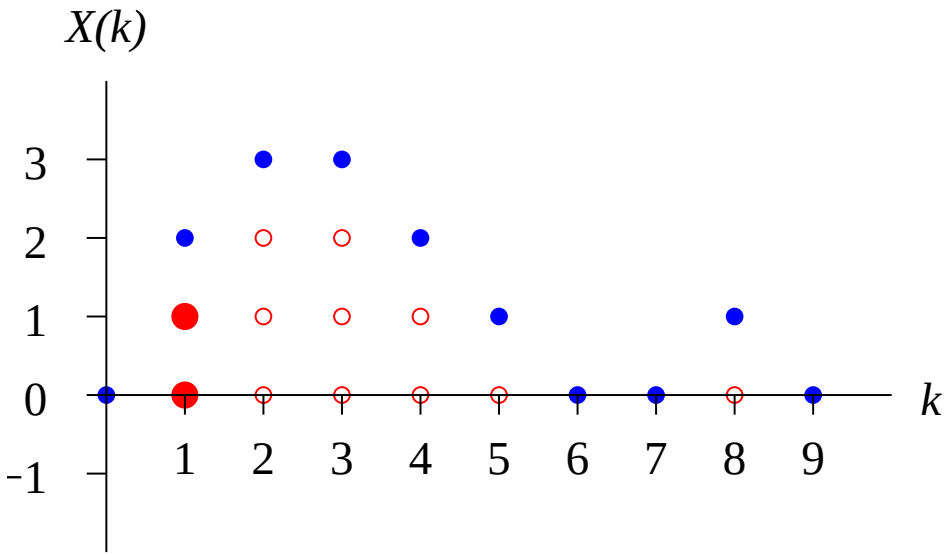




Step 0: $X(0) = 0$.

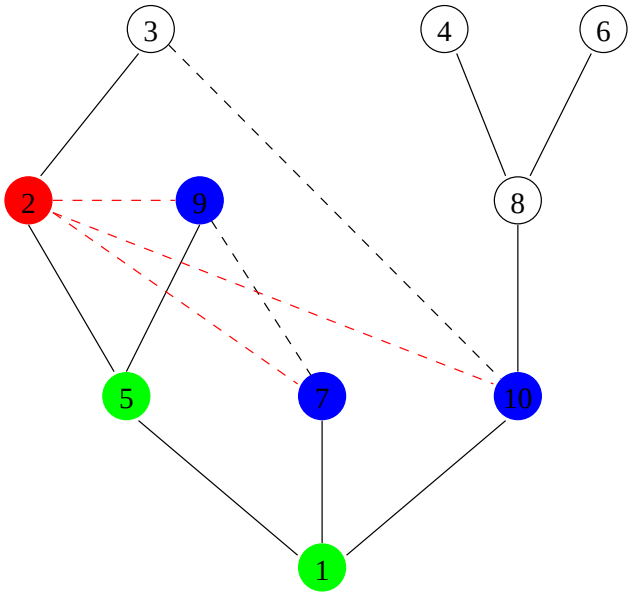
Depth-first walk and permitted edges

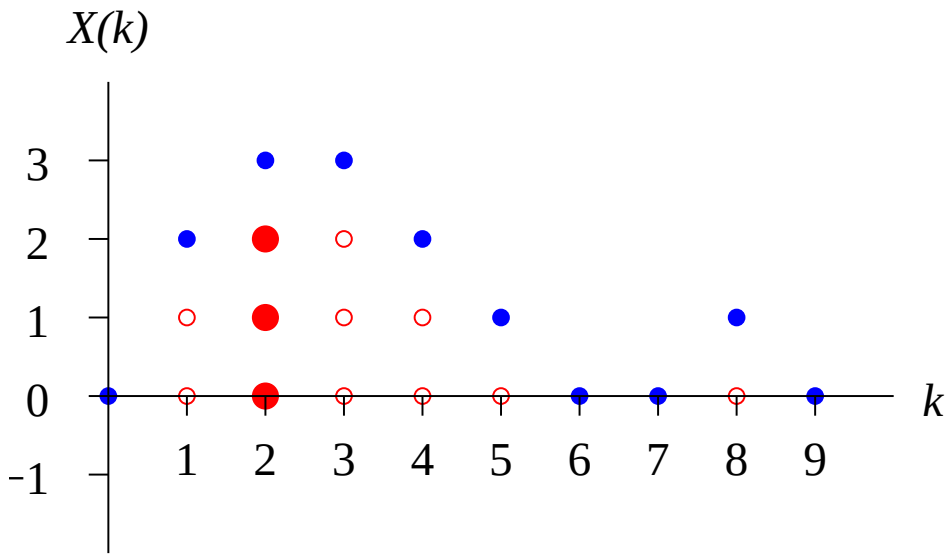




Step 1: $X(1) = 2$.

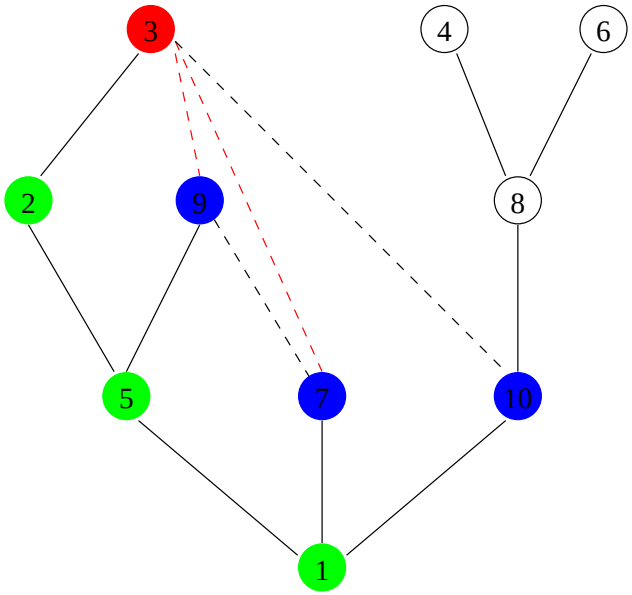
Depth-first walk and permitted edges

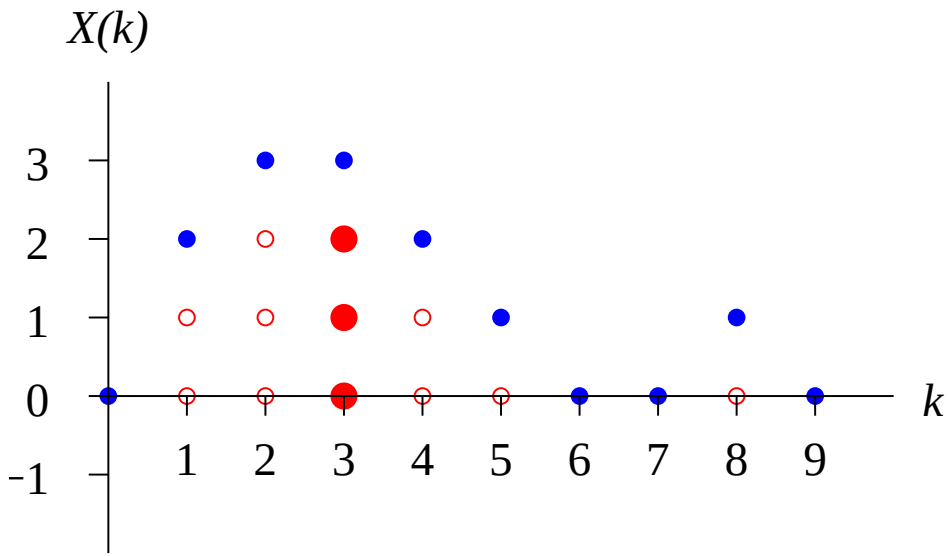




Step 2: $X(2) = 3$.

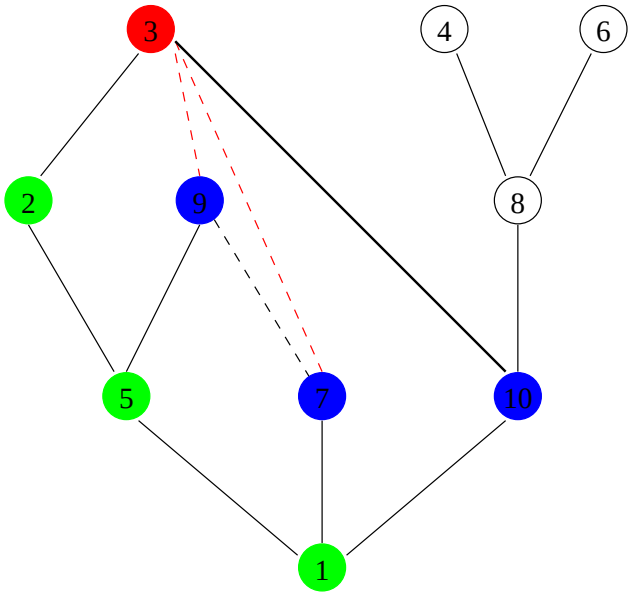
Depth-first walk and permitted edges

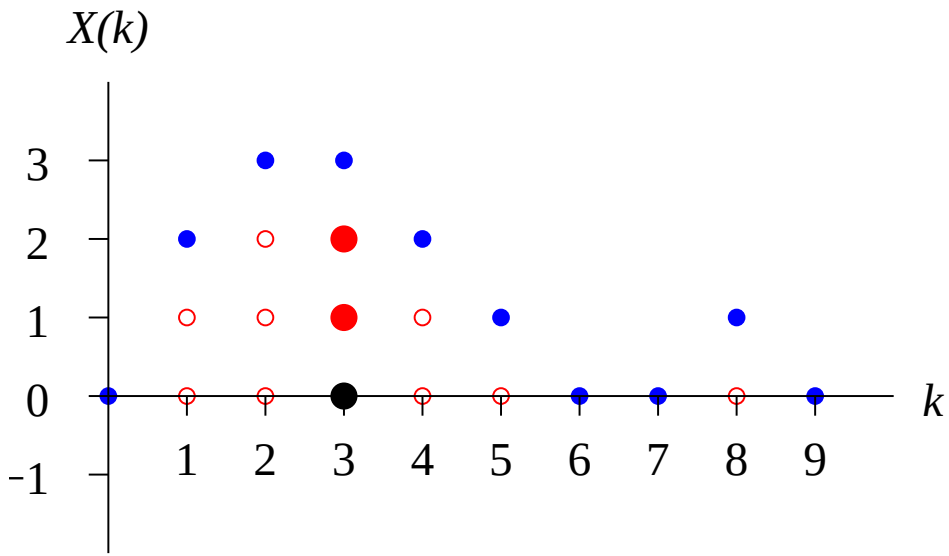




Step 3: $X(3) = 3$.

Depth-first walk and permitted edges





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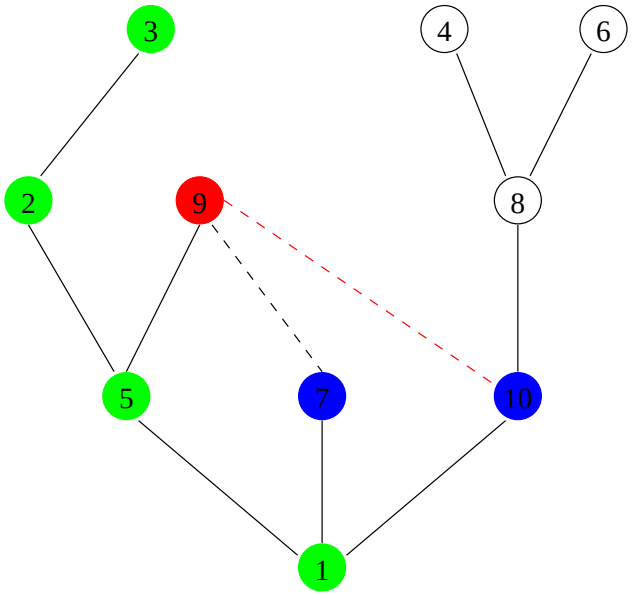
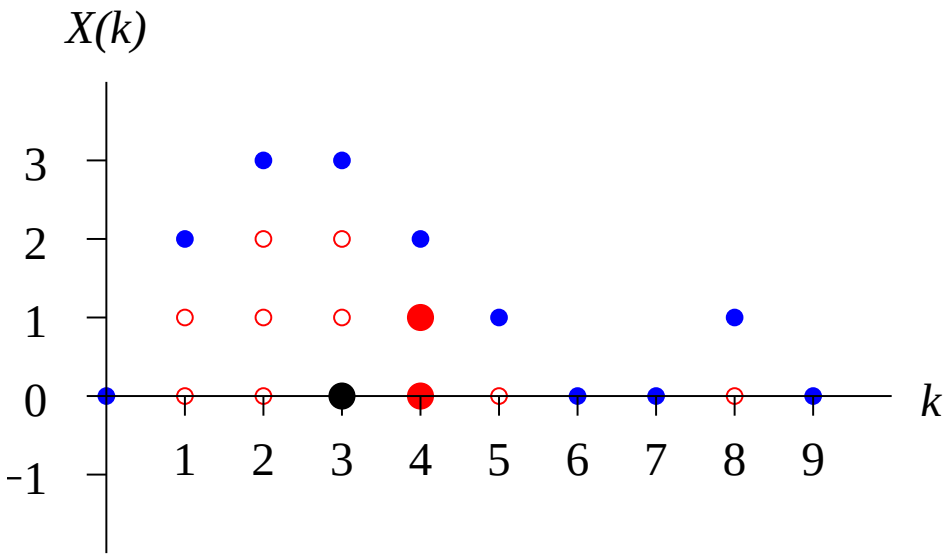
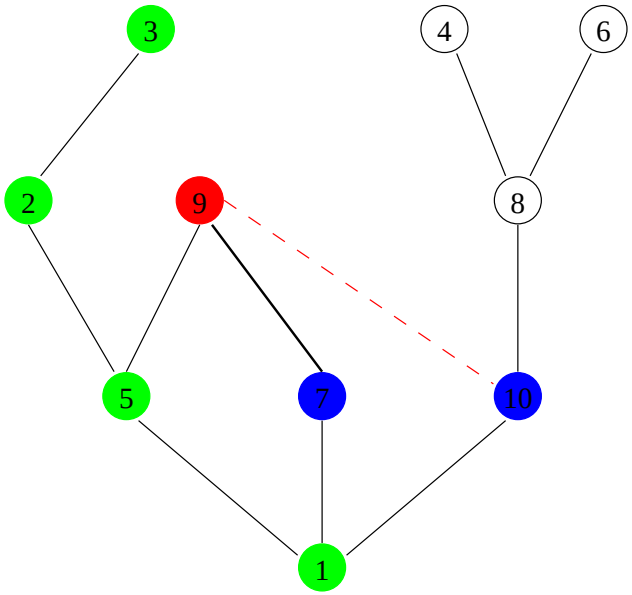
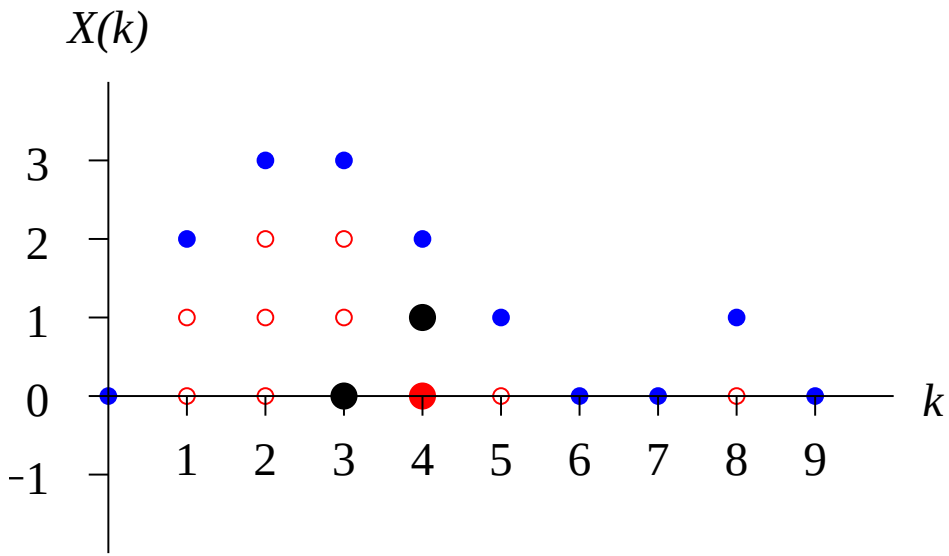




Step 4: $X(4) = 2$.

Depth-first walk and permitted edges





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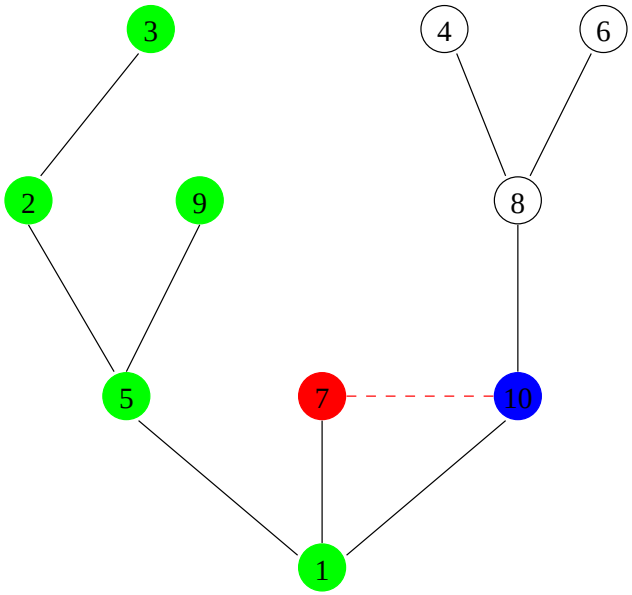
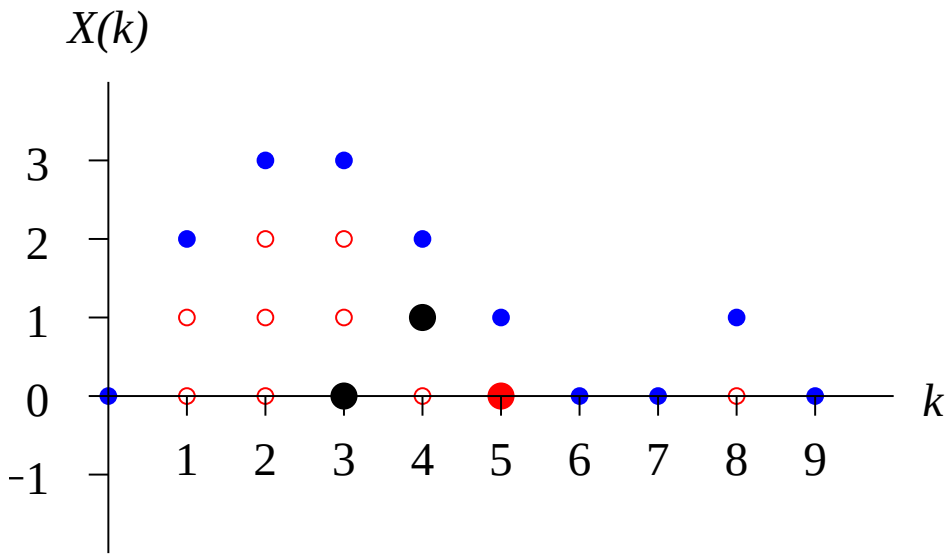
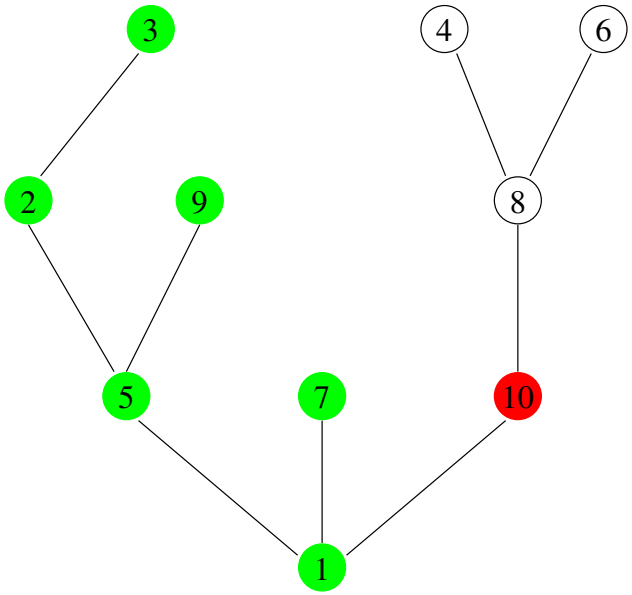
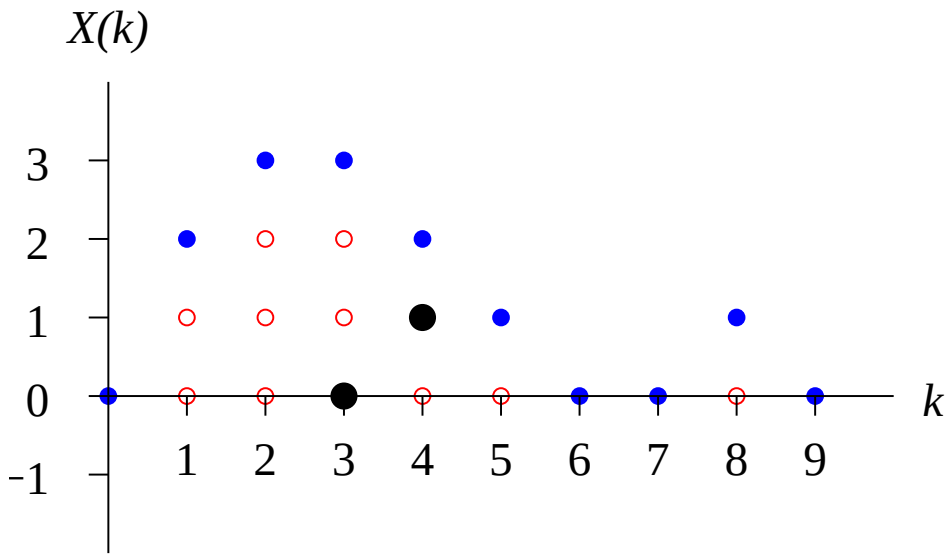




Step 5: $X(5) = 1$.

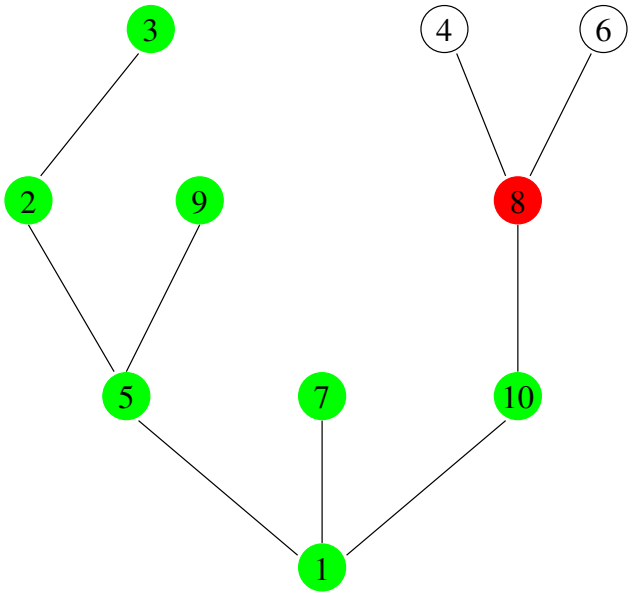
Depth-first walk and permitted edges

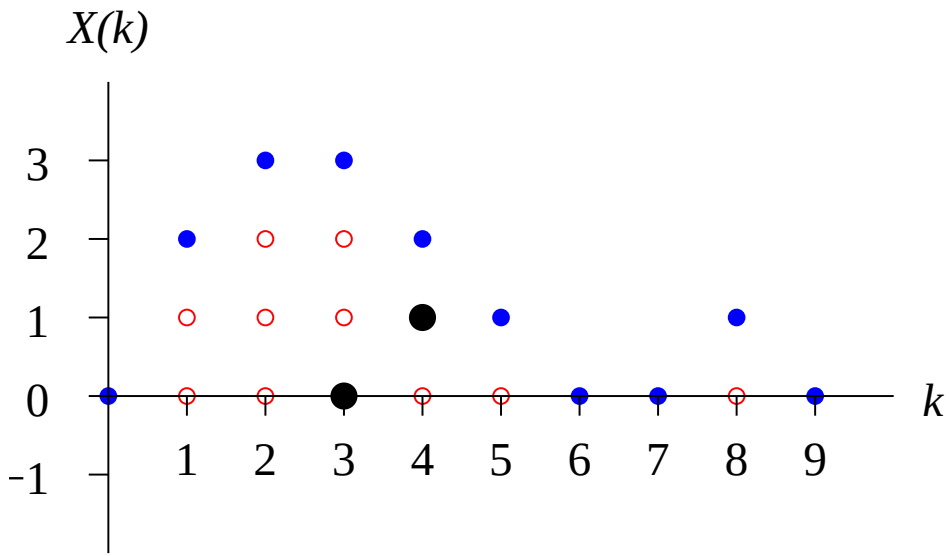




Step 6: $X(6) = 0$.

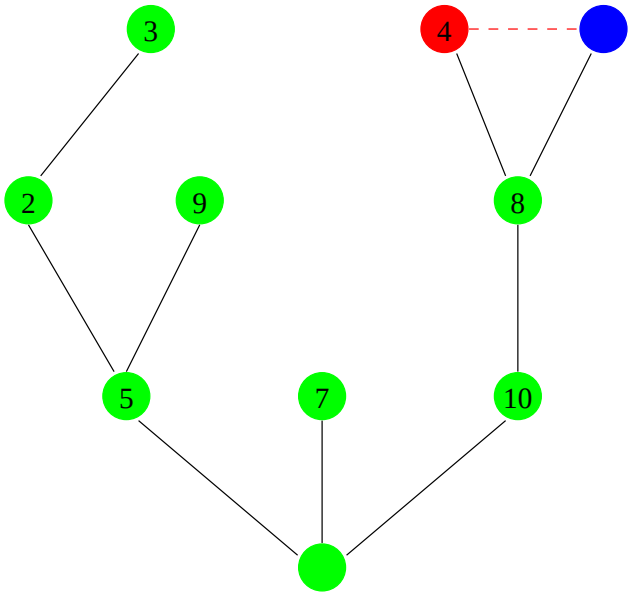
Depth-first walk and permitted edges

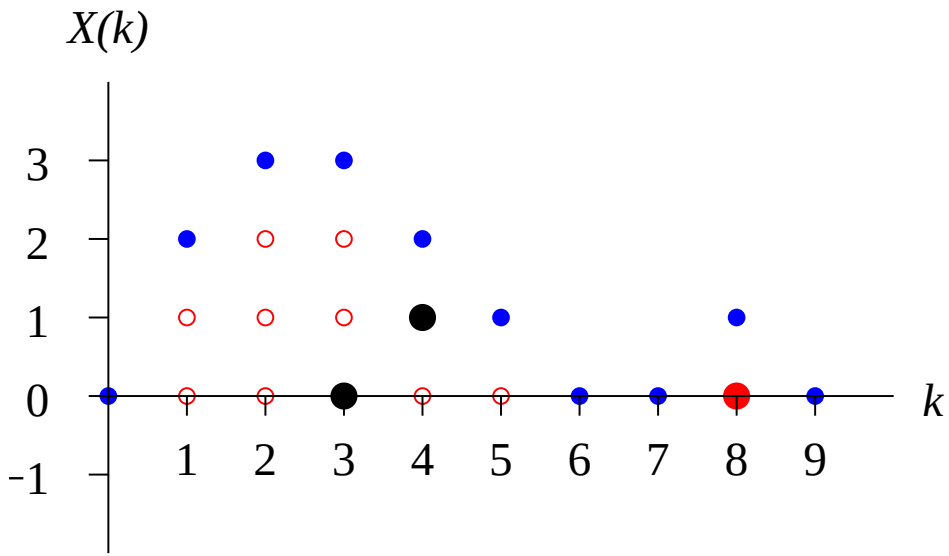




Step 7: $X(7) = 0$.

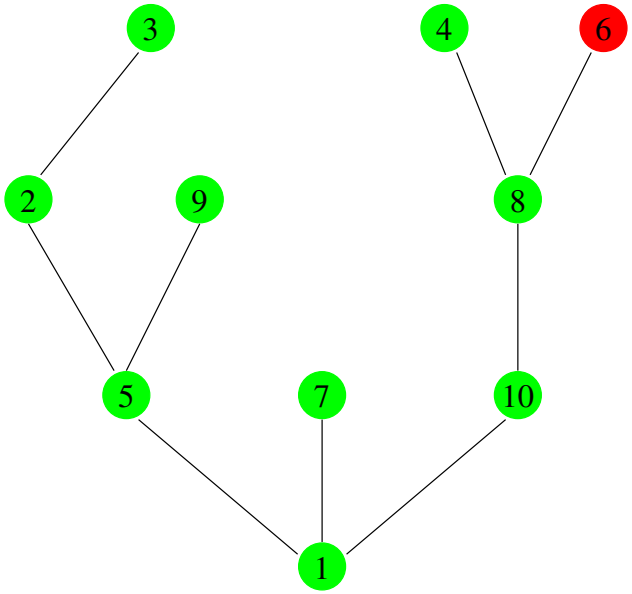
Depth-first walk and permitted edges

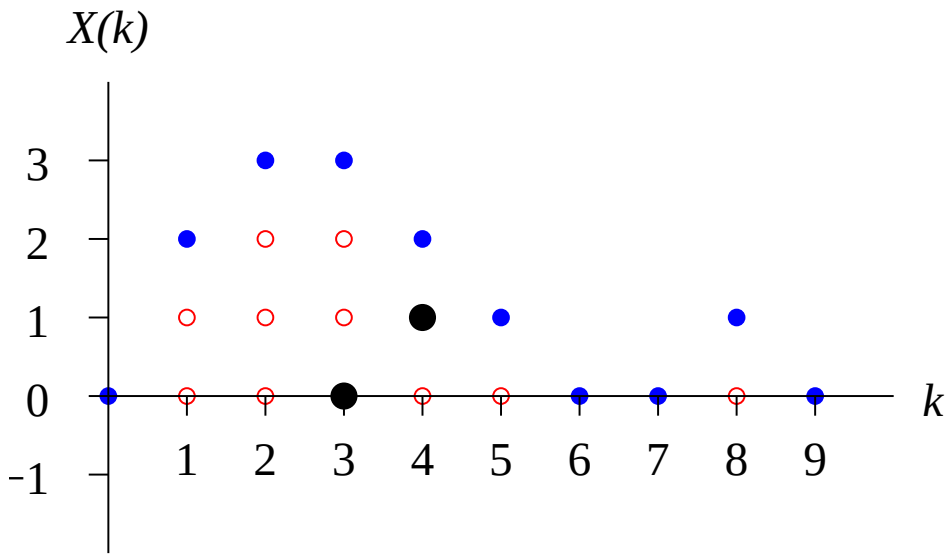




Step 8: $X(8) = 1$.

Depth-first walk and permitted edges





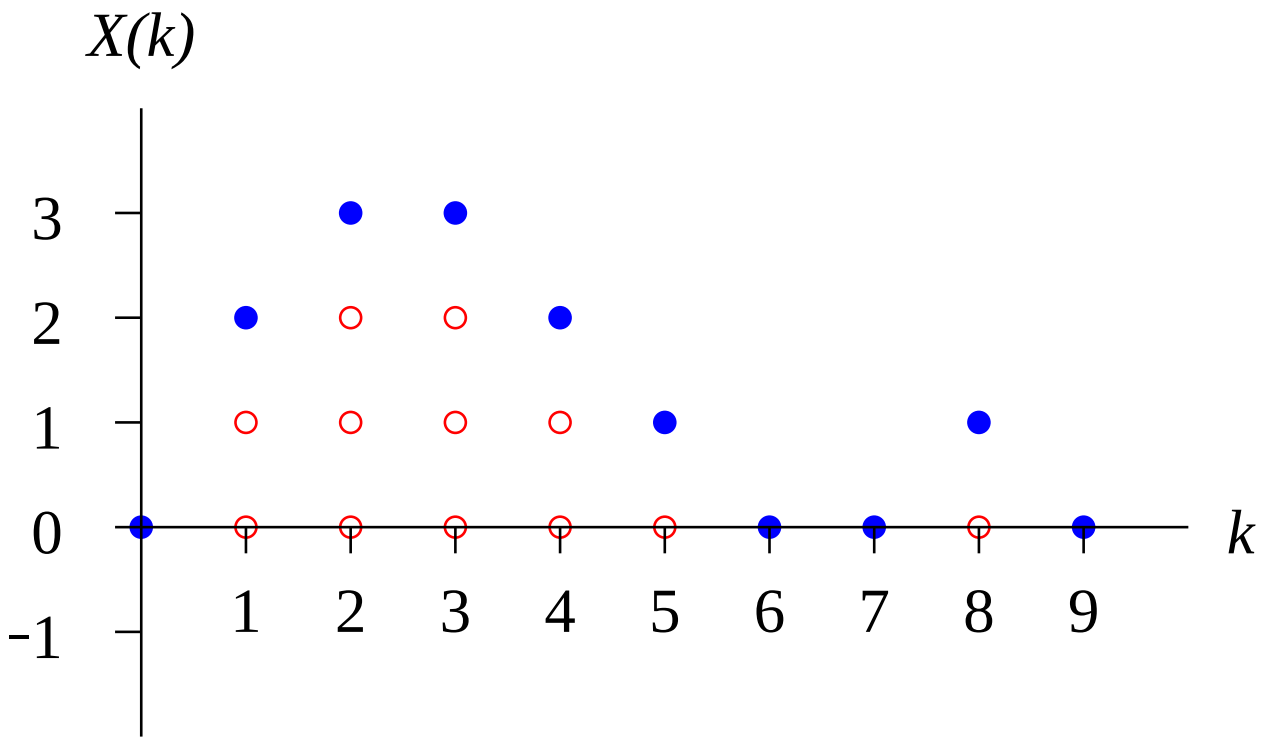
Step 10: $X(9) = 0$.

Area

At step $k \geq 0$ there are $X(k)$ permitted edges. So the total number is

$$a(T) = \sum_{k=0}^{m-1} X(k).$$

We call this the area of T.



Classifying graphs by depth-first tree

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Recall that $\mathbb{T}_{[m]}$ is the set of trees with label-set $[m] = \{1, 2, \ldots, m\}$. Then

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is a partition of the set of connected graphs on $[m]$.
Furthermore, for given T, an element of $\mathbb{G}_T$ corresponds to a set of marks (with integer coordinates) under the discrete excursion for T.

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Conditional on obtaining s surplus edges, the tree thus generated is a uniform random connected graph on m vertices with s surplus edges.

Tilting

Unconditionally, the above method generates a uniformly random connected graph on $1, \ldots, m$. When the desired surplus s is far from $m^2/2$, the conditioning is extremely unlikely. We can modify the method to make it more likely we get the desired surplus.

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For $p = 0$ we are just choosing a uniformly random tree.

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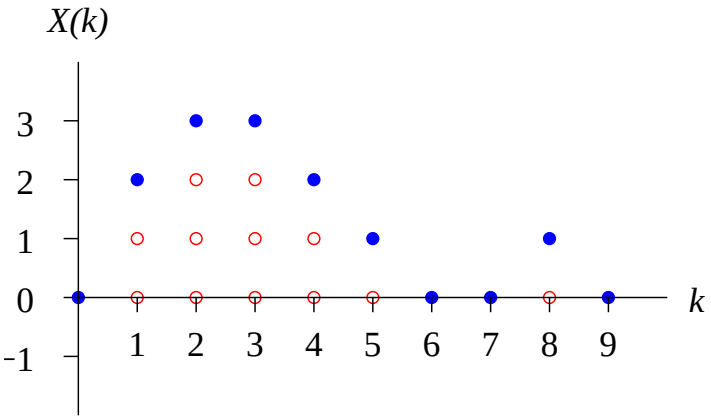
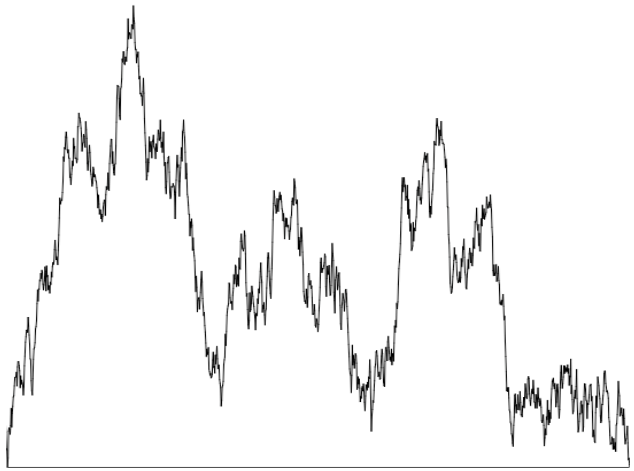
The result: a uniformly random connected component of $G_{n,1/n}$, conditioned upon having size m.

The scaling limit of depth-first walks.

When time is scaled by $1/m$, space is scaled by $1/\sqrt{m}$, the limit of depth-first walk of a *uniform* tree is *Brownian excursion*.





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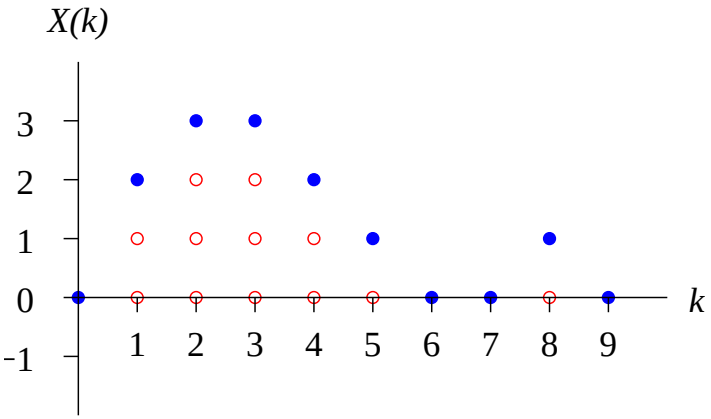

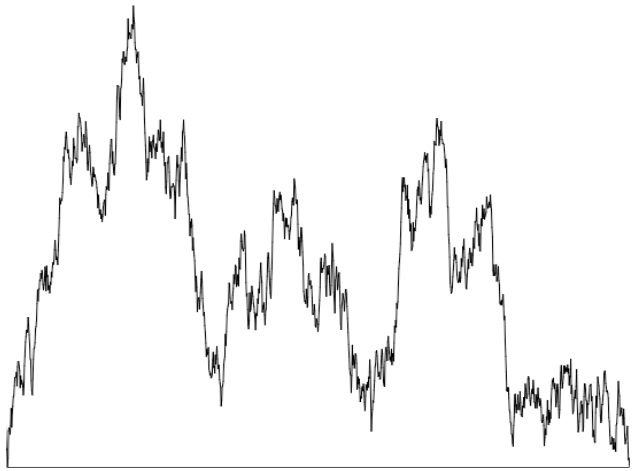

The *area* $a(T)$ is then $\Theta(m^{3/2})$, so if $m = n^{2/3}$ then $a(T) = \Theta(n)$ and

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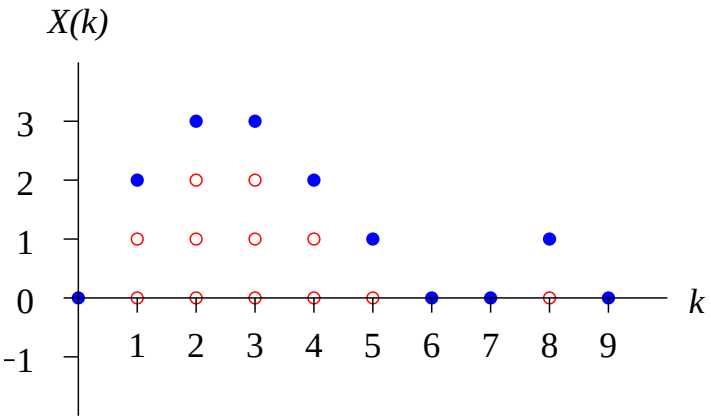

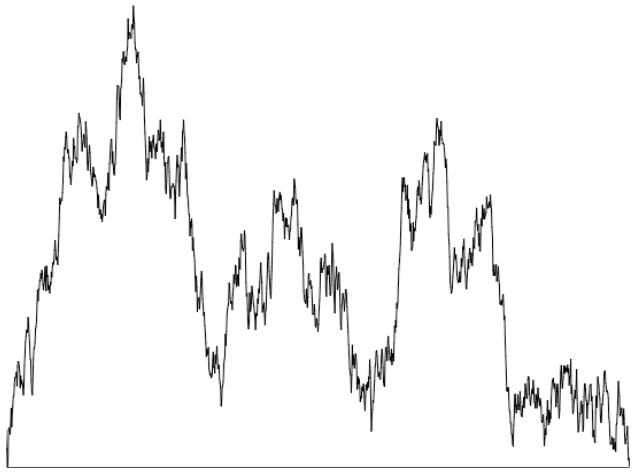

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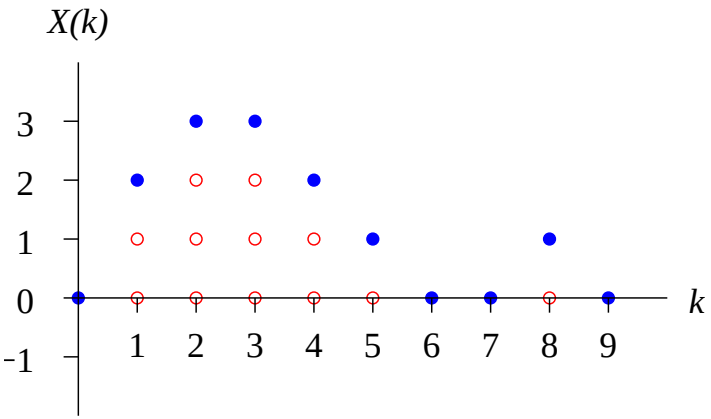

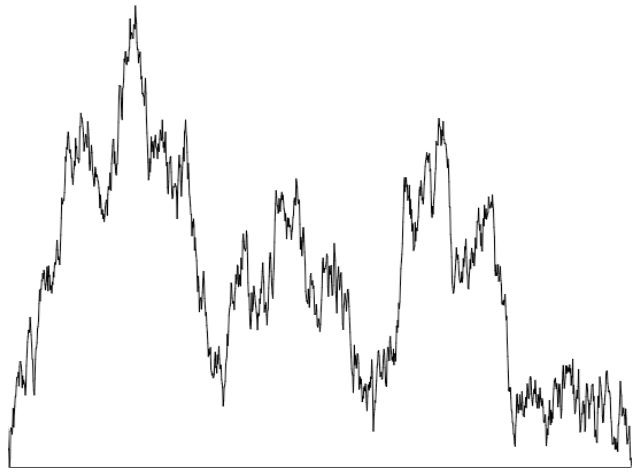

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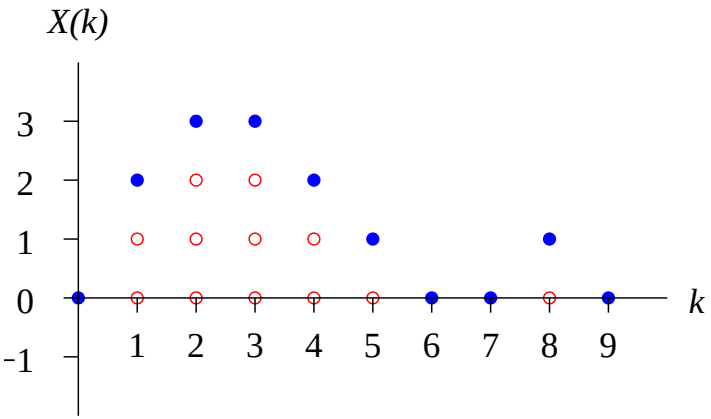

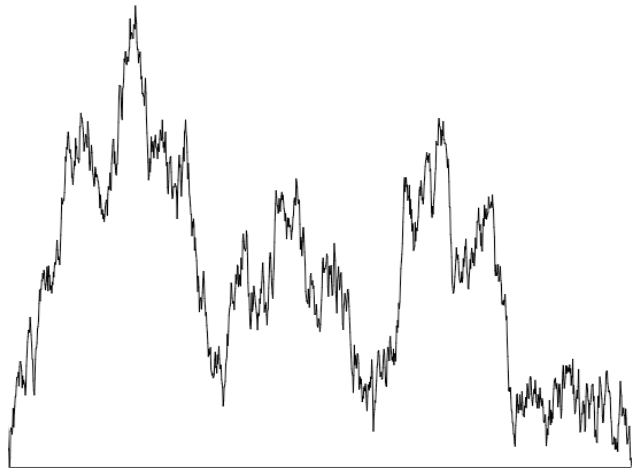

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This tilt can be shown to be exactly the effect of the quadratic drift. (Using Girsanov's theorem.)

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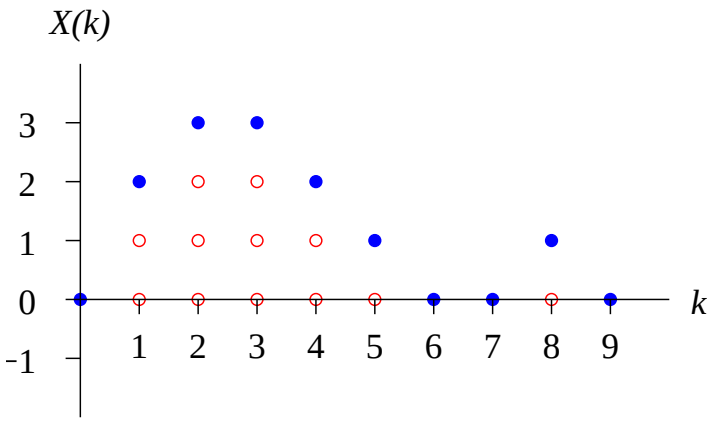

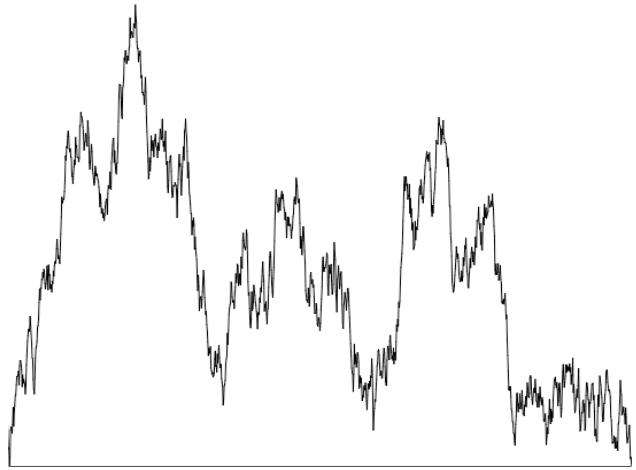

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In the limit, the binomial point process of marks under the depth-first walk becomes a Poisson process of marks under the excursion.

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Now have an asymptotic description of the structure of each "macroscopic" component of the critical random graph.

Brownian motion with drift?

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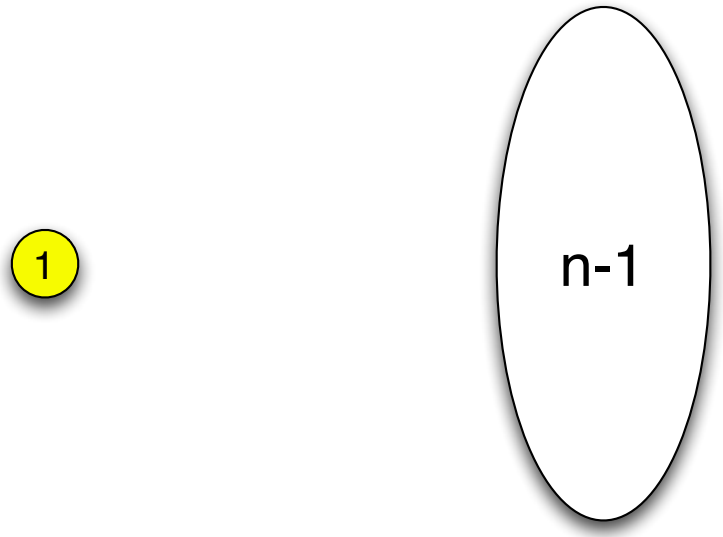

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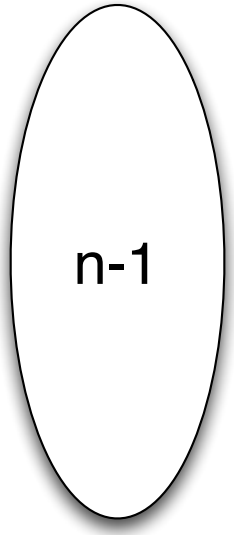

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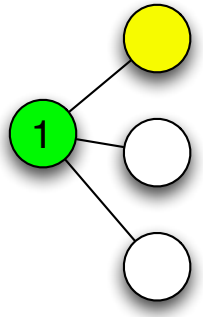

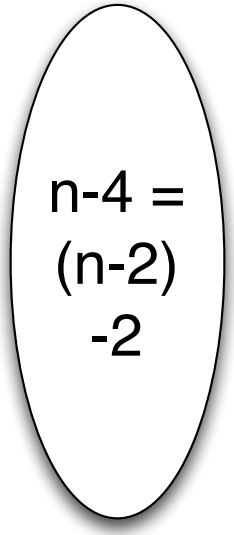

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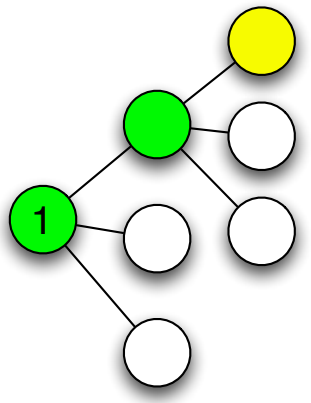

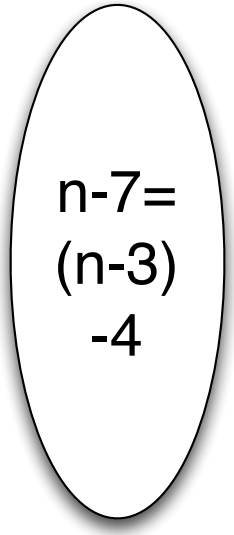

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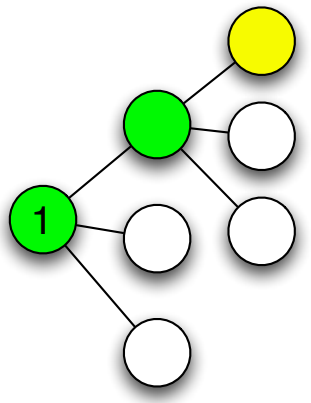

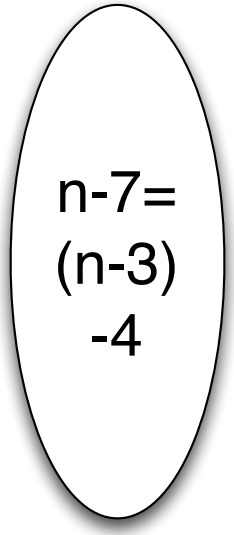

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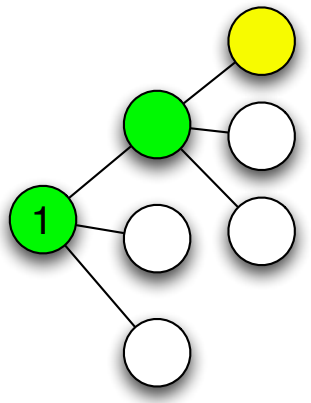

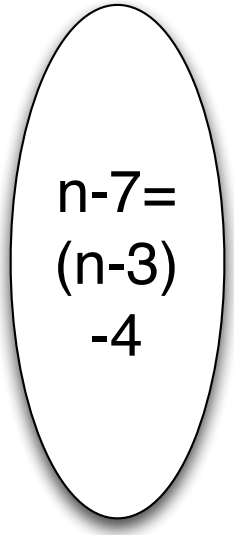

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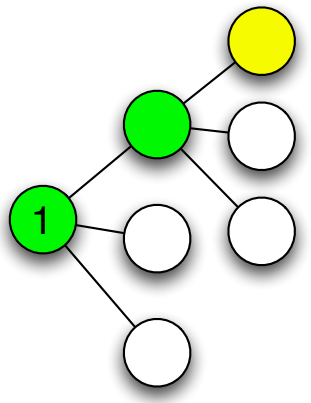

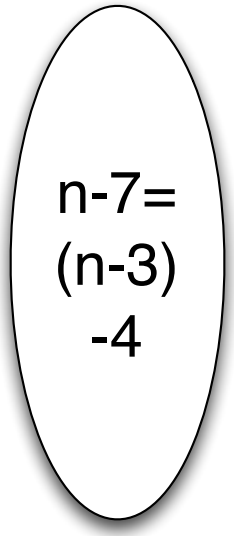

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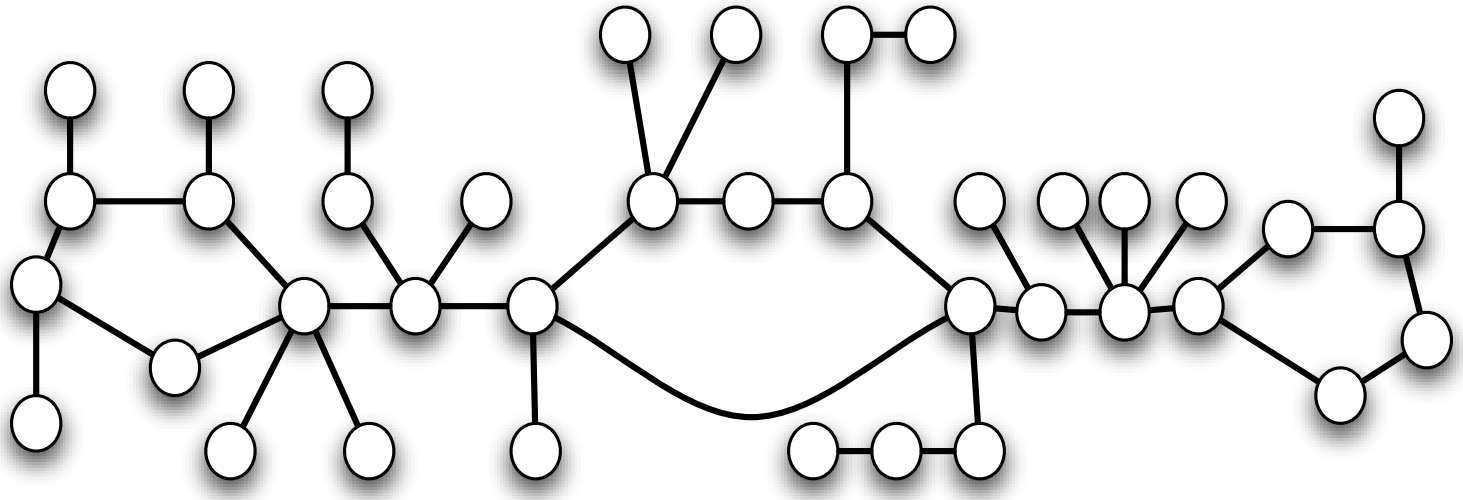
The cycle structure of a connected graph.



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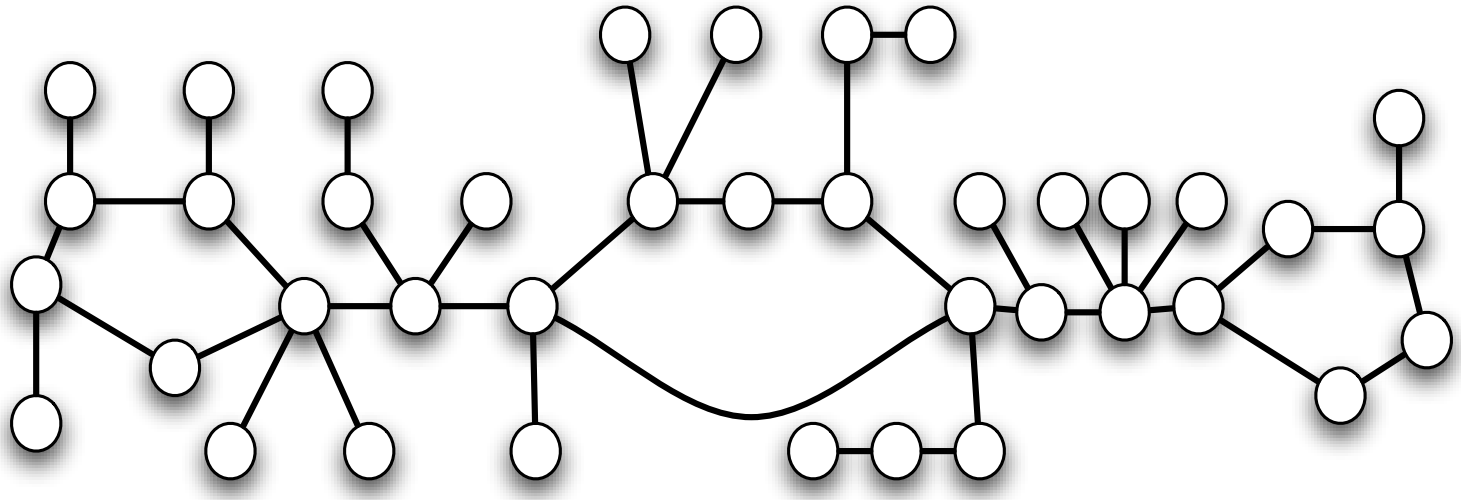

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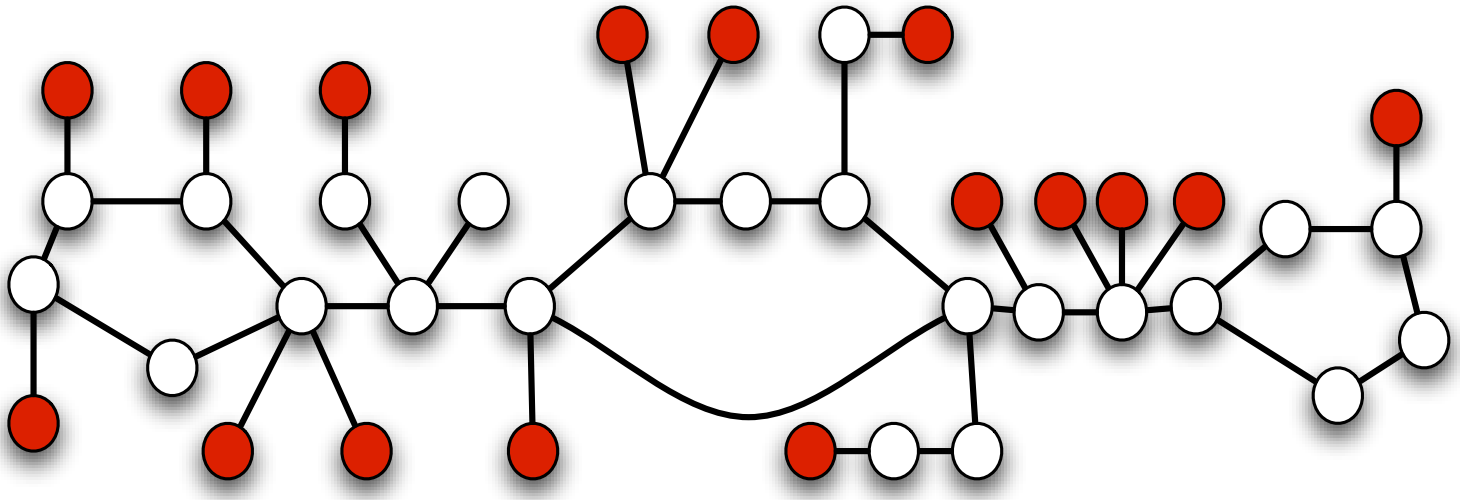

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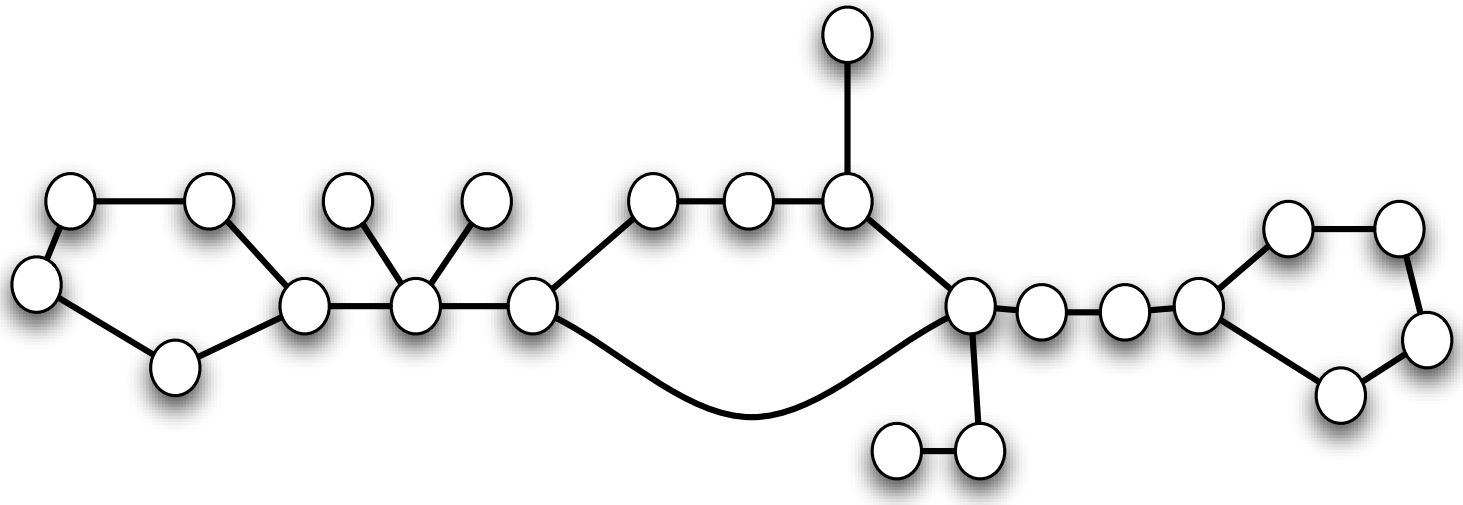

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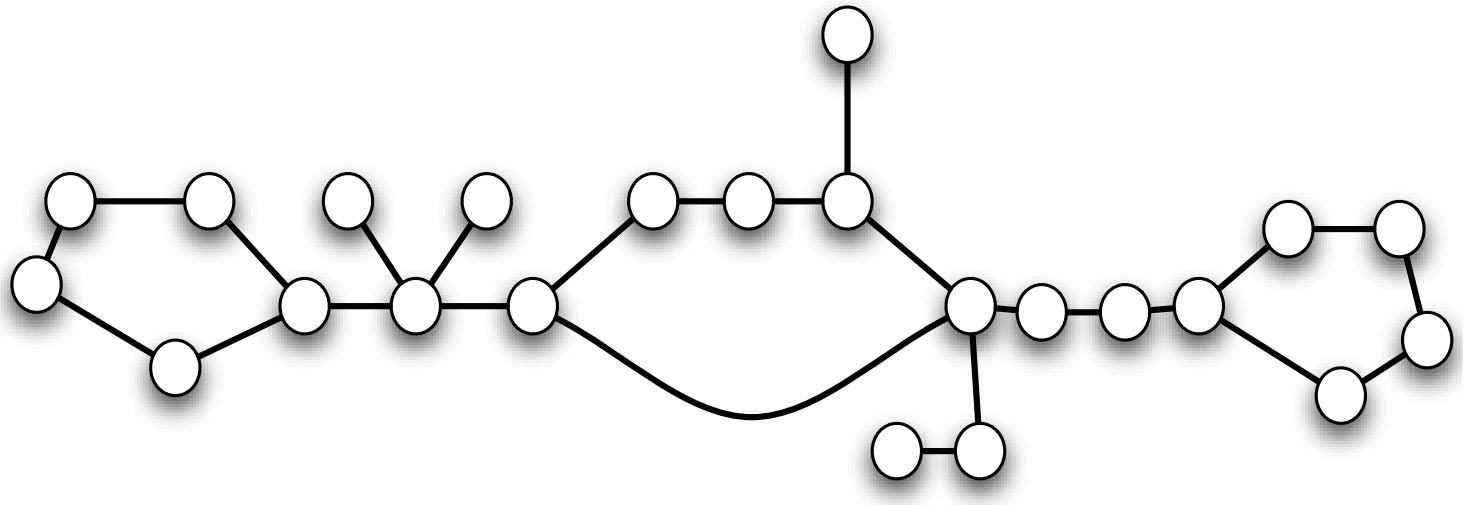

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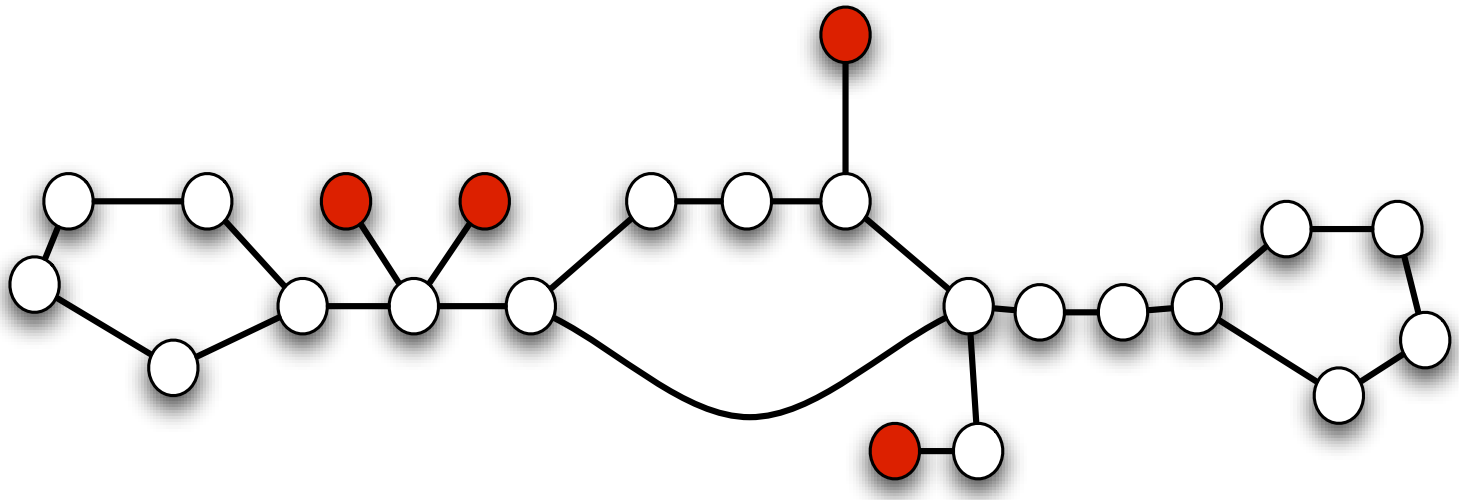

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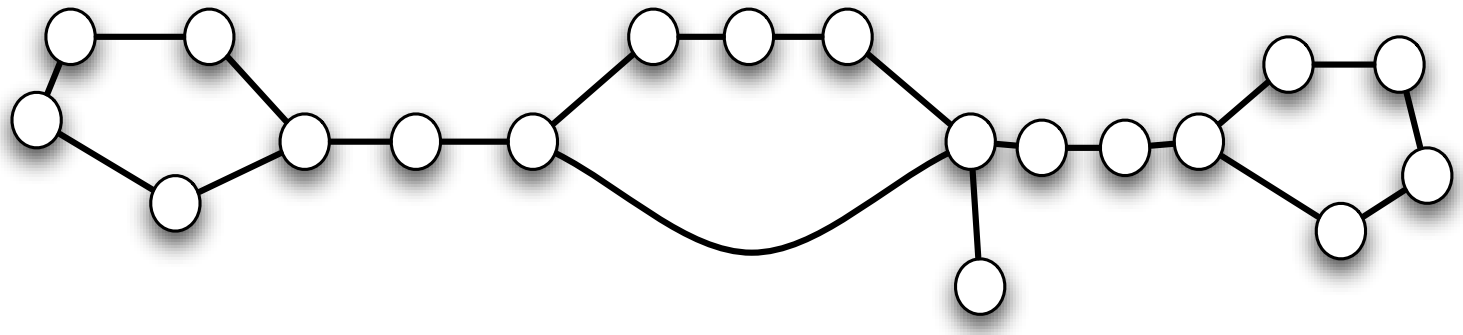

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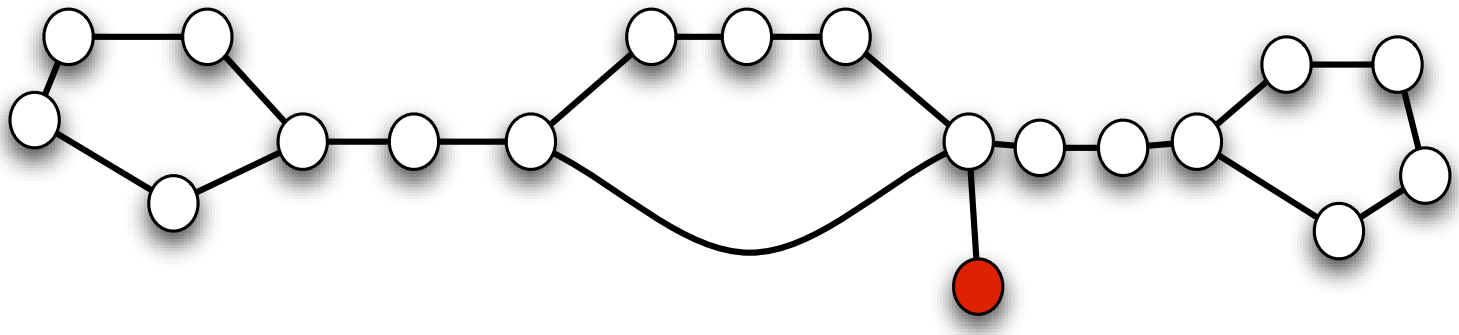

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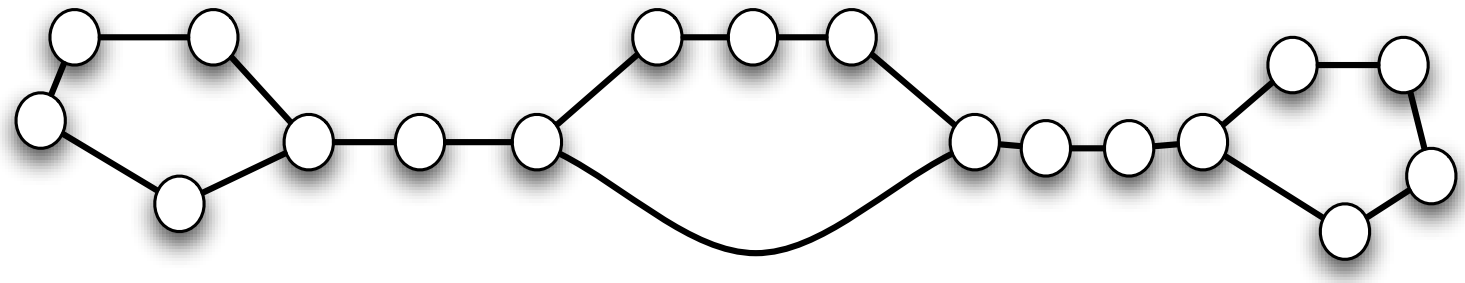

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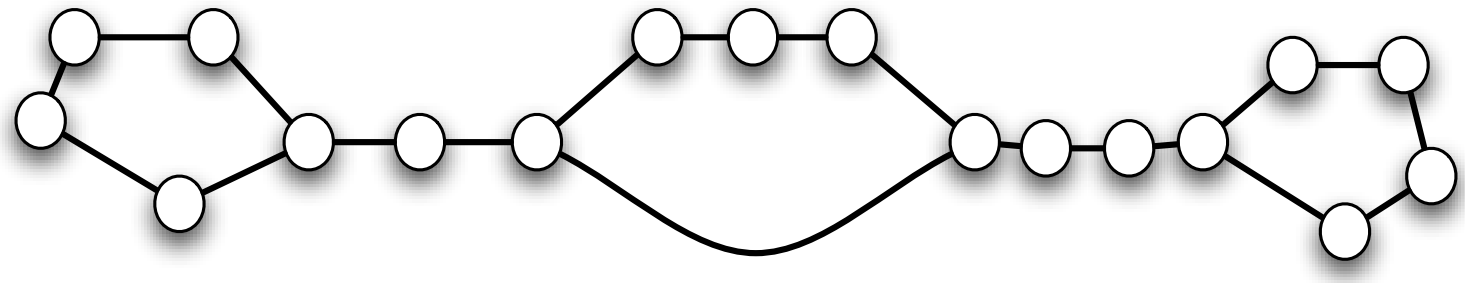

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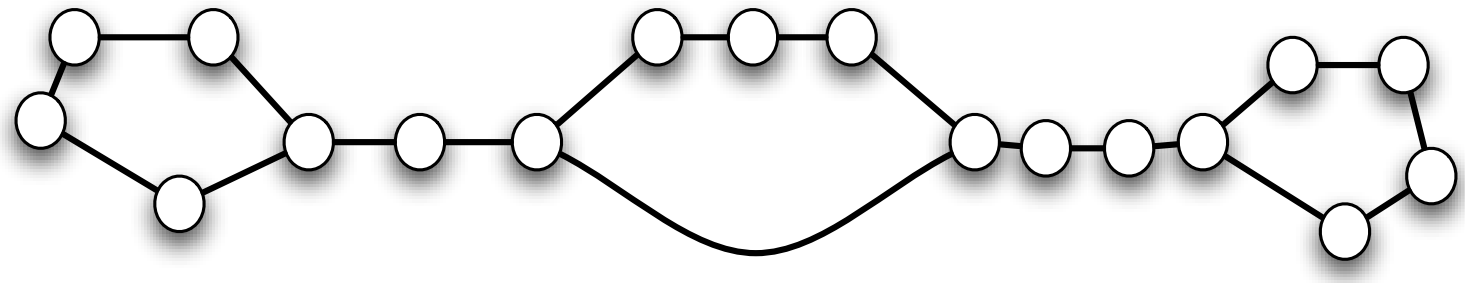

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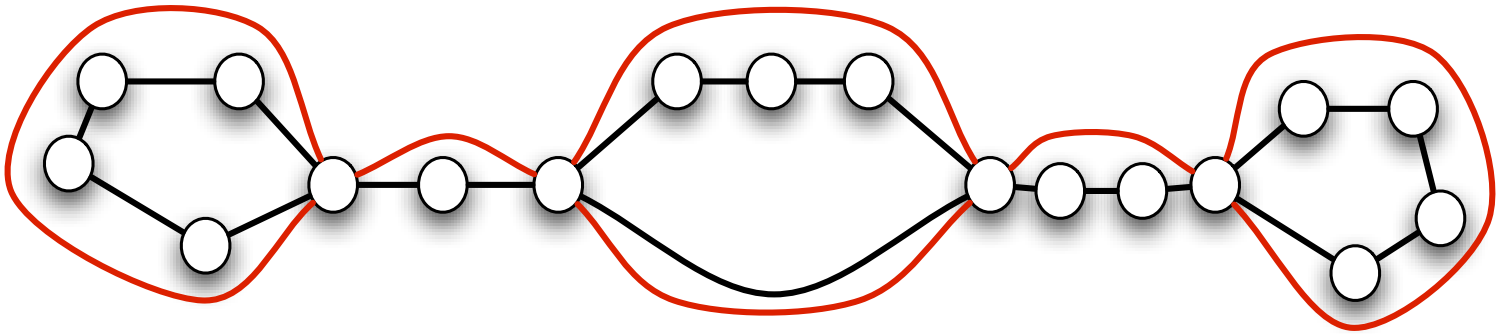

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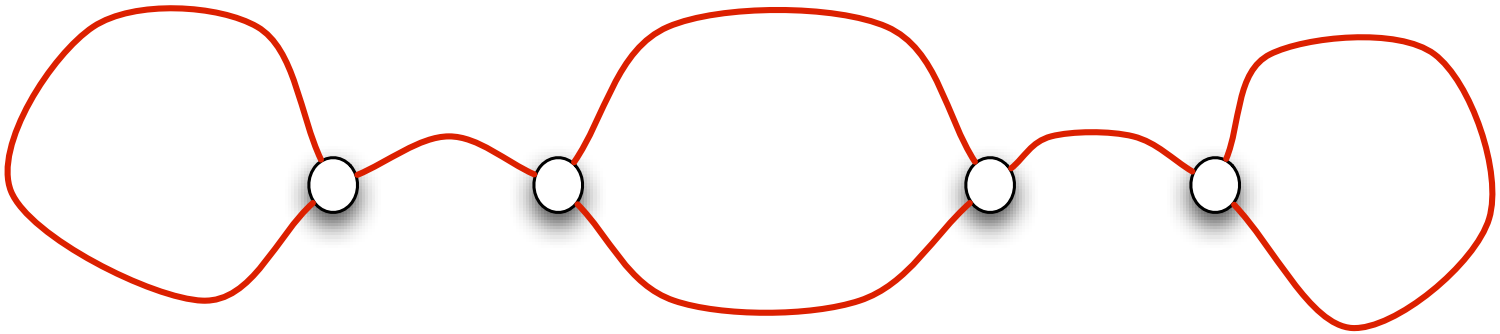

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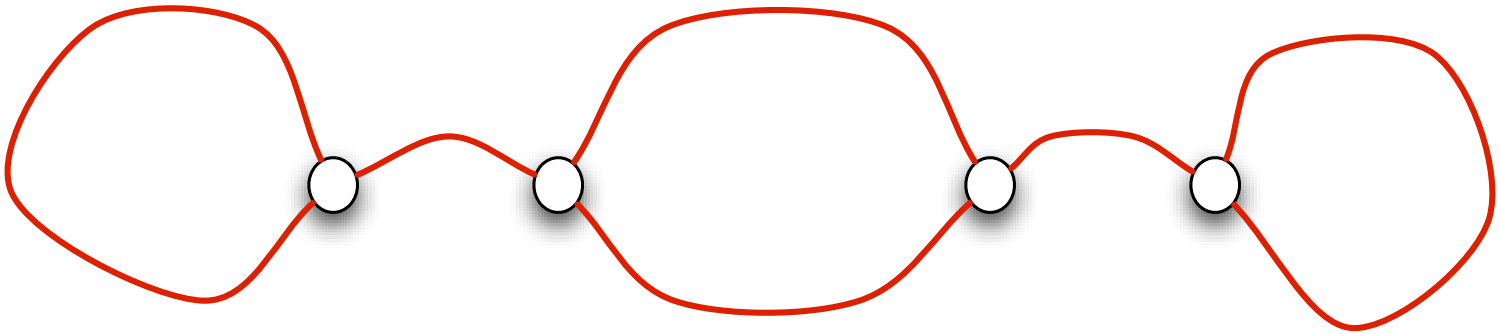

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- Result after suppressing degree-2 vertices is called the *kernel*.

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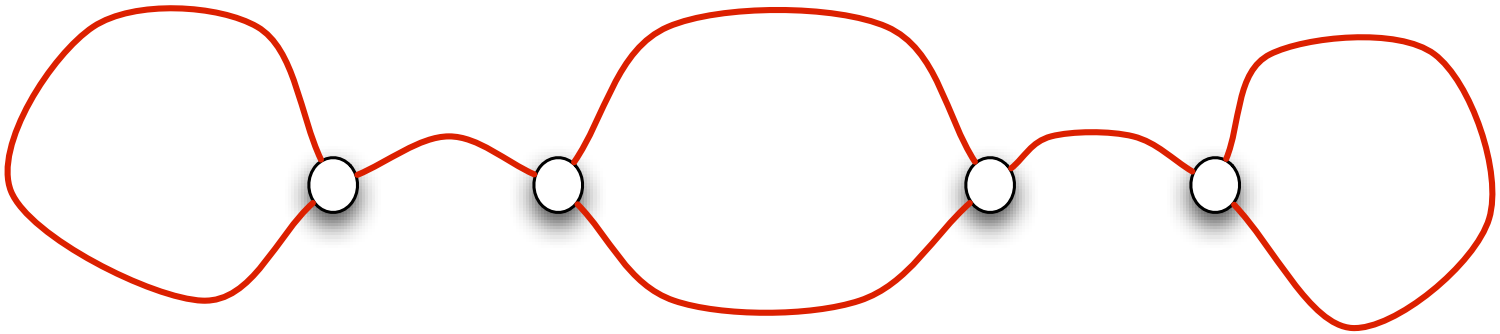

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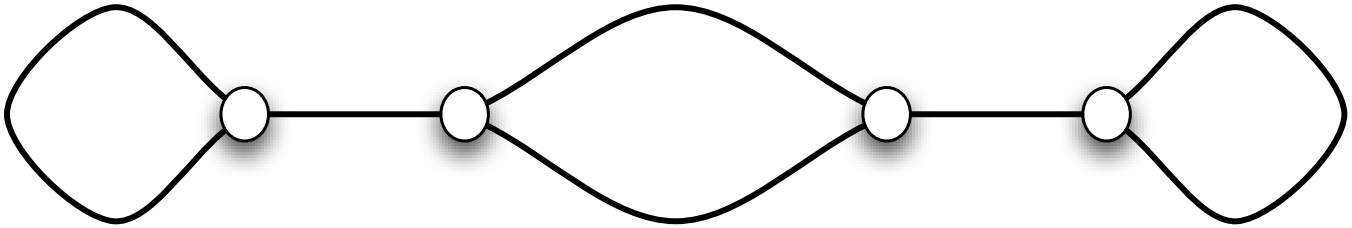
- Leaves (nodes of degree 1) aren't in cycles, so don't affect the cycle structure – remove them!
- What's left when all leaves are gone is called the *core*.
- Vertices of degree 2 in whats left just form paths – suppress them!
- Result after suppressing degree-2 vertices is called the *kernel*.
- Kernel has same surplus as the original graph.

The kernel



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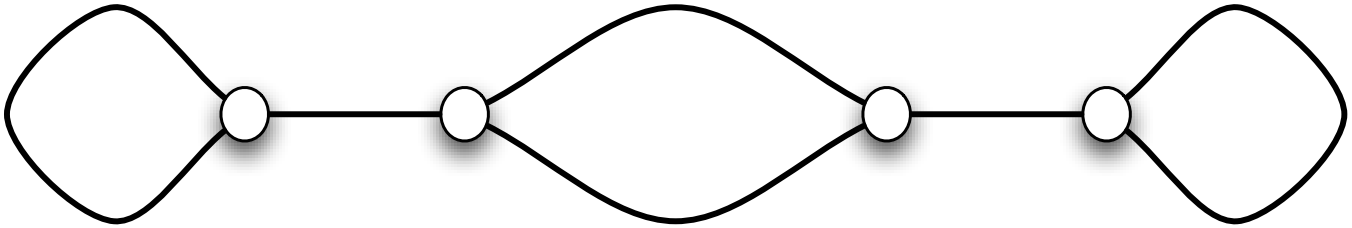

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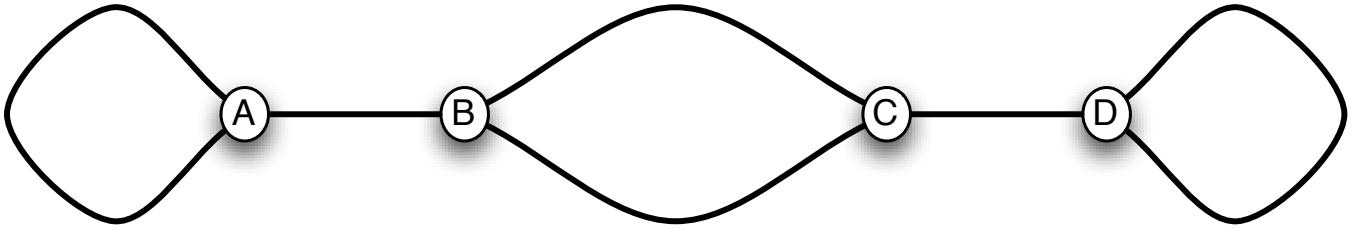

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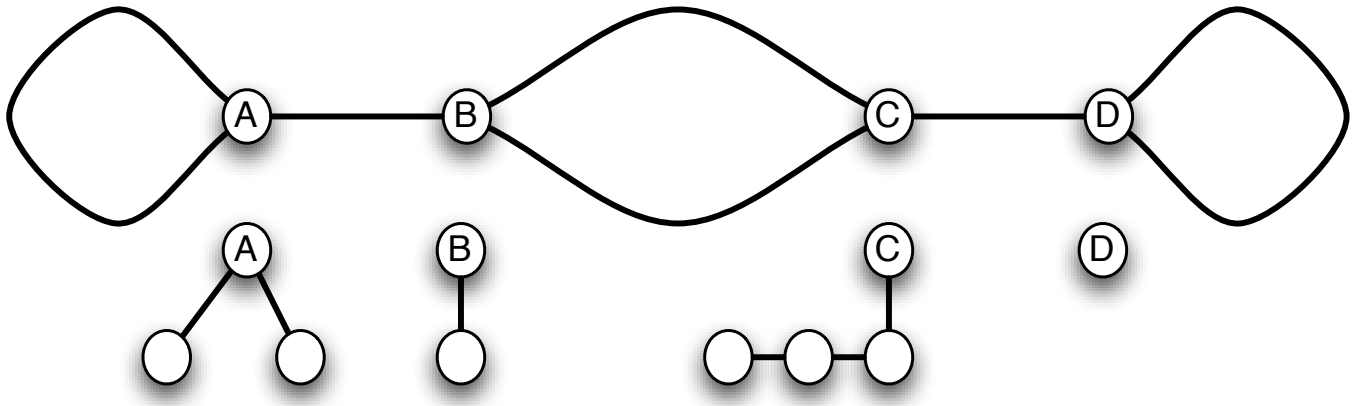

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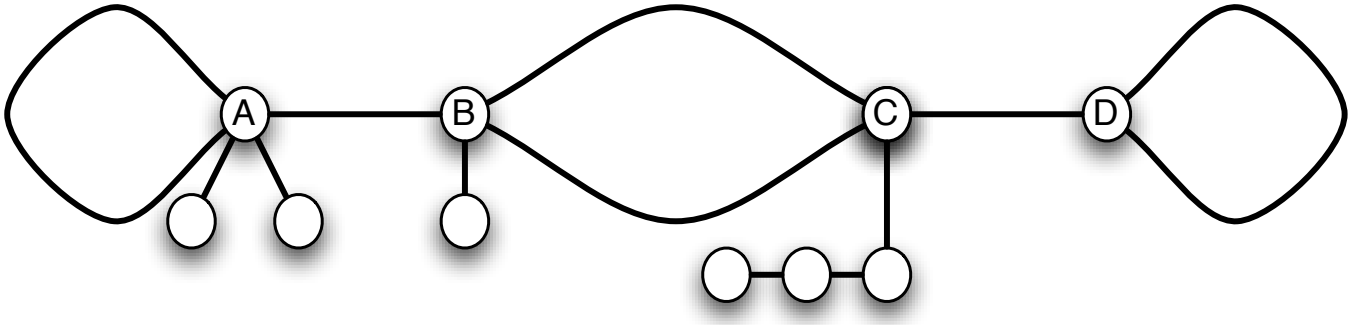

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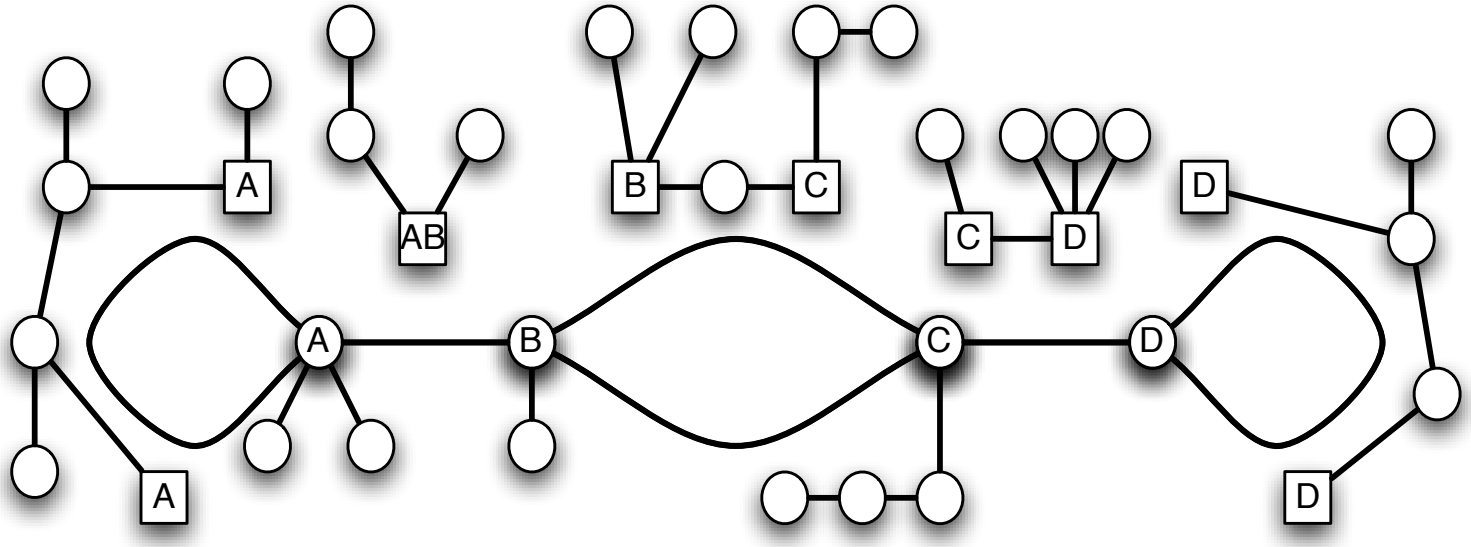

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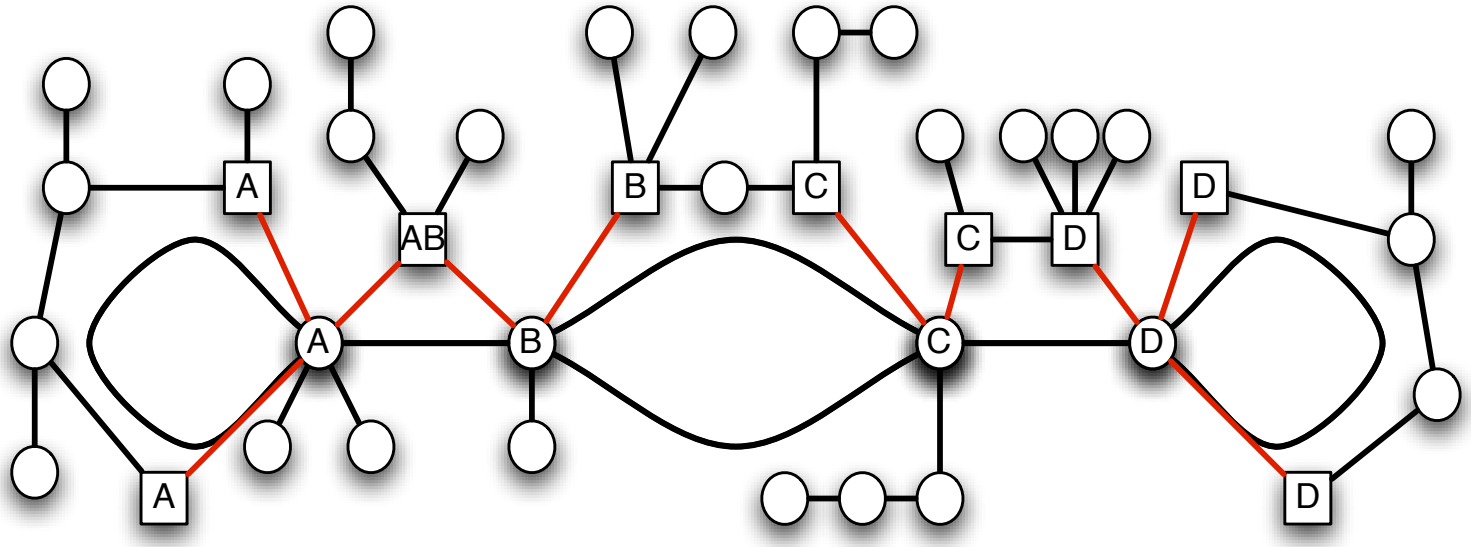

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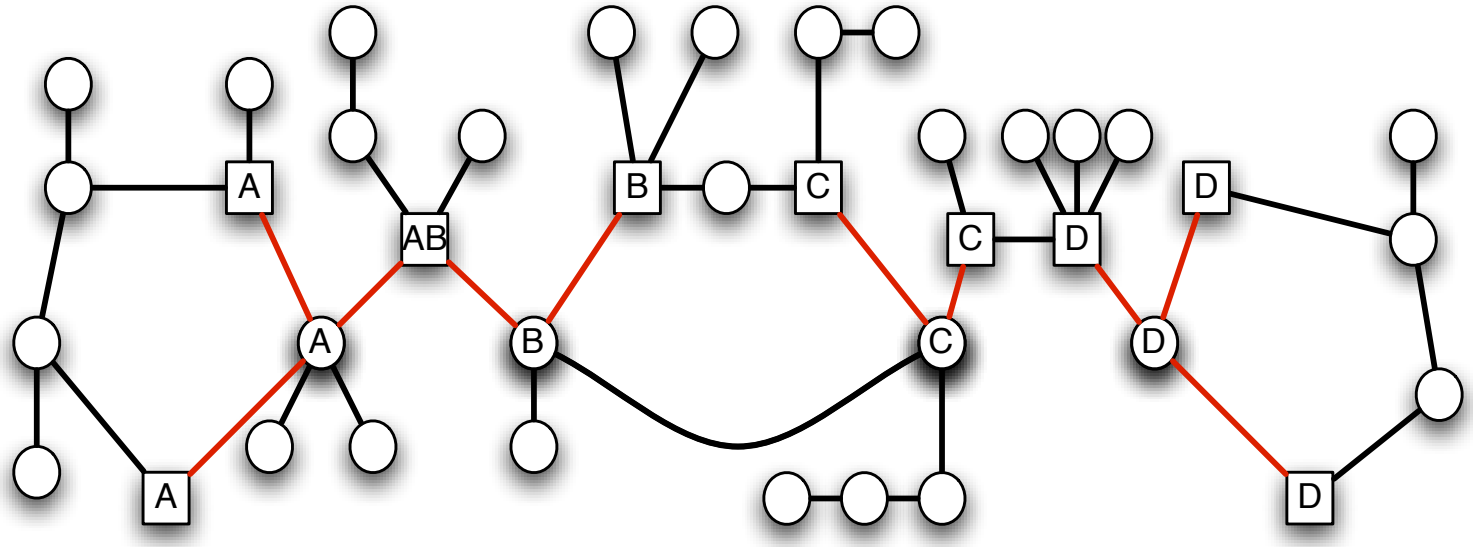

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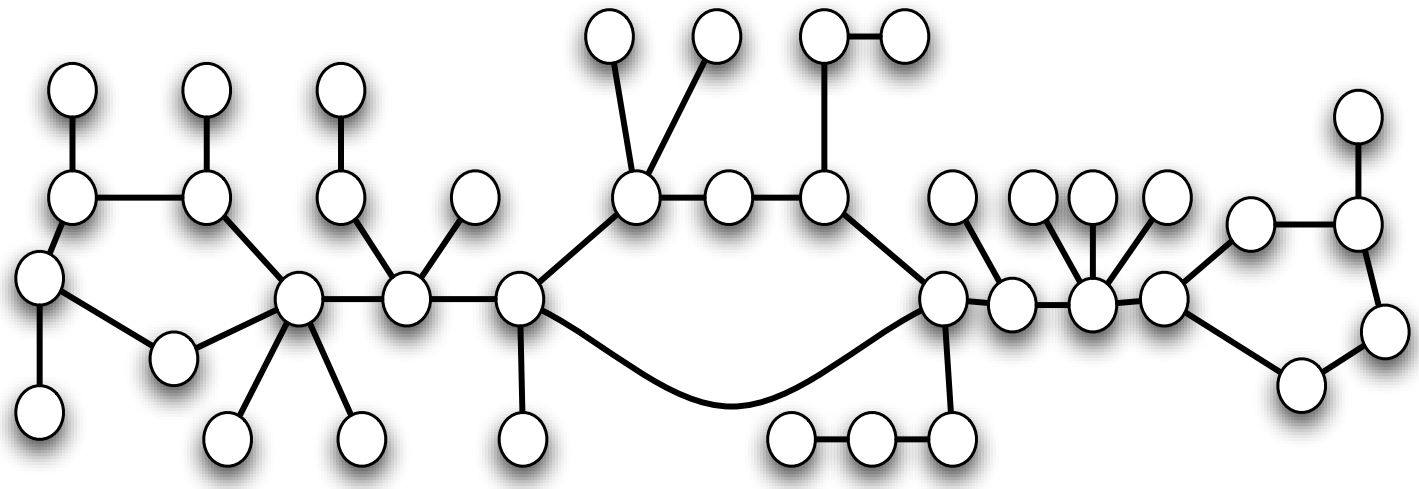

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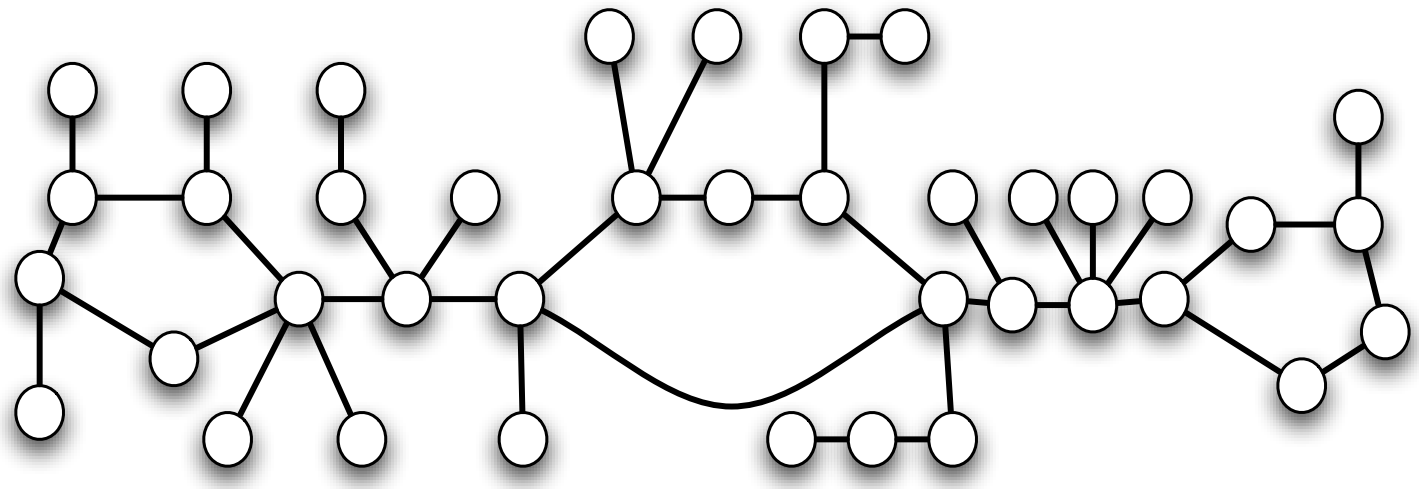

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- Symmetry factors: $1/2$ for loops, $1/k!$ for each k-multiedge.

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- The vector $(|T_1|/m, \ldots, |T_{3(s-1)}|/m)$ has distributional limit Dirichlet$(1/2, \ldots, 1/2)$.

The limiting picture

In the limit, then, a random component can be built by gluing *randomly rescaled Brownian CRTs* along the edges of a random kernel.

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- In other words, core path lengths are just given by splittings of an interval by uniforms.

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Repeat until all cycles are broken.

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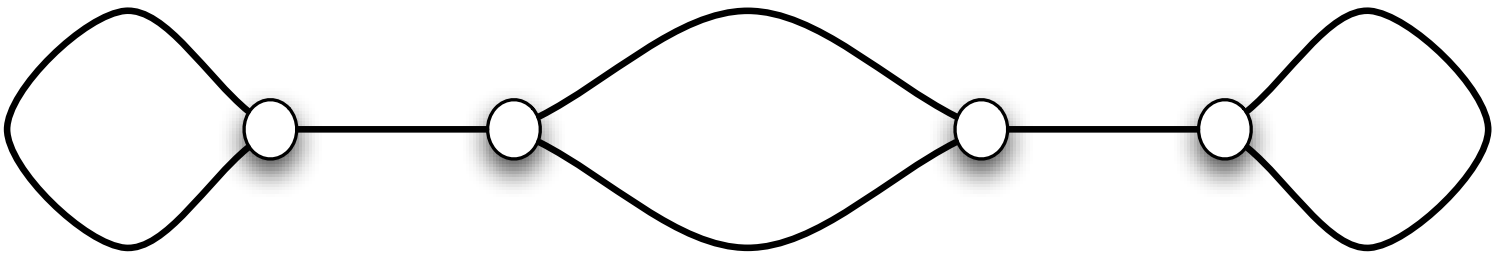

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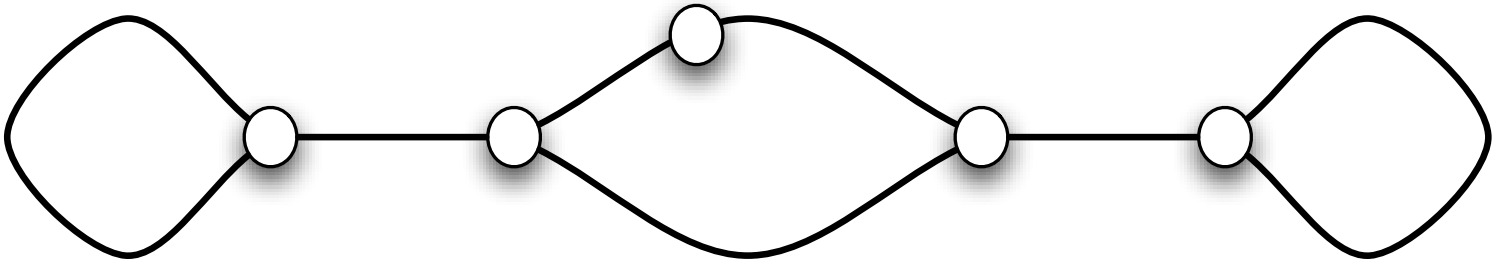

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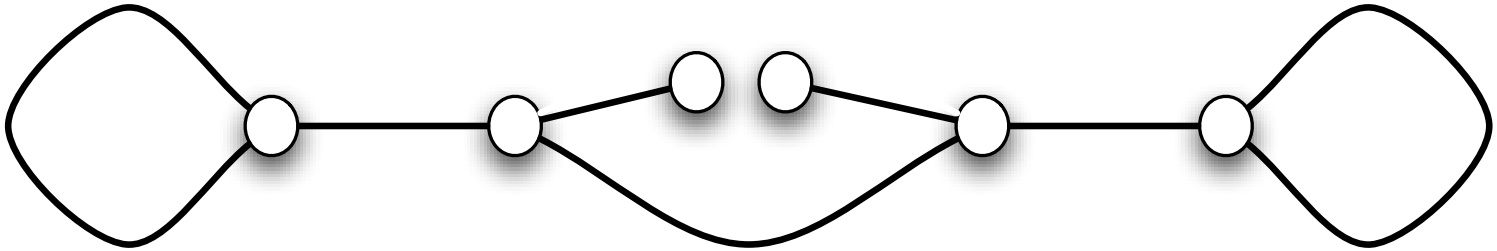

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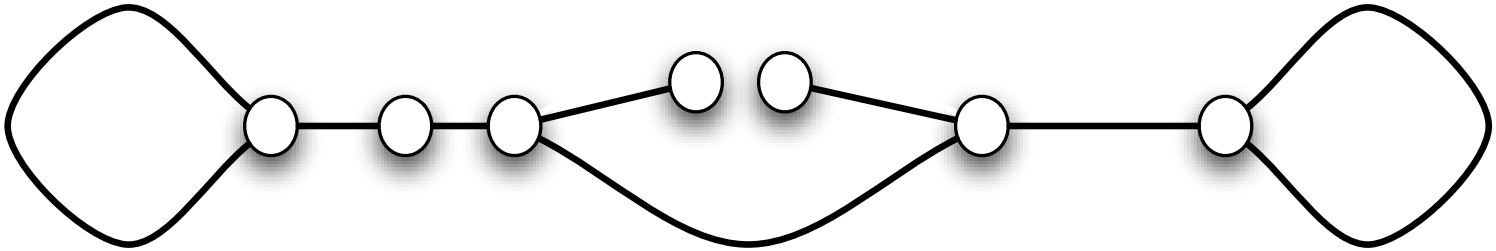

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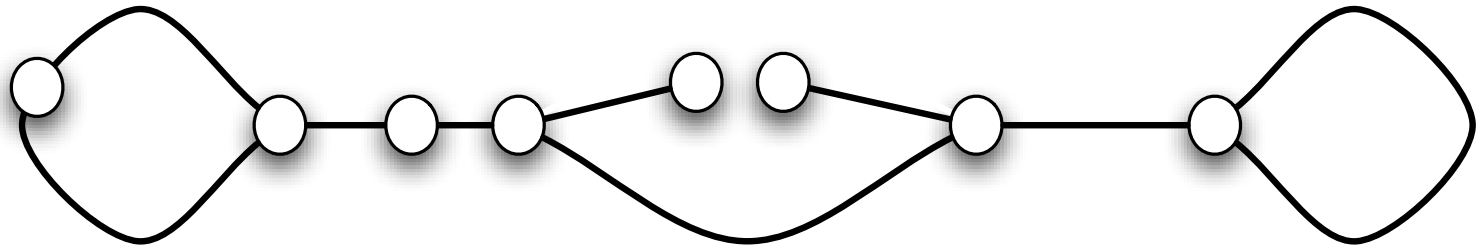

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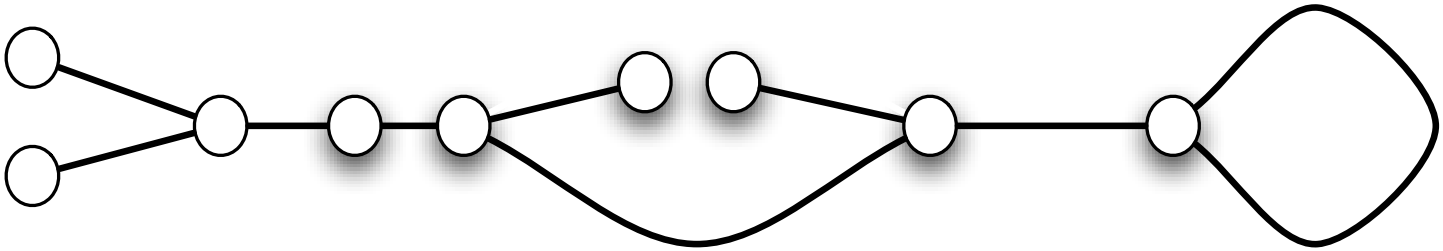

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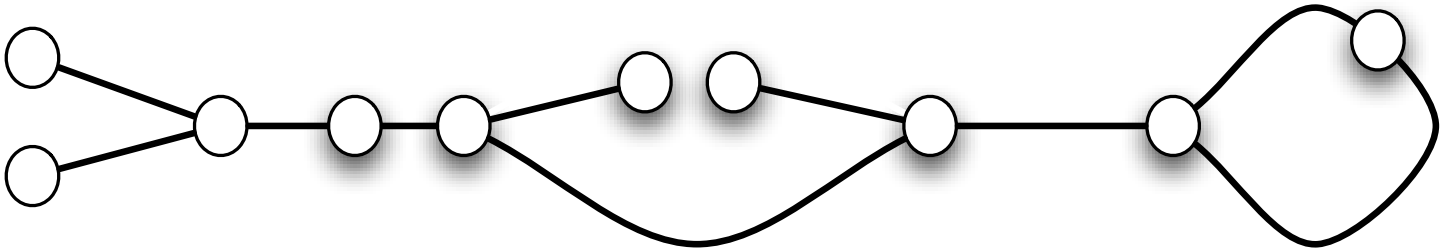

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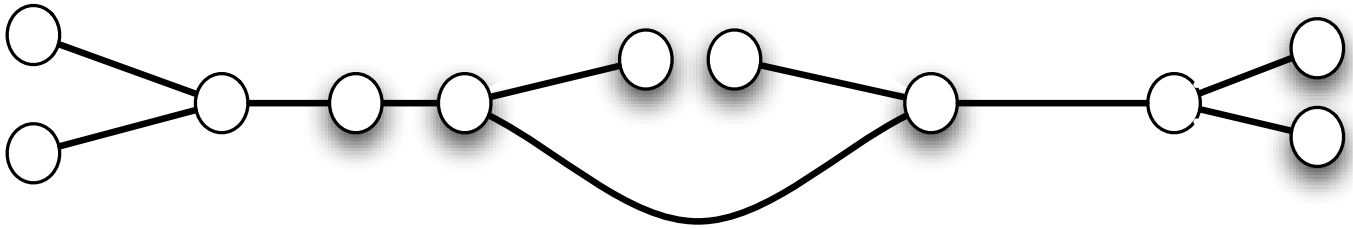

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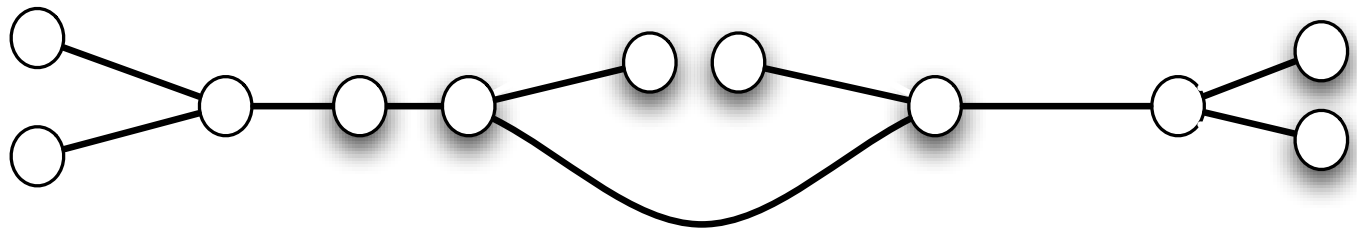

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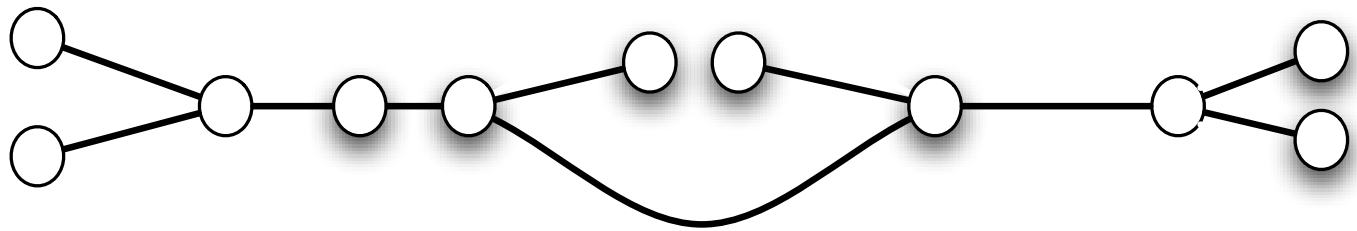

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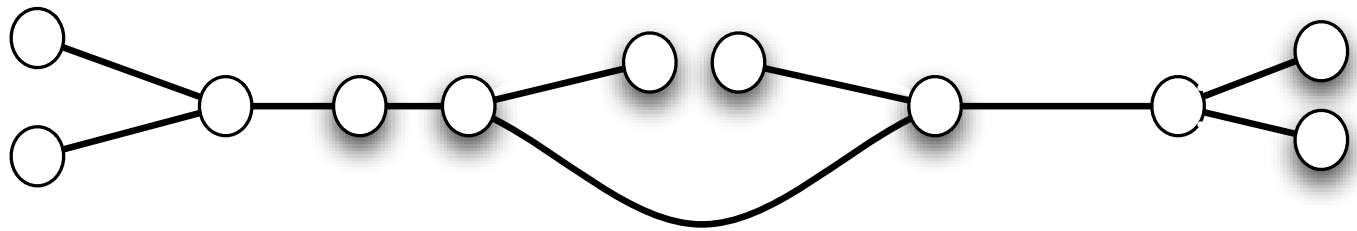

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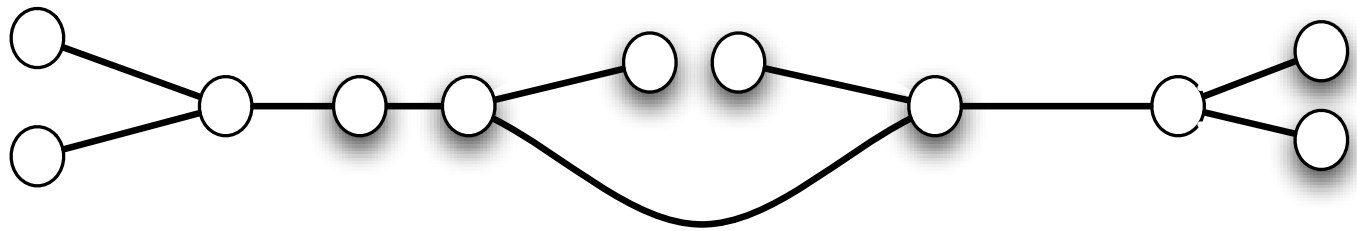

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In the $s \to \infty$ limit, it must have height of order $s^{1/3}$, because in the $s \to \infty$ limit it encodes the structure of the MST of K_n, and this is known to have order $n^{1/3}$.

Proof idea

Theorem

As $n \to \infty$, $T_{n,1/n} \xrightarrow{\mathrm{d}} \mathcal{T}$, *for some sequence* $\mathcal{T} = (\mathcal{T}_i, i \geq 1)$ *of random metric spaces.* (*Component−wise GHP, plus tail decay estimates.*)

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The principle of accompanying laws then yields that
$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \lim_{\lambda\to\infty} T_{n,1}^\lambda = \lim_{\lambda\to\infty} T_1^\lambda = \mathcal{M}$,
for some random metric space $\mathcal{M}$. □

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4. the law of $\mathcal{M}$ is mutually singular with that of the Brownian CRT (which has Minkowski dimension 2).

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$\star$ On the other hand, the proportion of points not covered by balls of radius $C\lambda^{-1}$ whp decays exponentially in C; together with estimates on the GH distance between $\mathcal{T}_\lambda$ and $\mathcal{M}$, this yields that the Minkowski dimension of the limit is ≤ 3.