RandNET Open Problems Workshop

LOUIGI ADDARIO BERRY

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Part 1: Four open problems.

- I present open problems
- Breakout rooms discuss problems
- Breakout groups give their feedback on the problems.

Which of these do you think are suitable for a RandNET workshop? Why or why not?

Do you understand the statements of the open problems?

Do you find the open problems compelling? Well-motivated? Why or why not?

Can you think of ways to improve the open problem statements?

- Come back with something to say
- Don't need to address all points or all problems.

When preparing an open question, think about who you are preparing it for: who will be reading, thinking about, and working on it.

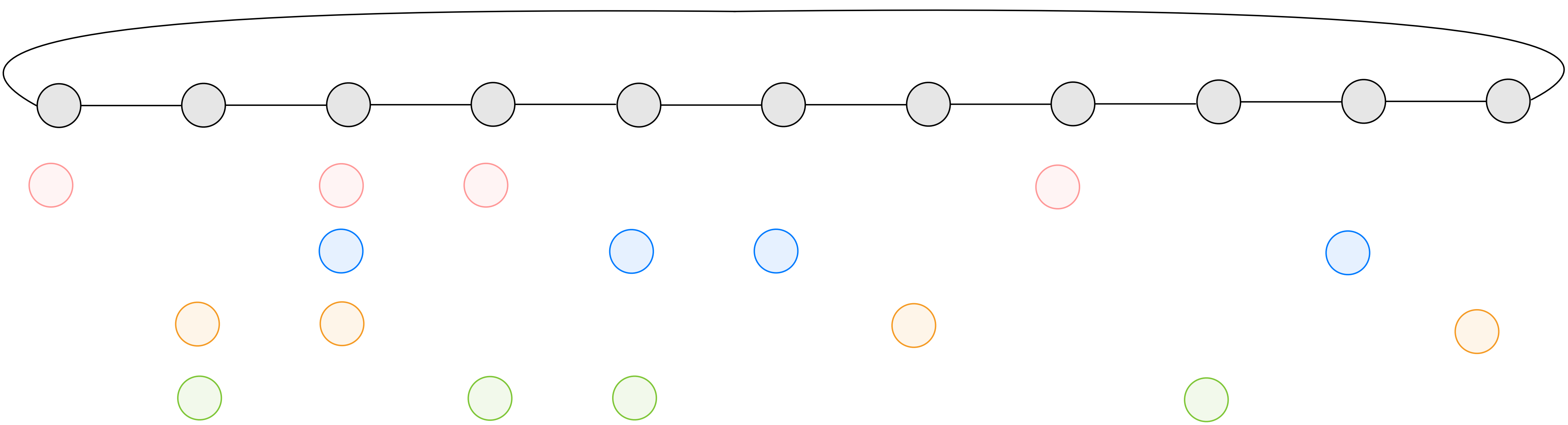
This should affect how you describe it, how much detail and background you give, and what sort of motivation you provide.

(The principle of taking the time to think about your audience goes a long way not just in coming up with open problems, but more generally in our line of work as researchers and educators.)

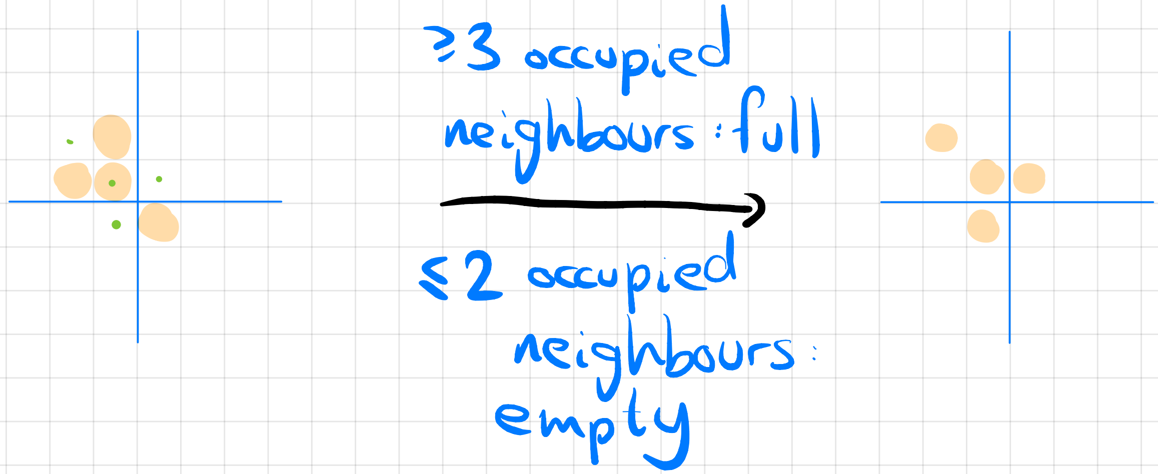
Covering with random translates

Fix $\alpha \in [0,1]$. Let $A \subset \mathbb{Z}/p\mathbb{Z}$ be a unif. random subset of size $\lfloor p^\alpha \rfloor$.

How many translates of A are needed to cover all of $\mathbb{Z}/p\mathbb{Z}$?



Statistical analysis of game of life



Do GoL on $(\mathbb{Z}/n\mathbb{Z})^2$ with IID Ber(p) initial condition.

Let $X_t(n,p) = \dfrac{\#\text{ full sites at time } t}{n^2}$

$X^-(n,p) = \liminf\limits_{t\to\infty} X_t(n,p)$

$X^+(n,p) = \limsup\limits_{t\to\infty} X_t(n,p)$

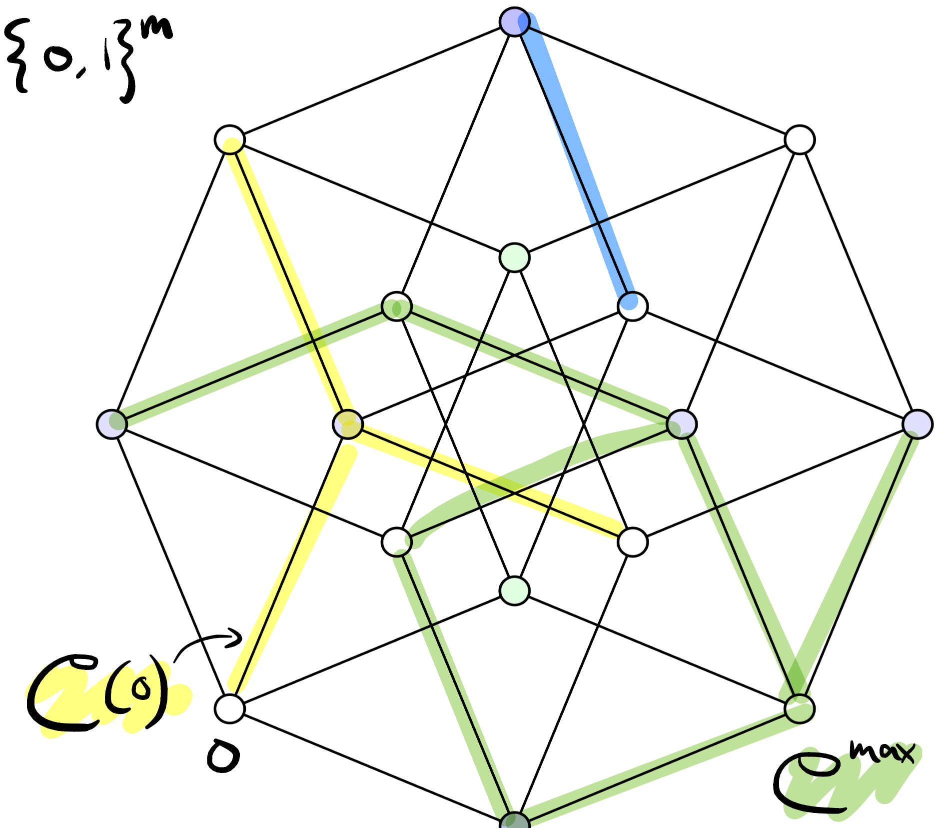
Conj: $\exists\, p_0 \in (0,1)$ s.t. for all $p > p_0$, $X^-(n,p)$ and $X^+(n,p)$ both converge in probability as $n\to\infty$, to the same (non-random) limit $f(p)$.

Large clusters in subcritical Hamming percolation

Bernoulli(p) bond percolation on $\{0,1\}^m$

$\mathcal{C}(v) \to$ Connected component of v

$\mathcal{C}^{max} \to$ Connected comp. with the most vertices.



Define $p_c = p_c(m)$ to be s.t.

Crit. Prob.

$$\mathbb{E}_{p_c} |\mathcal{C}(0)| = (2^m)^{1/3}.$$

Conjecture: If $\varepsilon_m \gg \frac{1}{(2^m)^{1/3}}$ and $p = p_c(1-\varepsilon_m)$

then $\dfrac{|\mathcal{C}^{max}|}{\varepsilon_m^{-2}\log(\varepsilon_m^3 2^m)} \longrightarrow 2$ in probability.

Hodge Conjecture

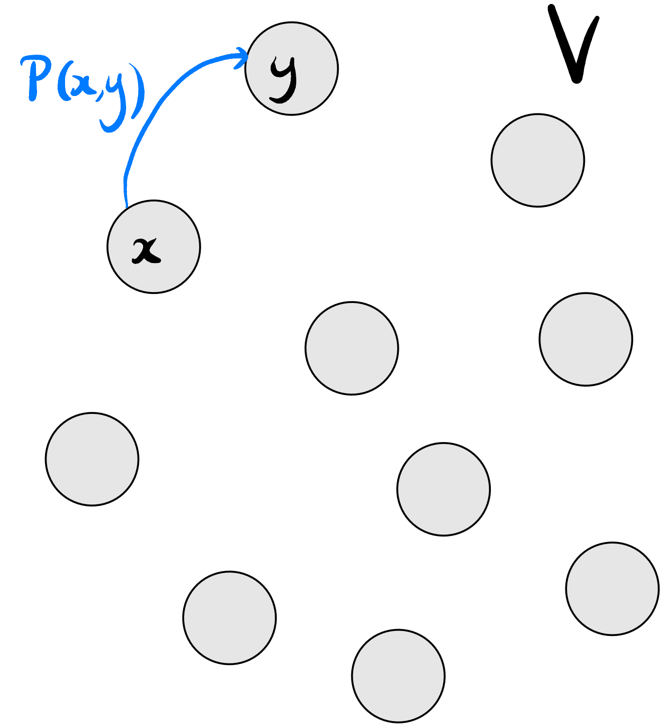
Let X be a nonsingular complex projective manifold. Then every Hodge class on X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

Part 2

Pre-circulated problems.

I will remind you of the problem statements; then you critique/improve/offer feedback on them

Meeting in a maze



Markov Chain $P = (P_{x,y})_{x,y \in V}$

Aperiodic, irred $\to$ stat. distribution $\pi = (\pi(v), v \in V)$

X, Y indep. copies of Markov chain with

$X_0 \sim \pi, Y_0 \sim \pi$ independent.

Write M for first time X, Y meet $= \min(t : X_t = Y_t)$

> Conjecture: X and Y usually find each other before exploring too much of the chain.

↳ Let T be the trace of the chain up to time $t-1$:

$T = \{X_s, s<t\} \cup \{Y_s, s<t\}$.

Then $\mathbb{E}\,\pi(T) < q < 1$

q an absolute constant.

If more time: examples (star, clique, cycle)

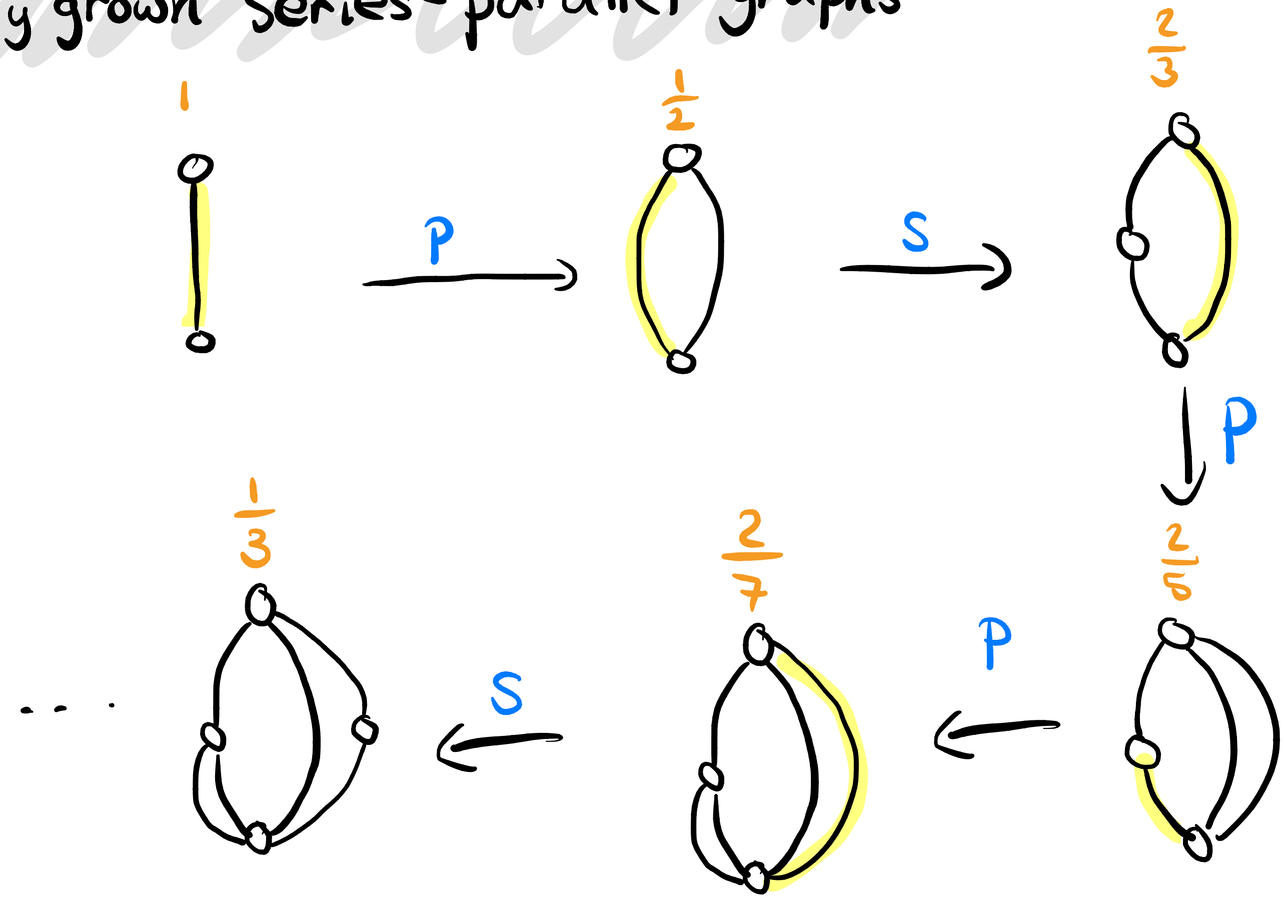
If even more time: context, motivation

Be ready with that stuff anyway in case there is interest!

(lots of prep work is "invisible")

Feedback on OP statement

Randomly grown series-parallel graphs



R_n := effective resistance at step n.

Question:

- Does R_n converge? (In dist/in prob/a.s.)

Open problem: Find $f_n : \mathbb{R} \to \mathbb{R}$ s.t. $f_n(R_n)$ converges in distribution.

If more time

- Duality observation
- Tree perspective
- Variants of the dynamics

Feedback on OP statement

Part 3 Advice

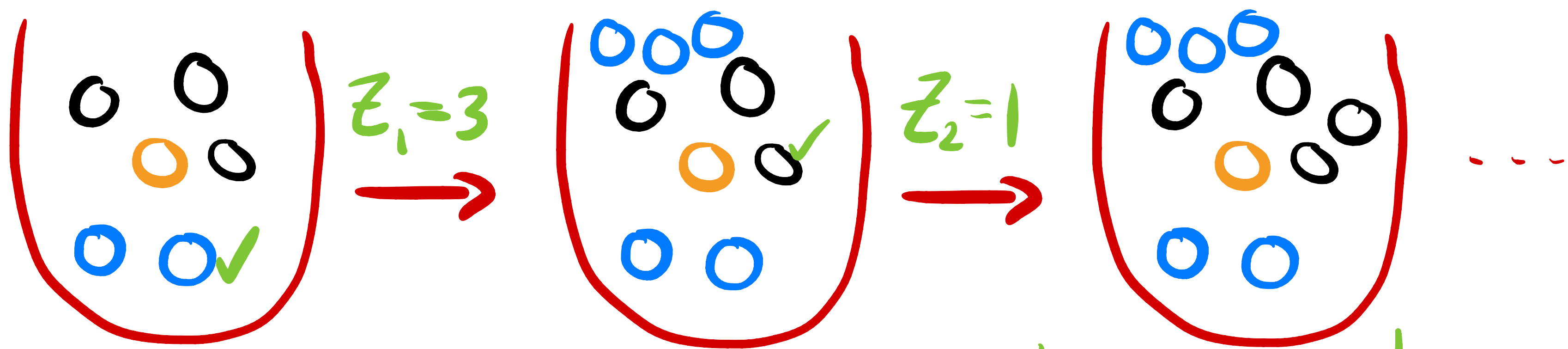
- My thoughts/guidance on coming up with and preparing/presenting open problems
- Your ideas for the same!
- Questions/comments/feedback.

Guidance and principles

- Know your audience
  - Can be complicated
  - Particularly for a new setting.
  - Required background
  - Required "mathematical sophistication"
  - Subject matter
- Know your (time) limits
  - Going over your time is disrespectful to others - both presenters and non-presenters.
- Know your (other) limits
  - Intermediate goals?
  - Can you get the ball rolling?

- Be ready to share
  - If you don't want to work on it in a group, don't ask it!
  - Posing means re-explaining/bringing people up to speed, sometimes repeatedly.
- Aim for problems where progress is possible.
  - Great to have an idea of how hard the problem is (but sometimes we get this wrong!)
  - See example

Random-replacement Polya urn



$Z_1, Z_2, \ldots$ IID non-neg. integer random variables.

Composition at time n: $(\mathcal{U}_1(n), \ldots, \mathcal{U}_d(n))$

$$\mathcal{U}(n+1) = \mathcal{U}(n) + Z_n \mathbb{1}_{A_n}$$

$$A_n = i \text{ with prob. } V_i(n) = \frac{U_i(n)}{U_1(n)+\cdots+\mathcal{U}_d(n)}$$

$$\mathbb{1}_a = (0, \ldots, 0, 1, 0, \ldots, 0)$$

a'th position

Questions

- What is asymptotic behaviour of $U(n)$ as a f^n of law of $(Z_i, i \geq 1)$?
- Finite vs. infinite mean?
- Heavy-tailed distributions?
- Growth rates of largest, second largest, ...?
- $V(n) \to V(\infty) \leftarrow$ Properties?

How to find an open problem

- Steal one.
- Collaborate
- Change the setting
- Randomize
  - Randomize initial condition.
  - Add random dynamics.
- Steal one →
  - Look @ science literature for inspiration.
  - Look @ the world for inspiration!
- Your ideas?