The critical random graph

L. Addario-Berry

Random Structures and Dynamics

Oxford

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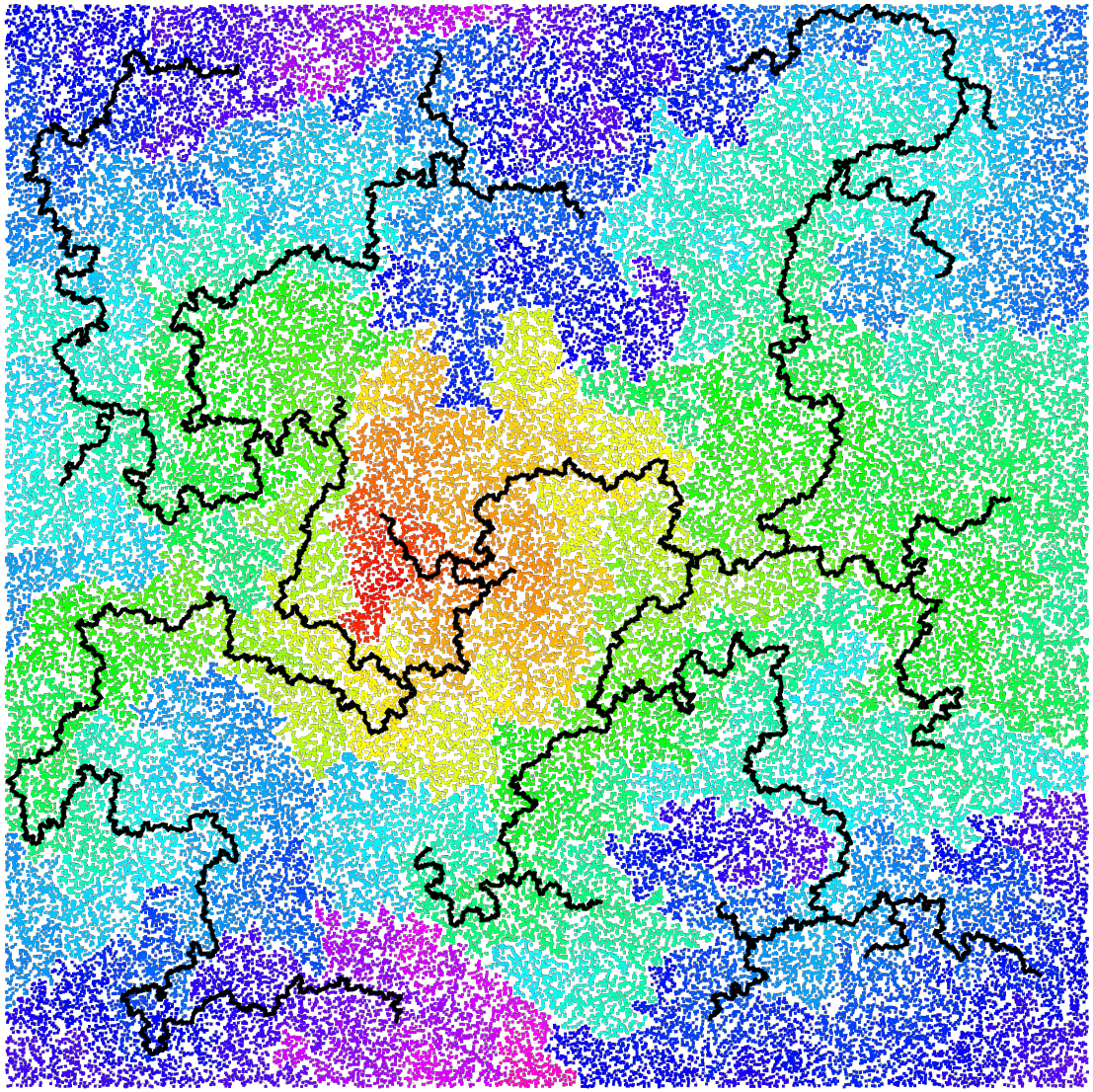
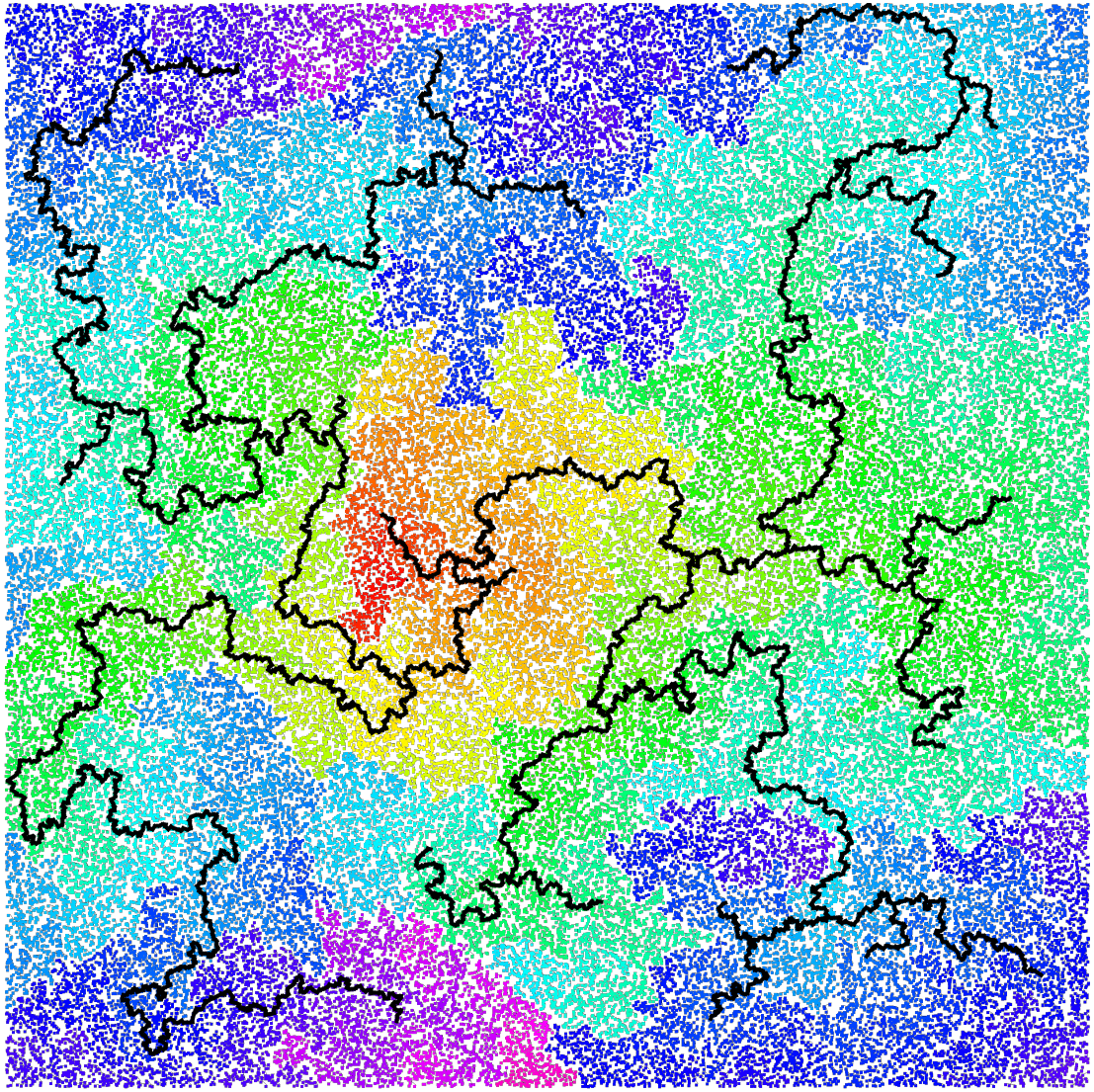


Motivation

Understand the following picture. (The (Euclidean) Minimum Spanning Tree of 100,000 uniformly random points.)



Motivation: minimum-weight spanning trees

Caveat: this is what motivates (originally motivated) *me*.

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- Minimum spanning tree is *algorithmically simple* but *probabilistically challenging*.

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- Given points $x_1, \ldots, x_n \in \mathbb{R}^d$.

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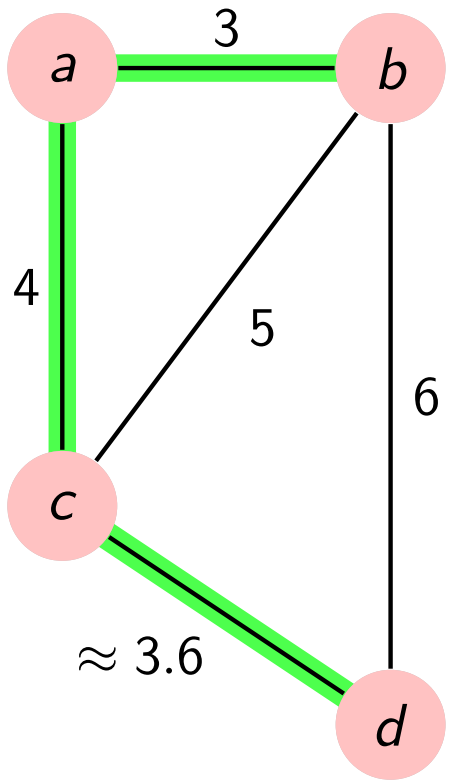

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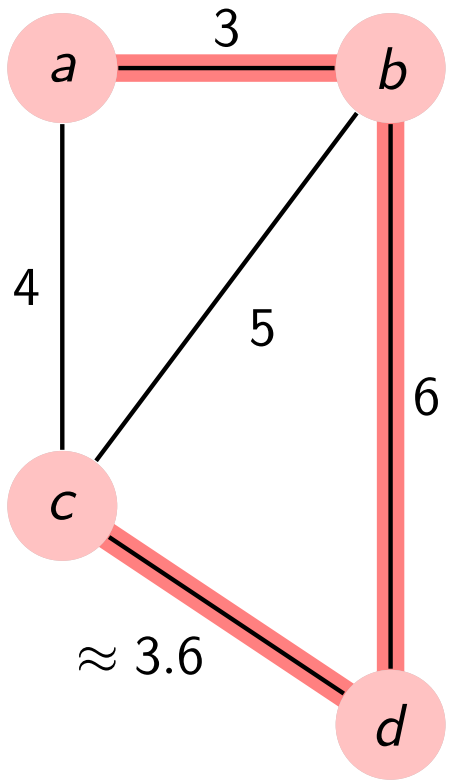

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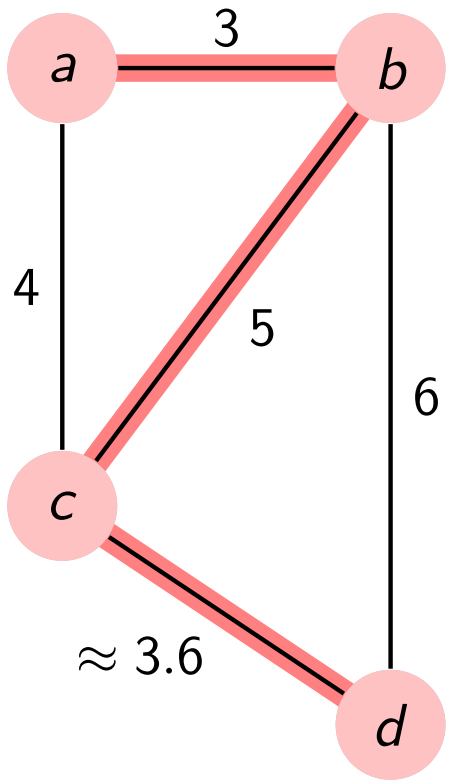

Intended learning outcomes

Goal for today:

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Goal for today: Explain the connection between minimum spanning trees and critical percolation.

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A small aside!

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- Also: always assume all edge weights distinct!

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How does one find the MST of a given (finite) graph $G = (V, E)$ with edge weights $\{w_e\}$?

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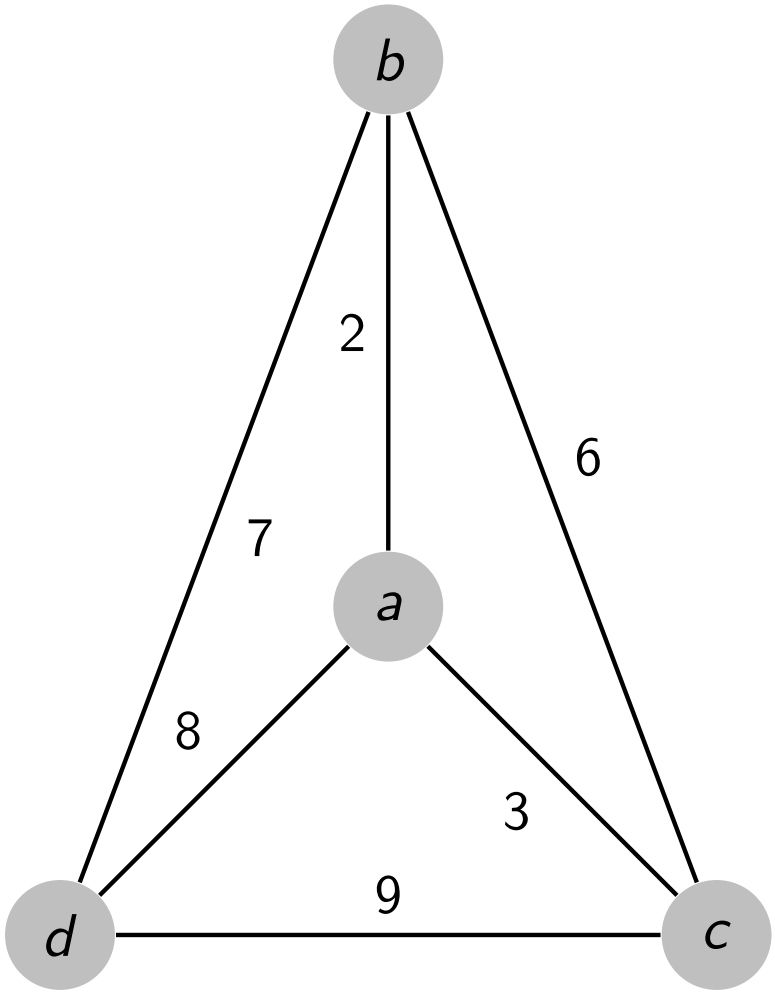
There are three natural algorithms.

Each turns out to expose a different probabilistic aspect of MSTs.

Kruskal's algorithm/Borůvka's algorithm/Percolation process

- Order edges in increasing order of weight as $e_1, \ldots, e_m$.



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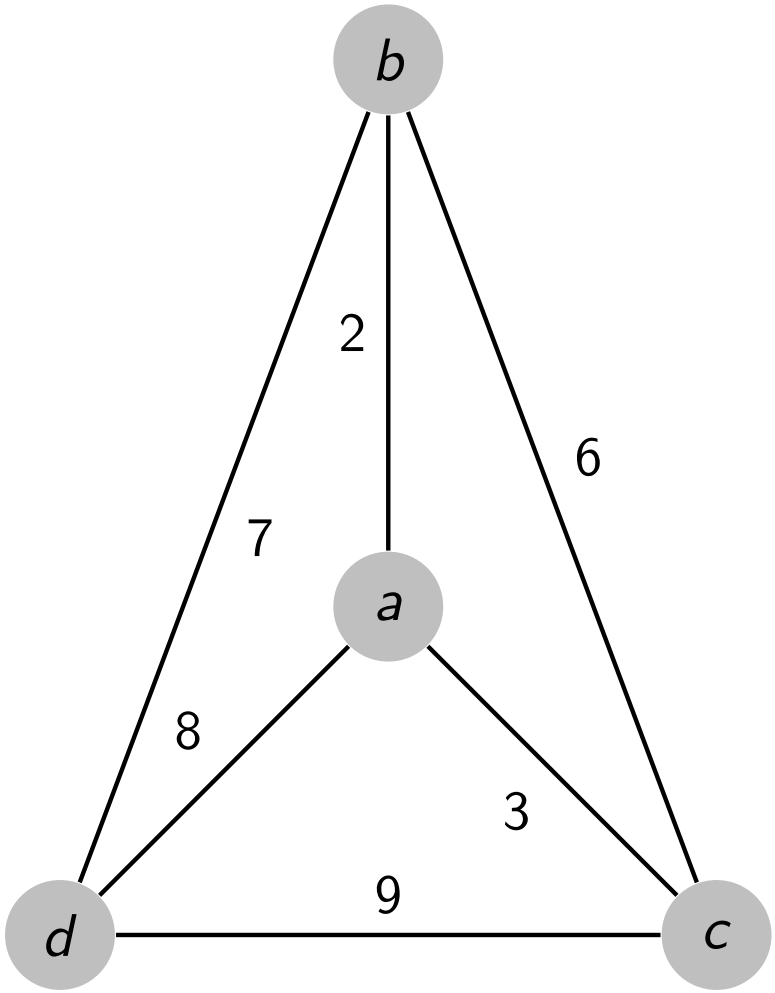

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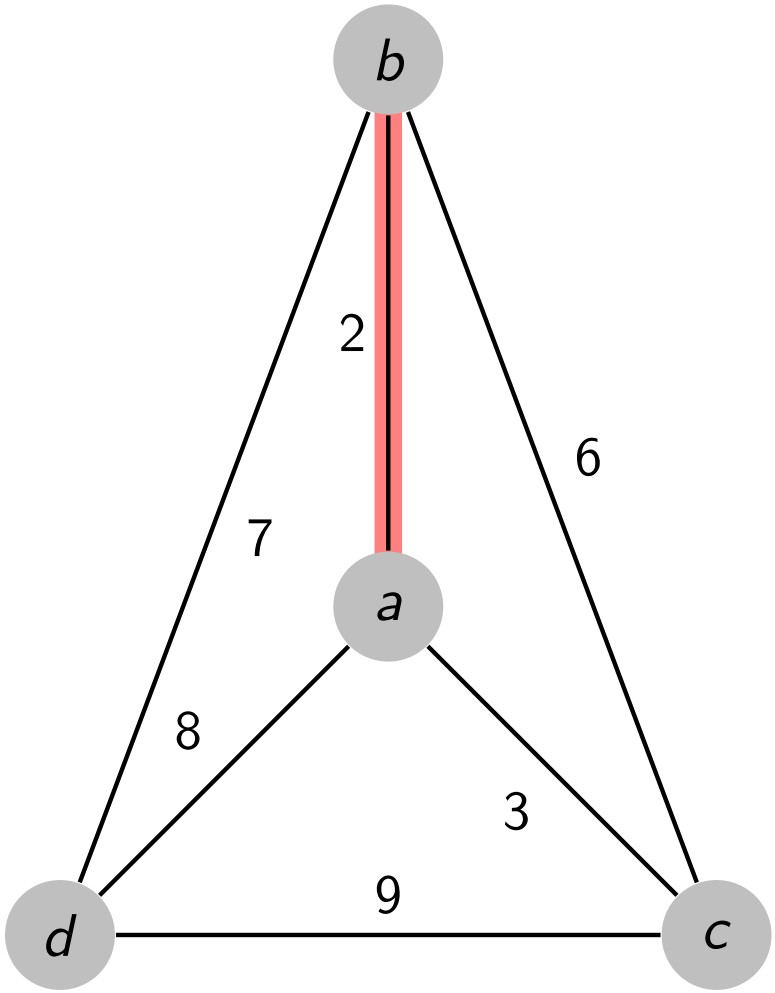

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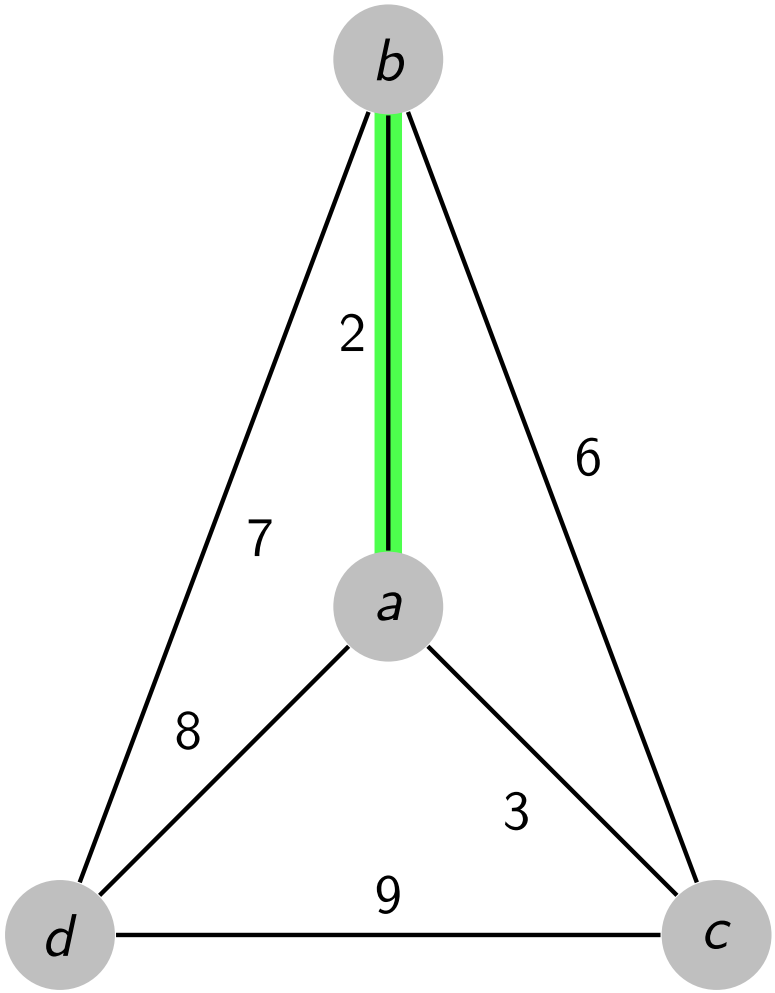

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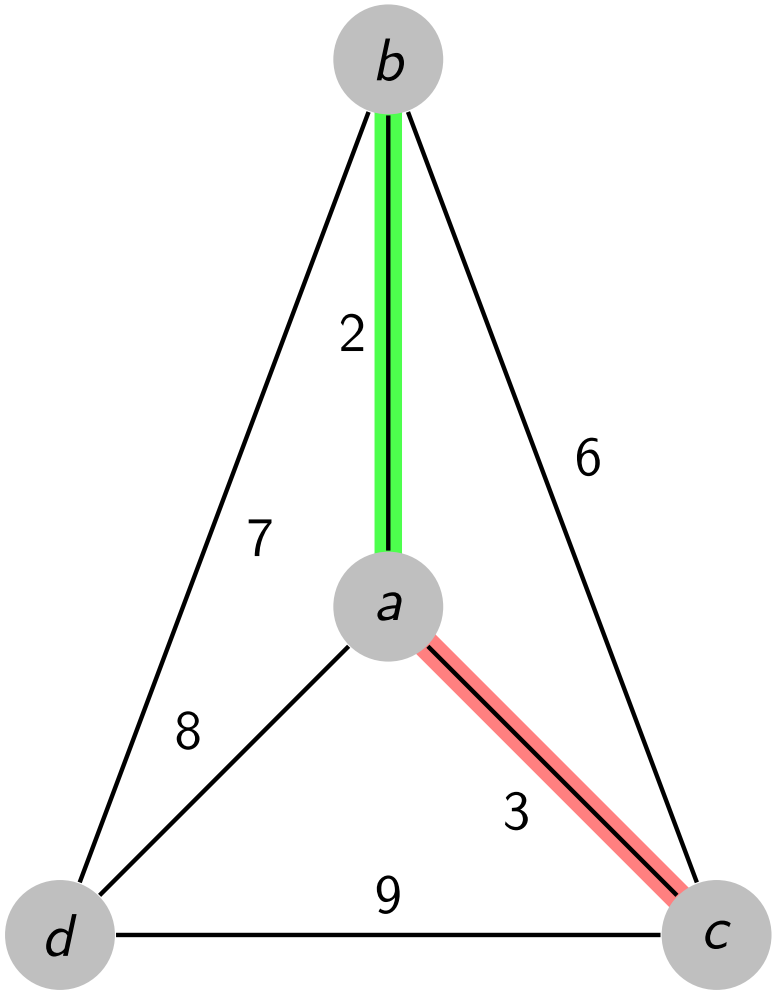

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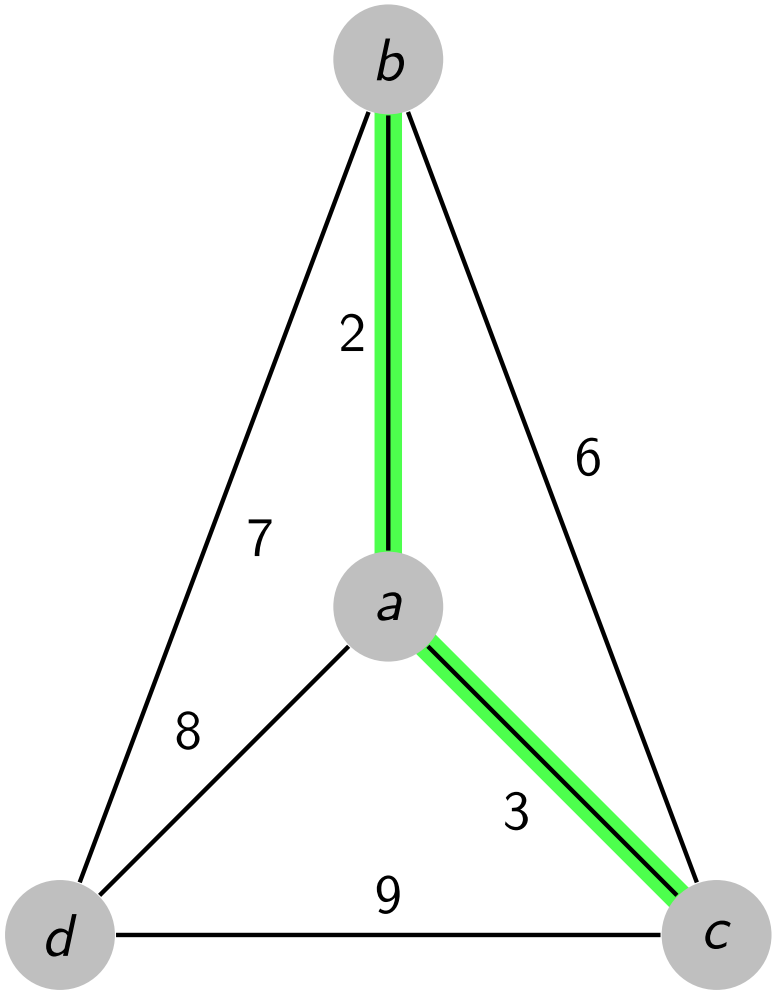

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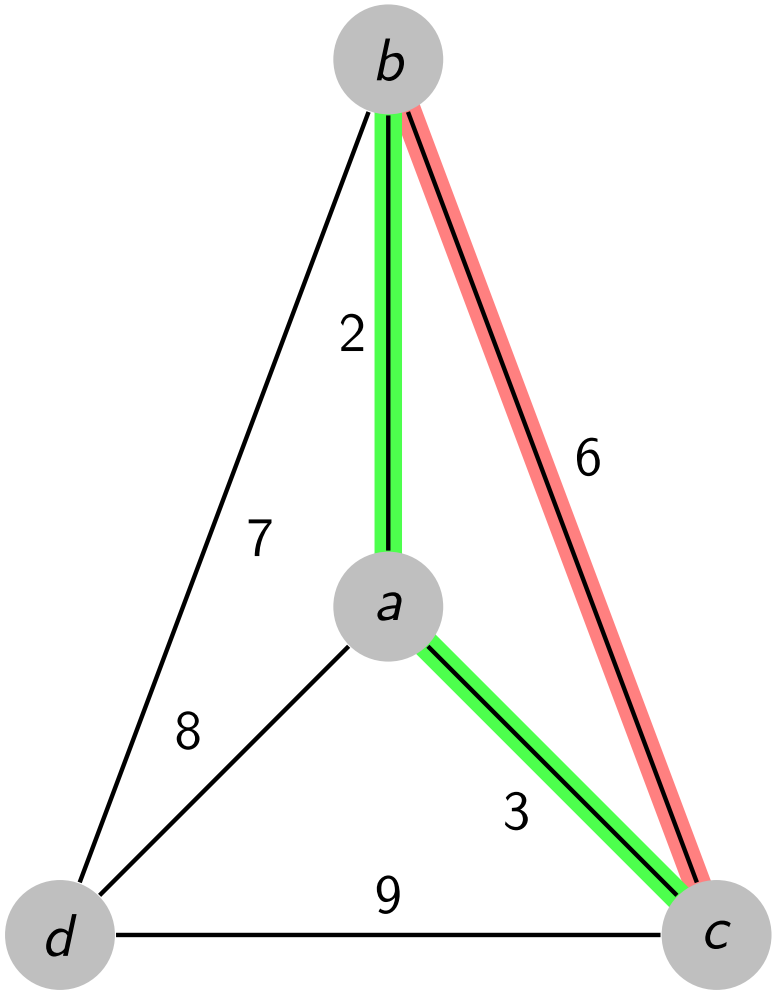

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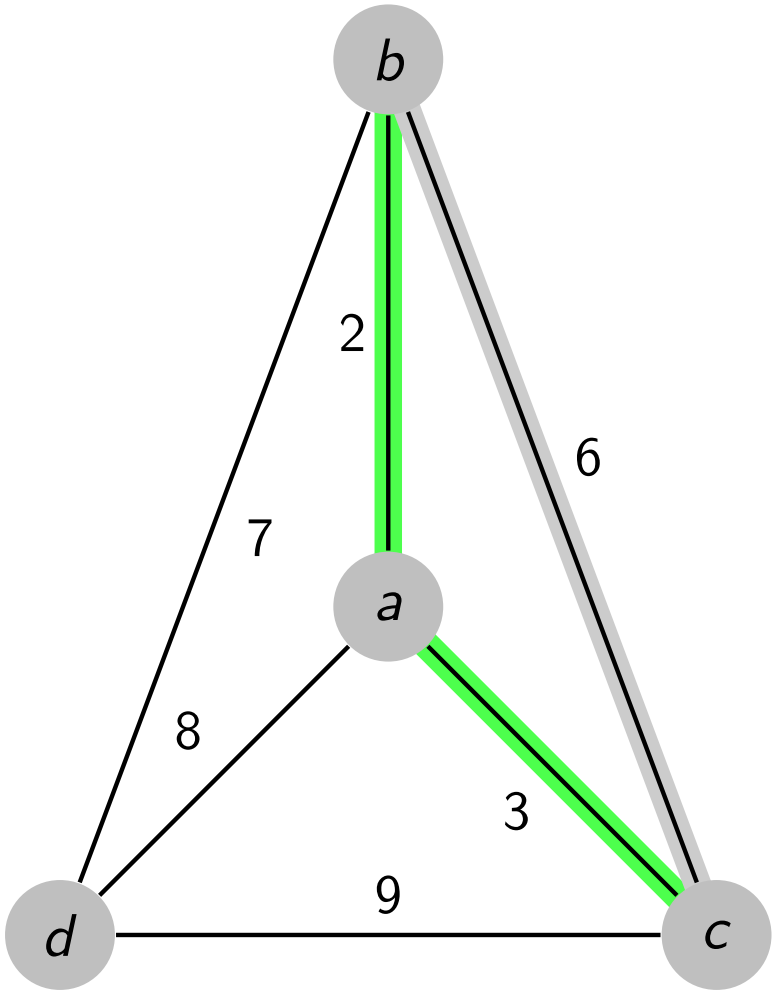

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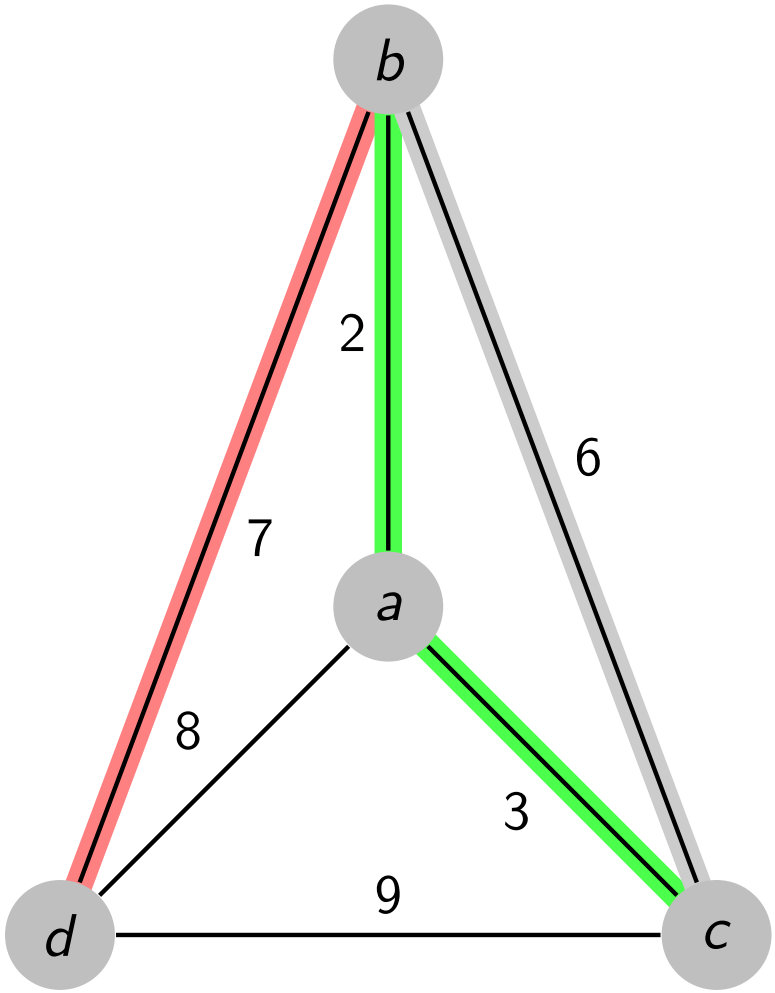

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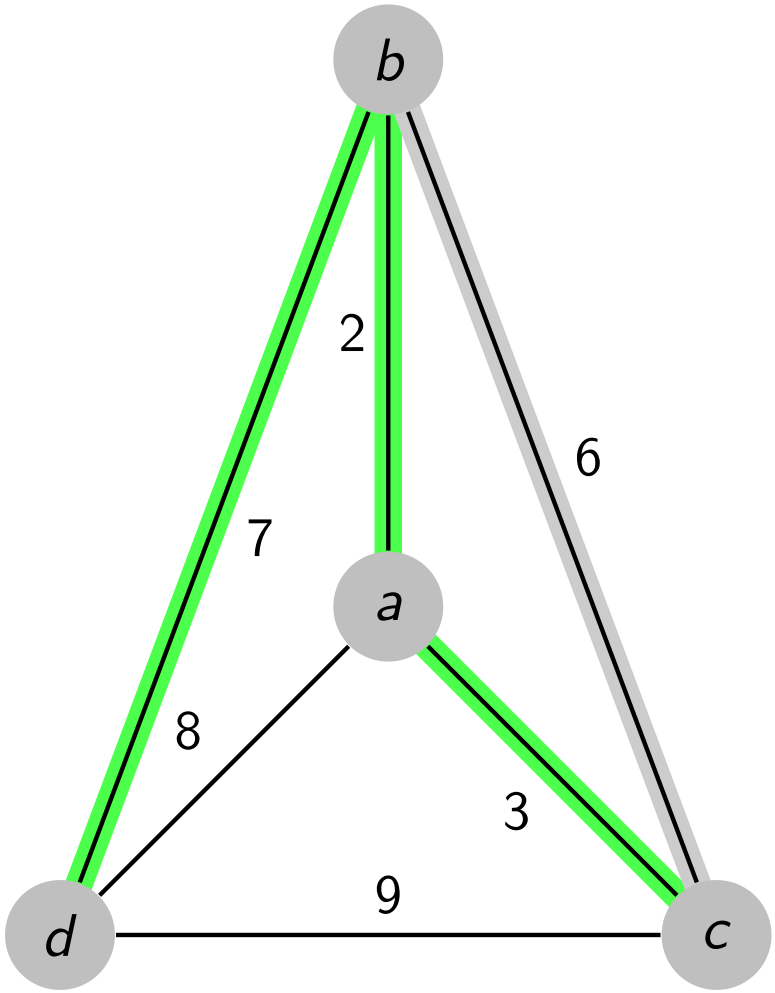

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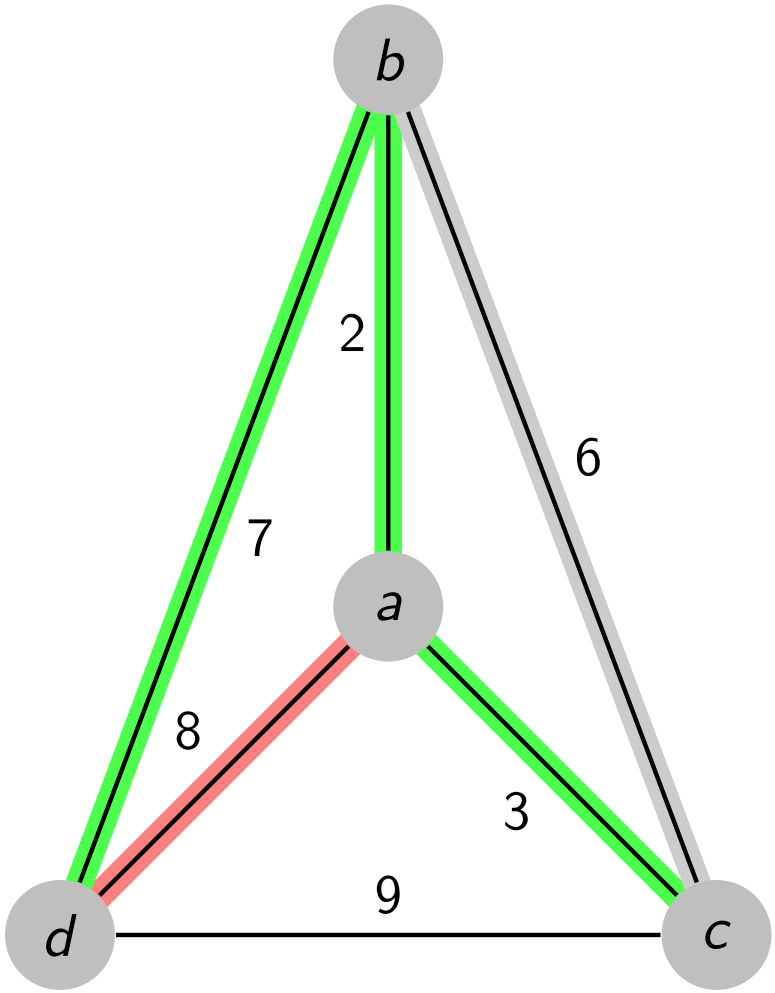

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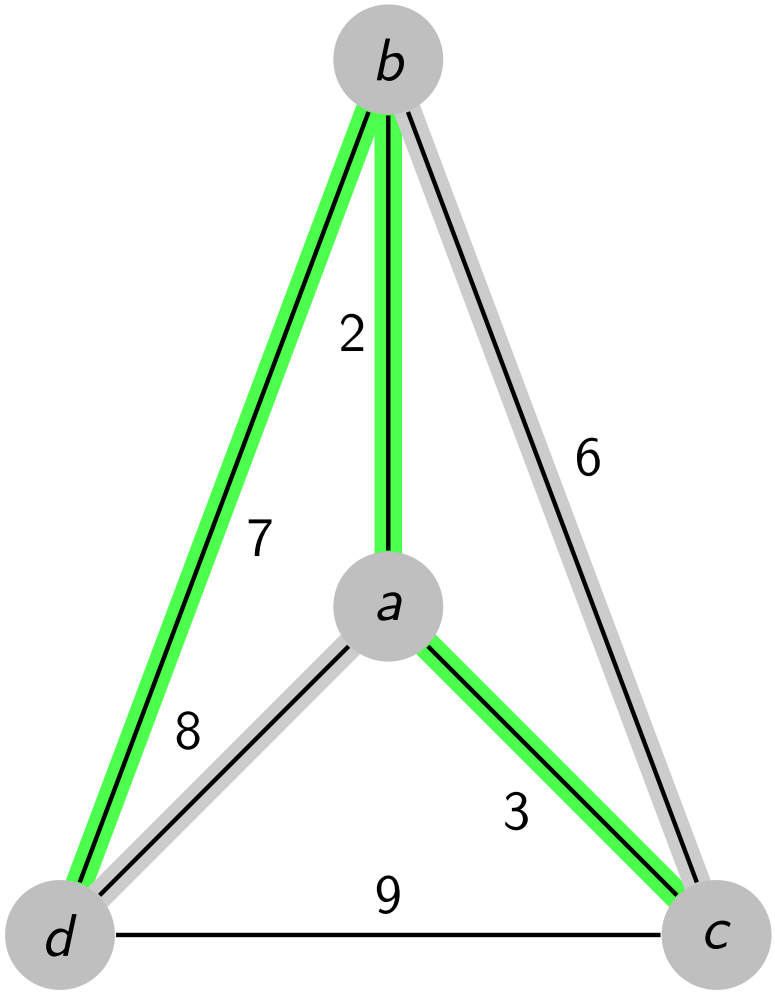

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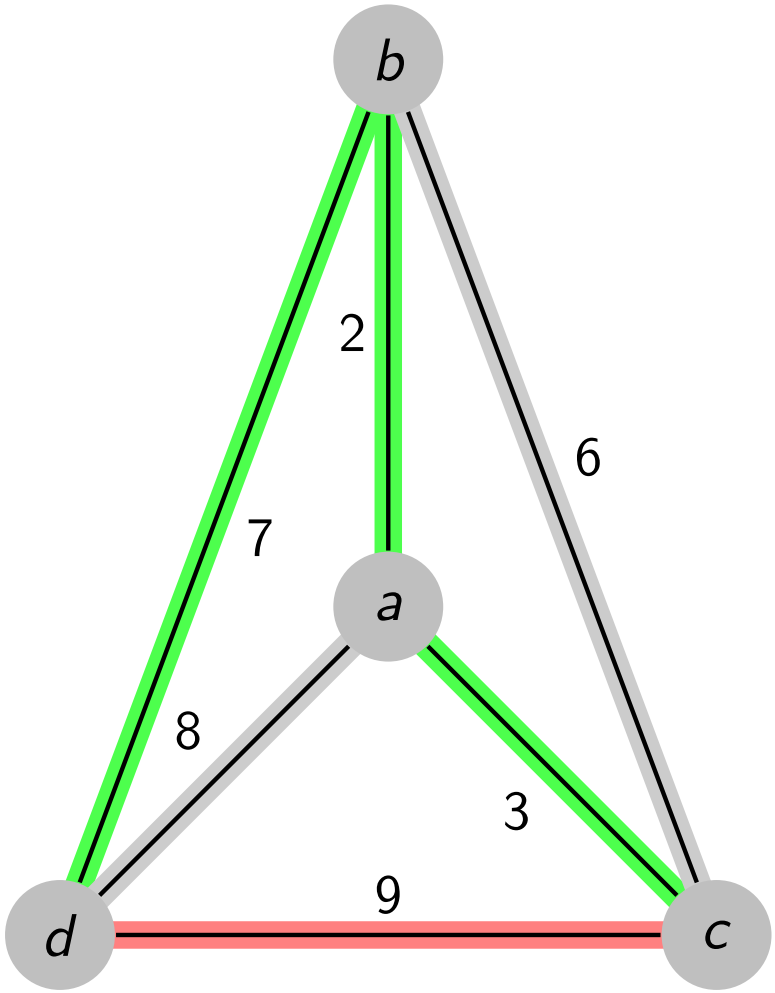

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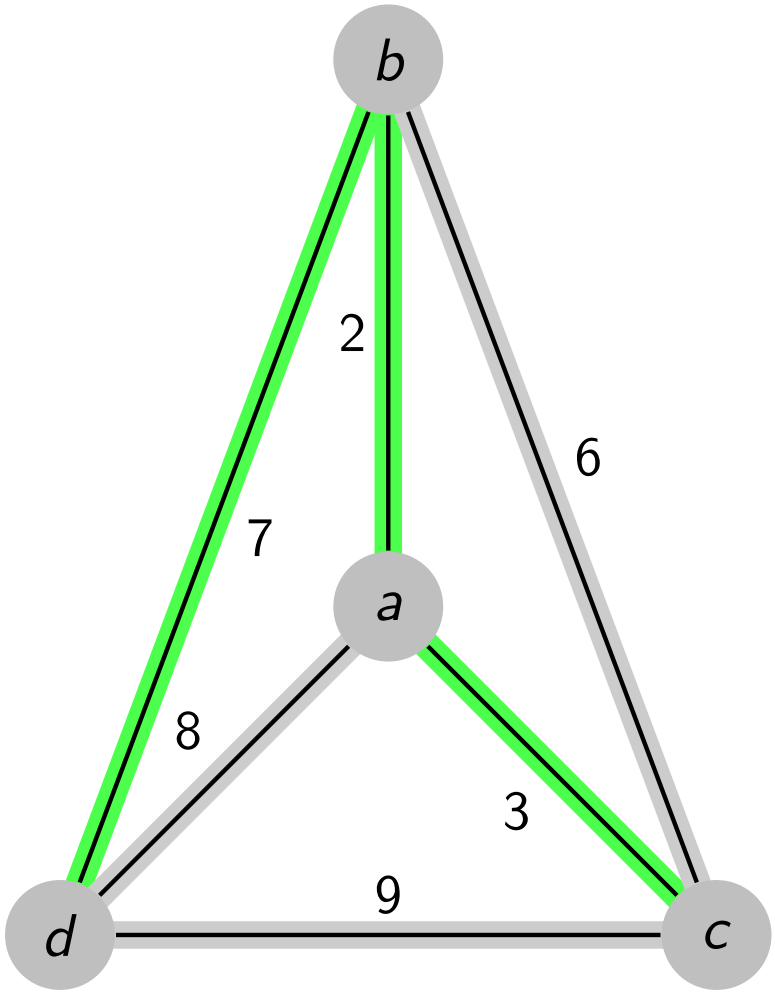

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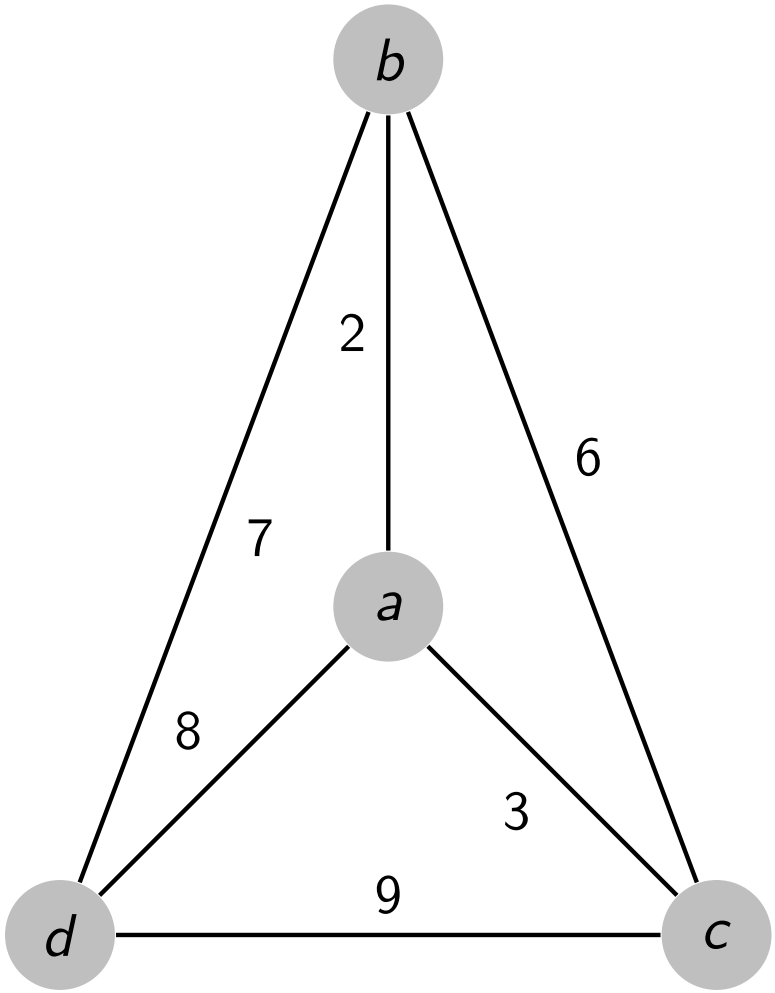

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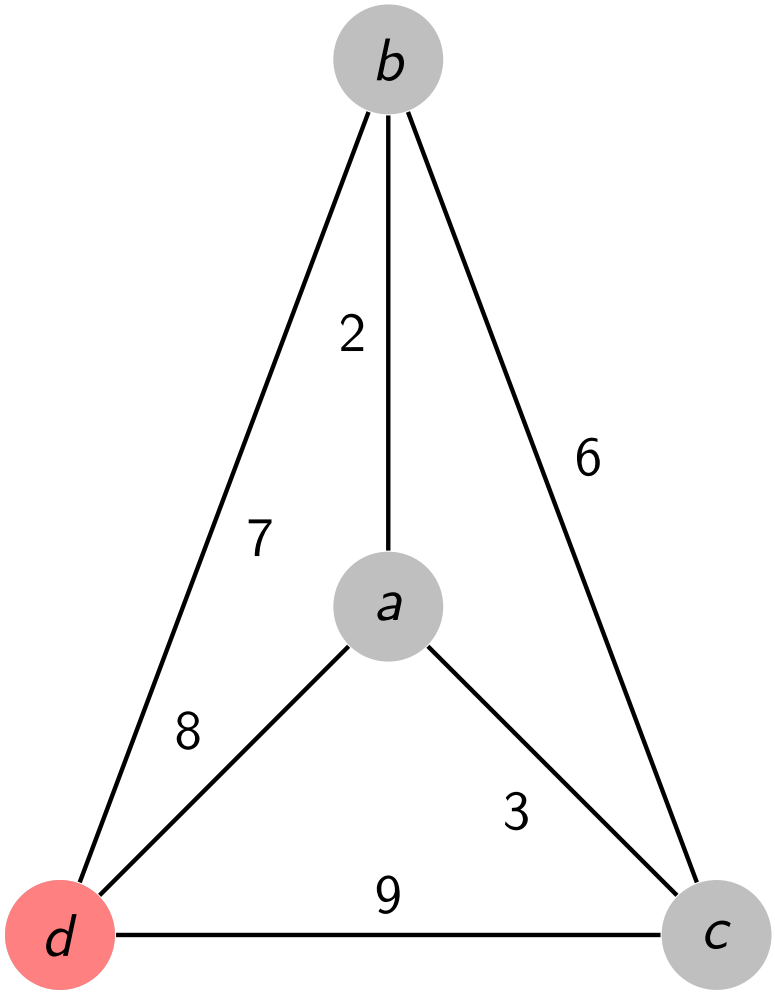

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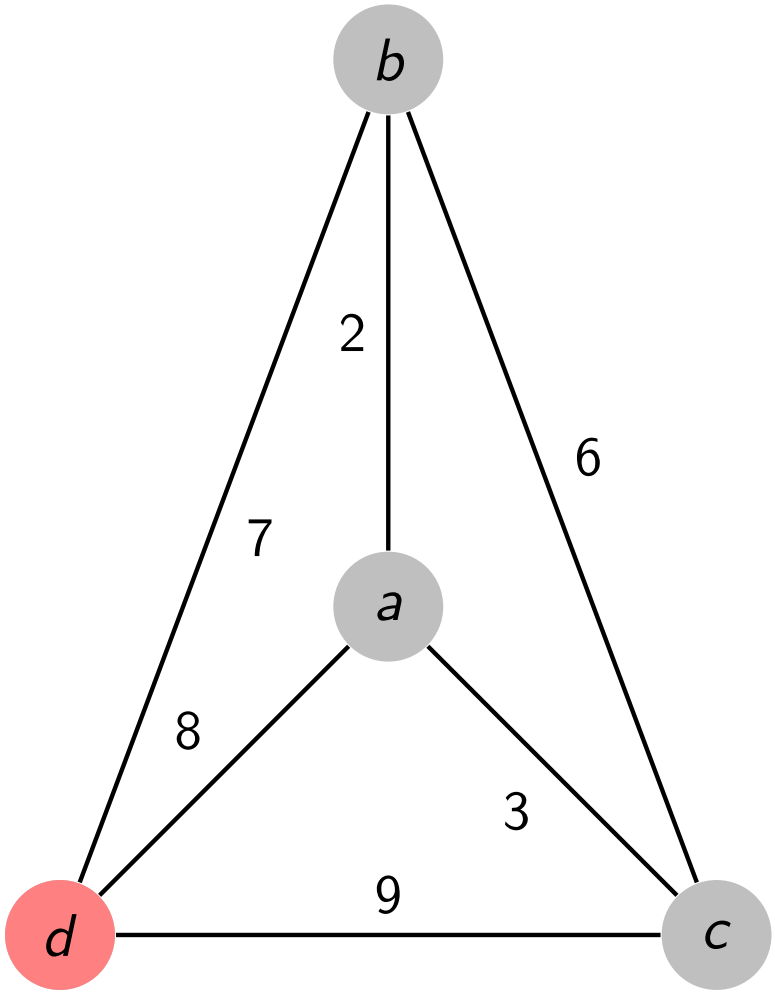

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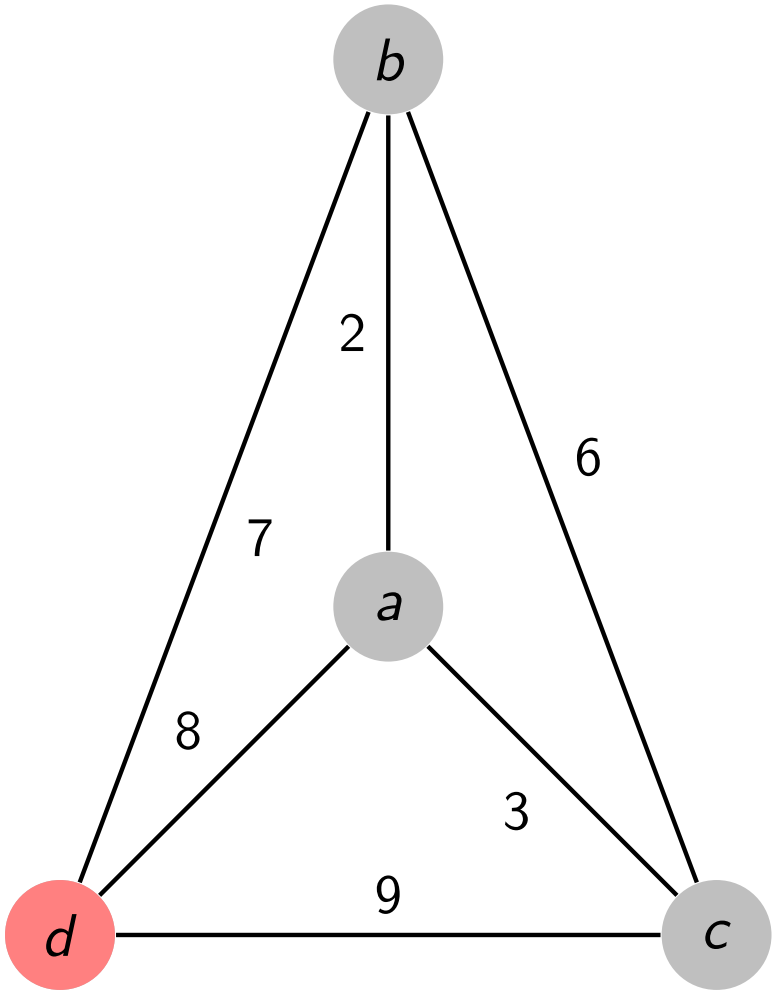

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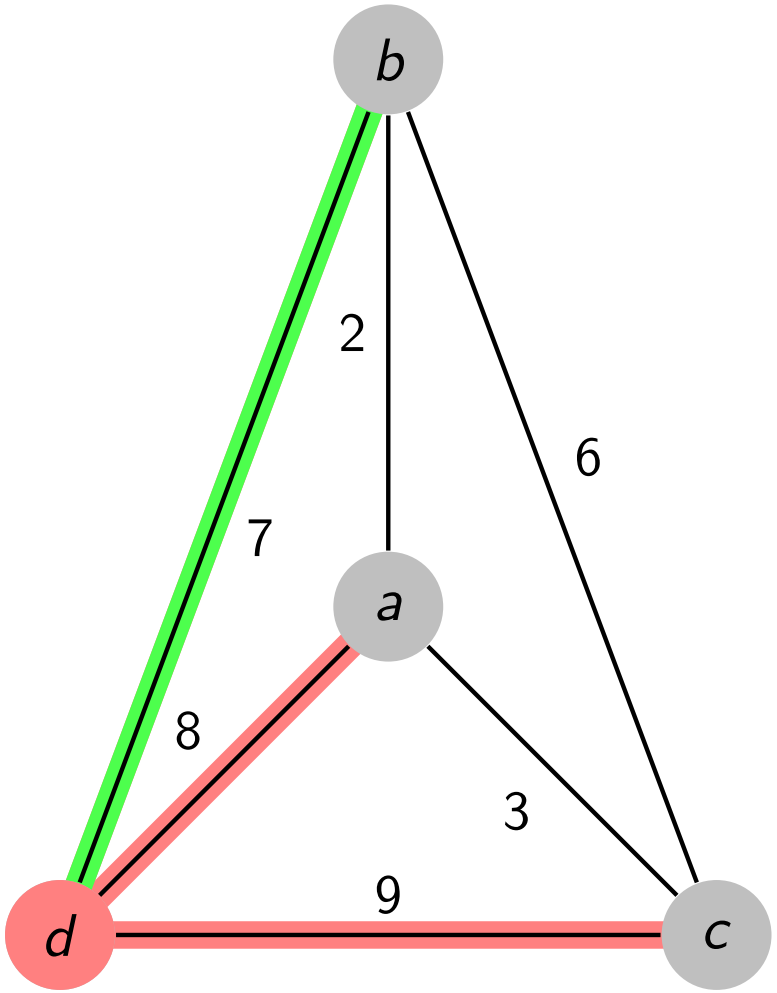

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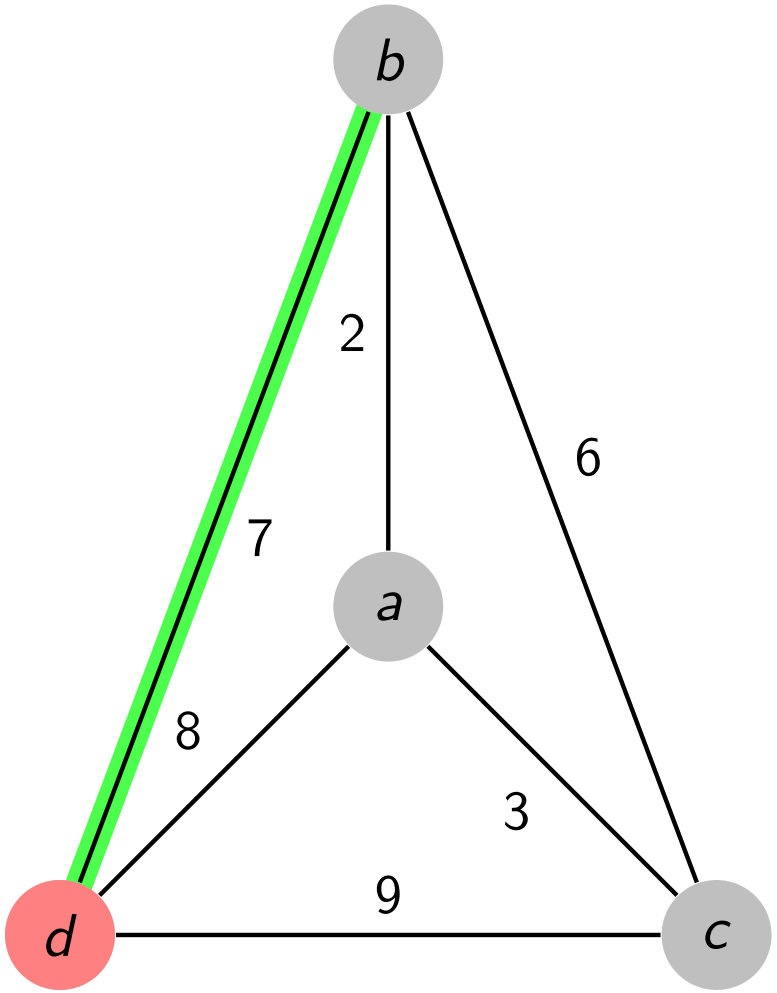

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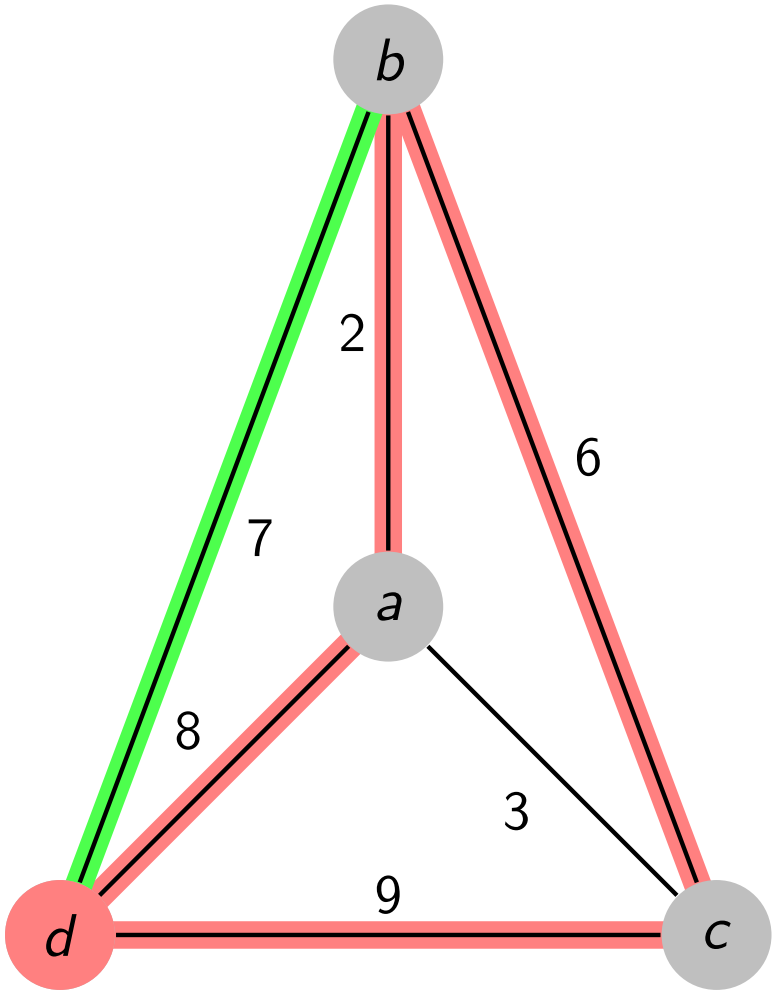

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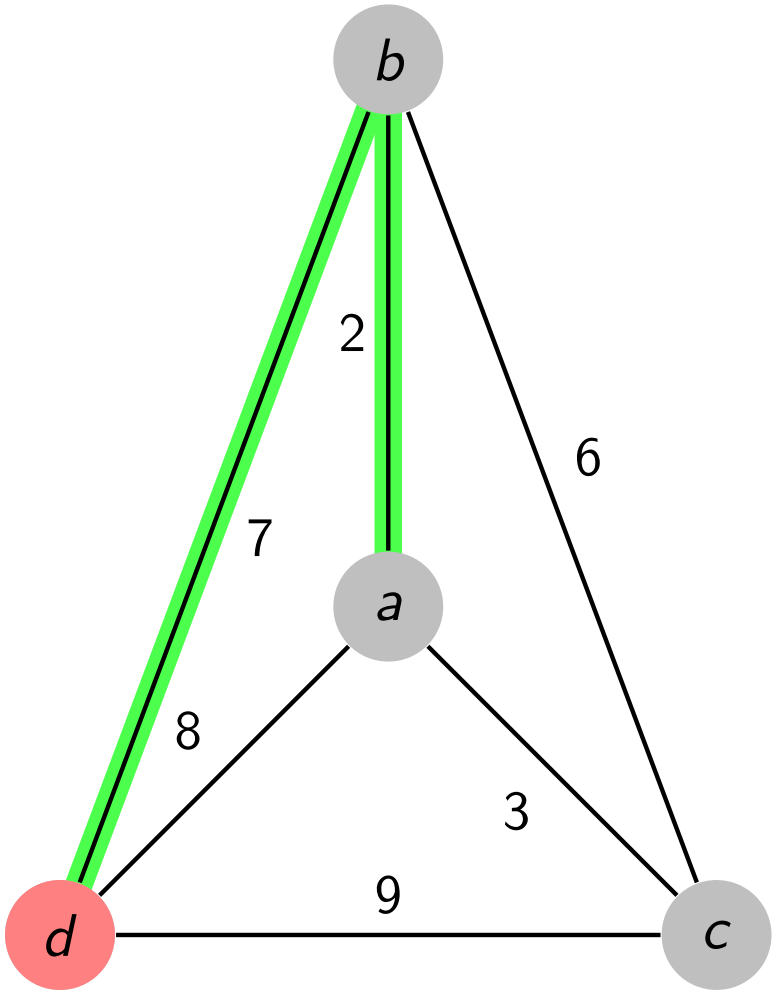

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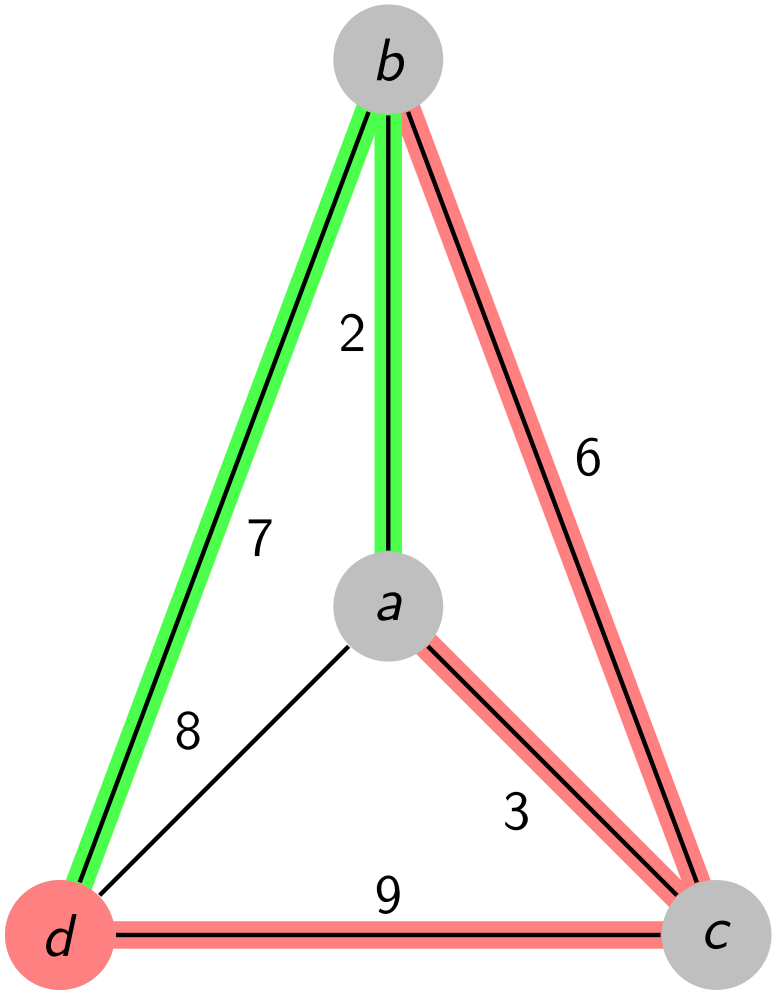

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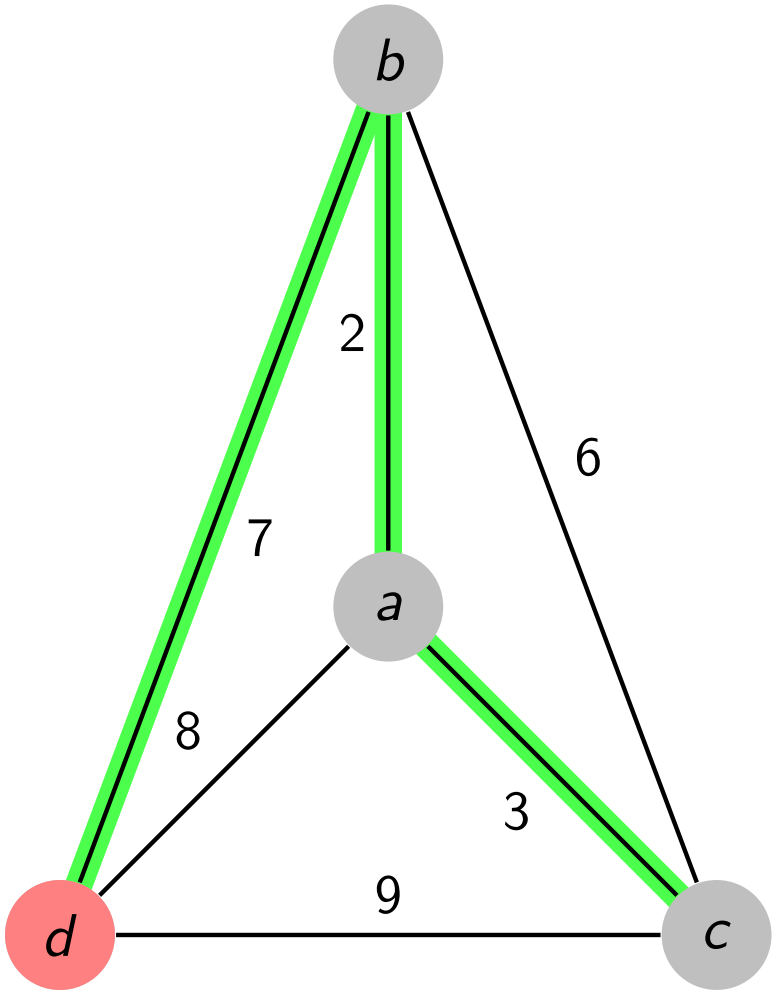

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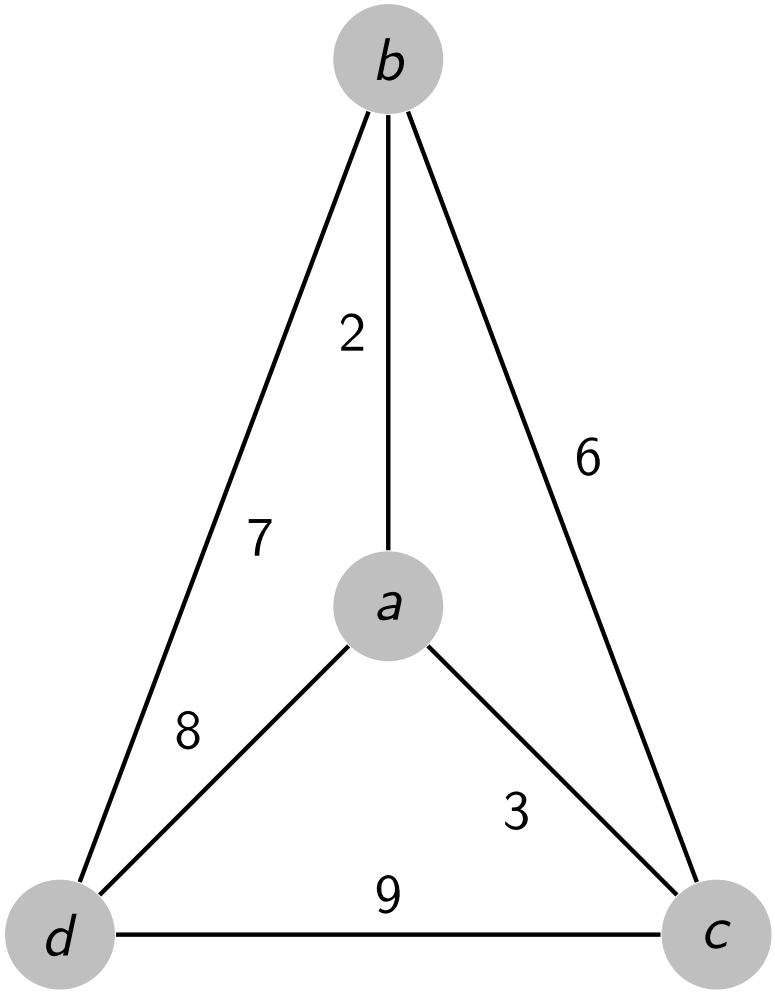

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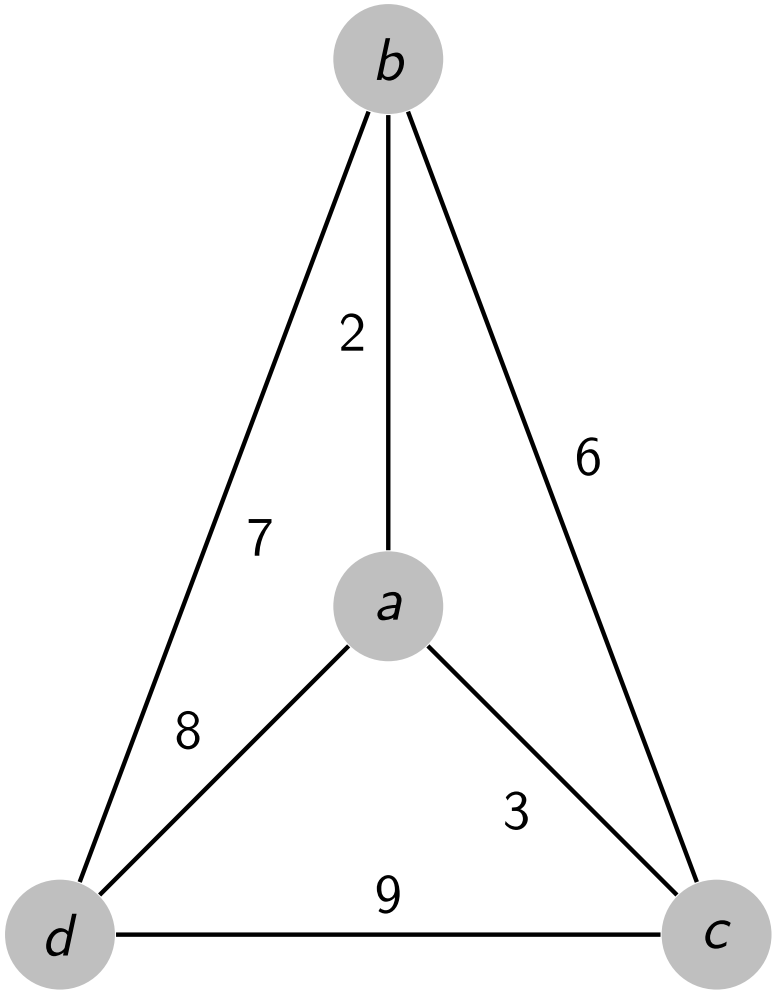

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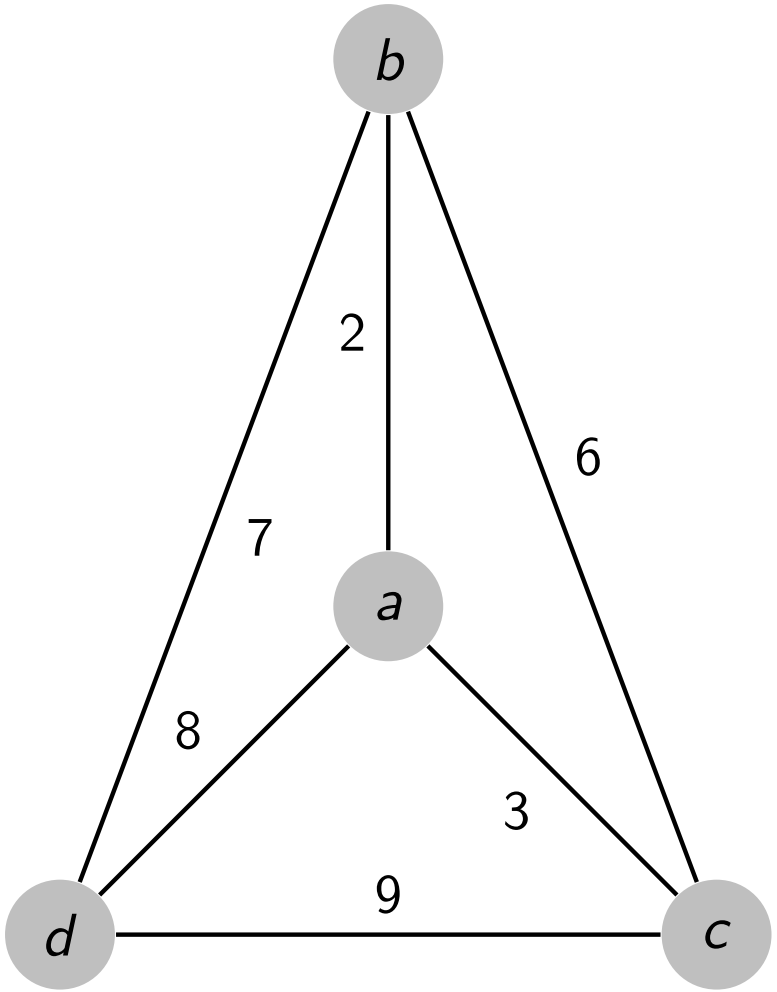

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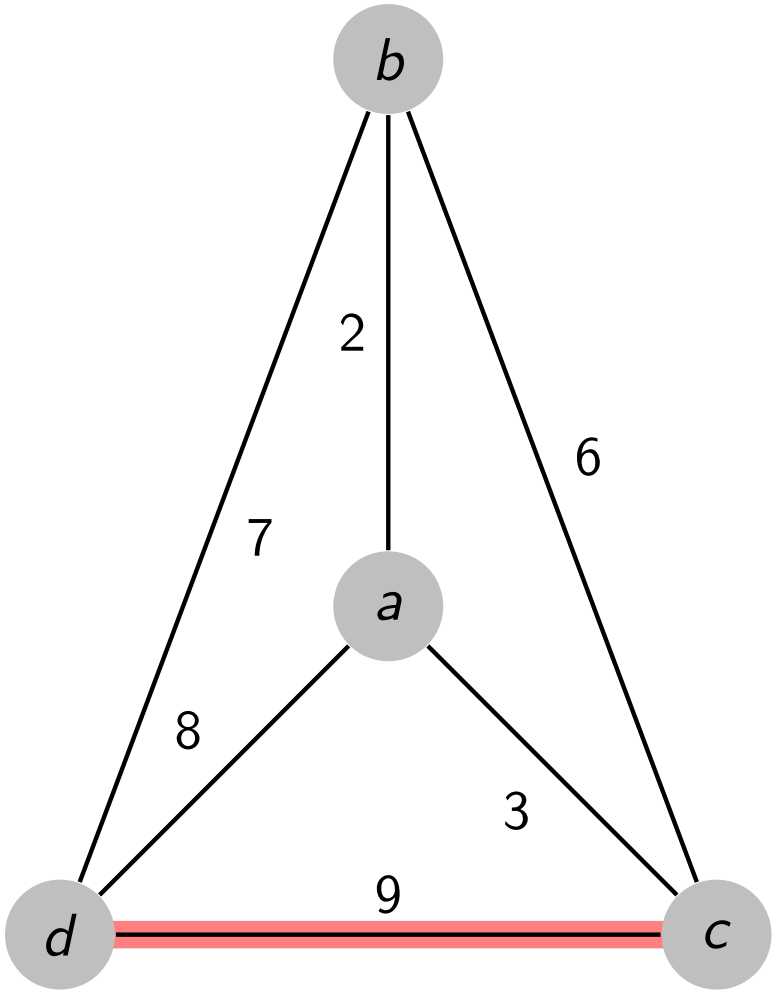

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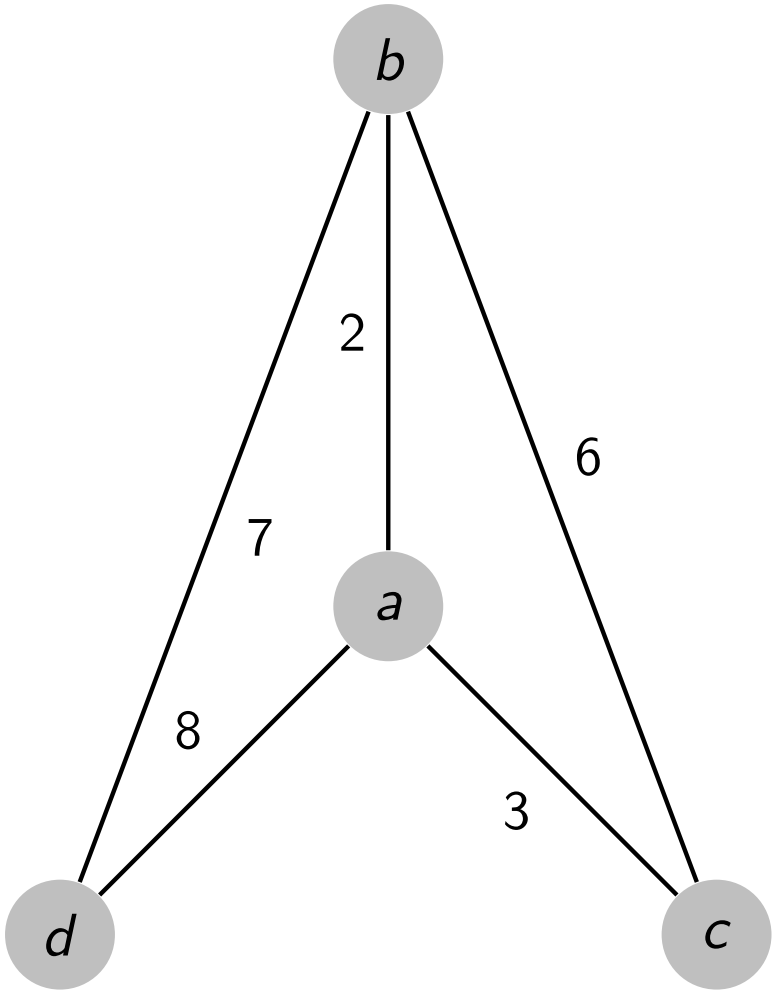

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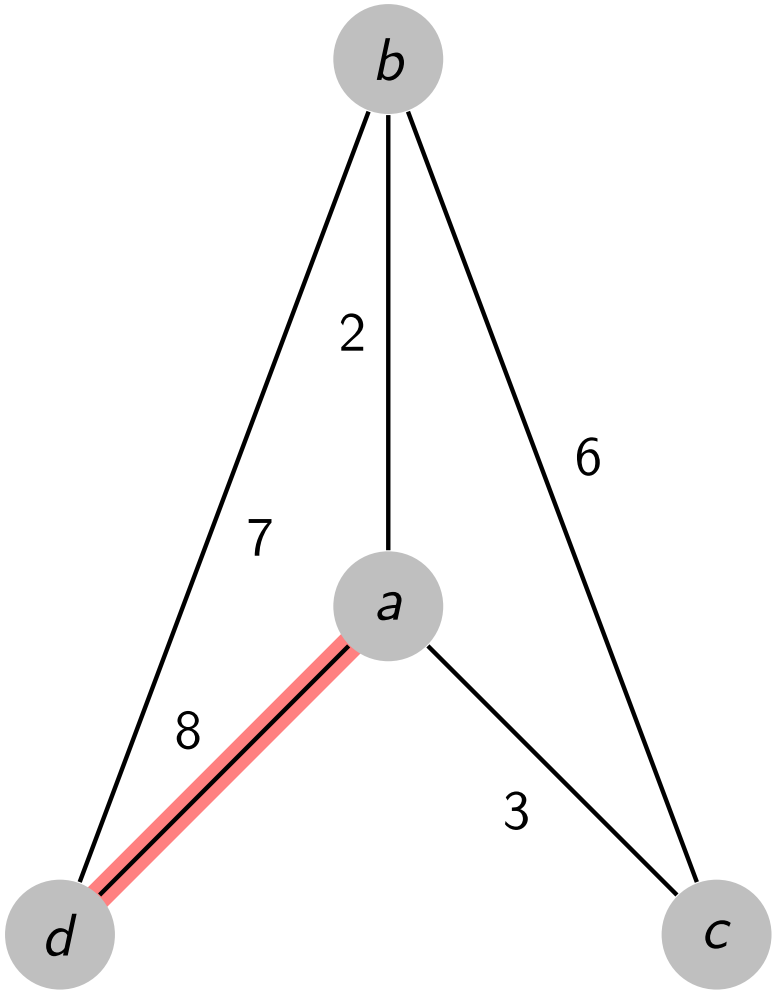

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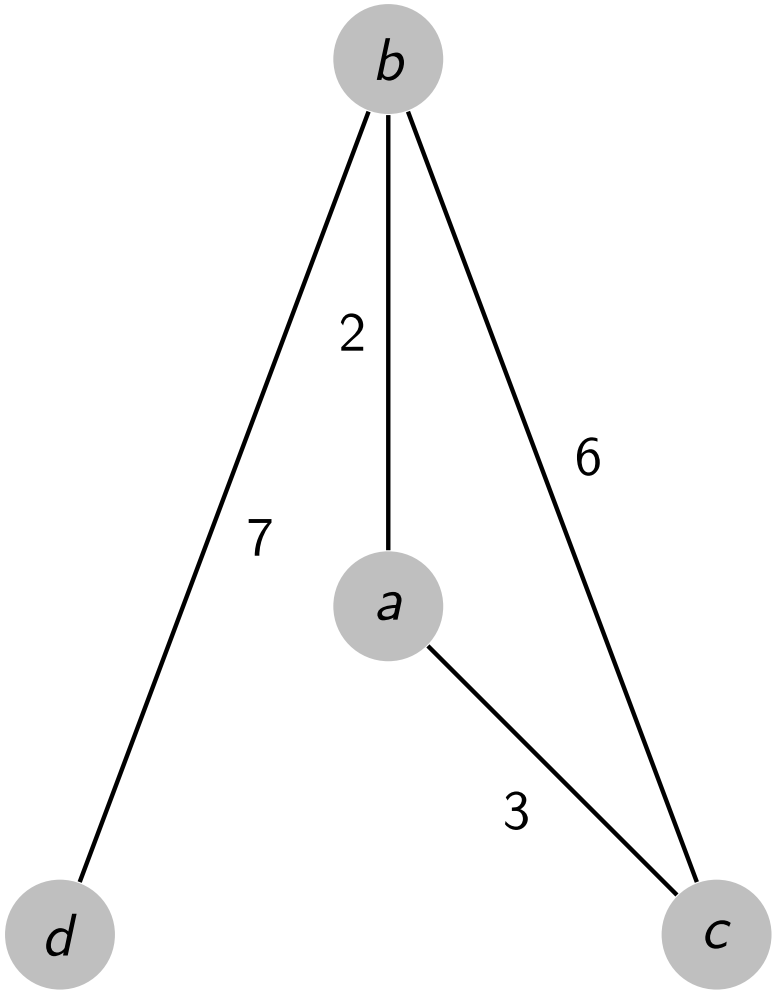

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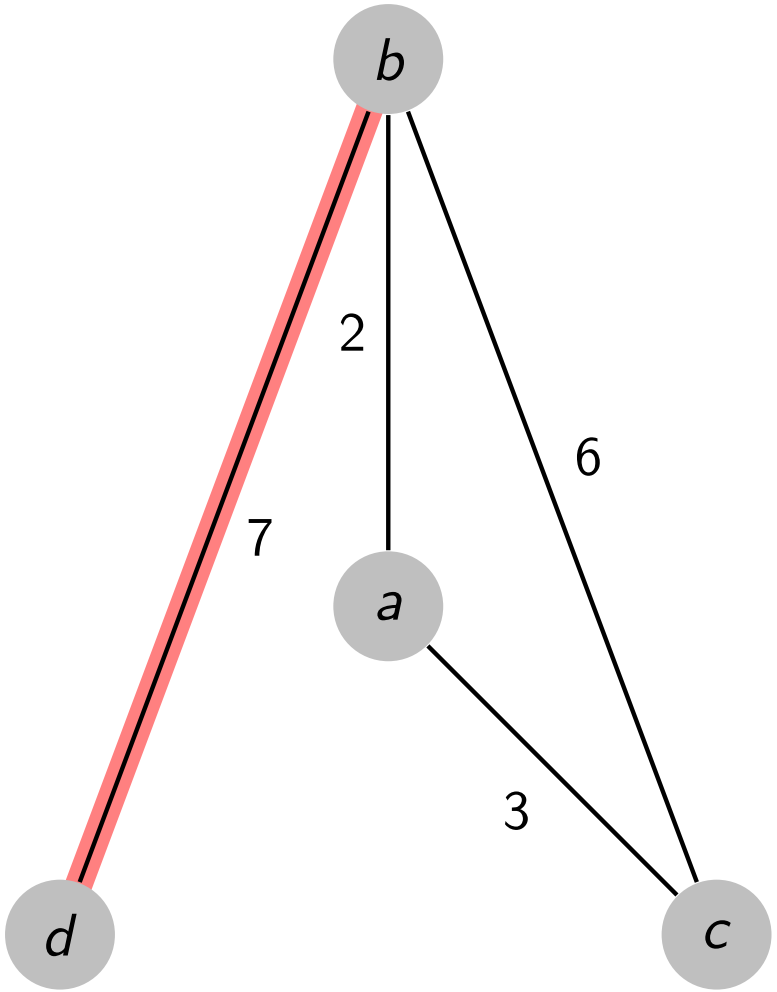

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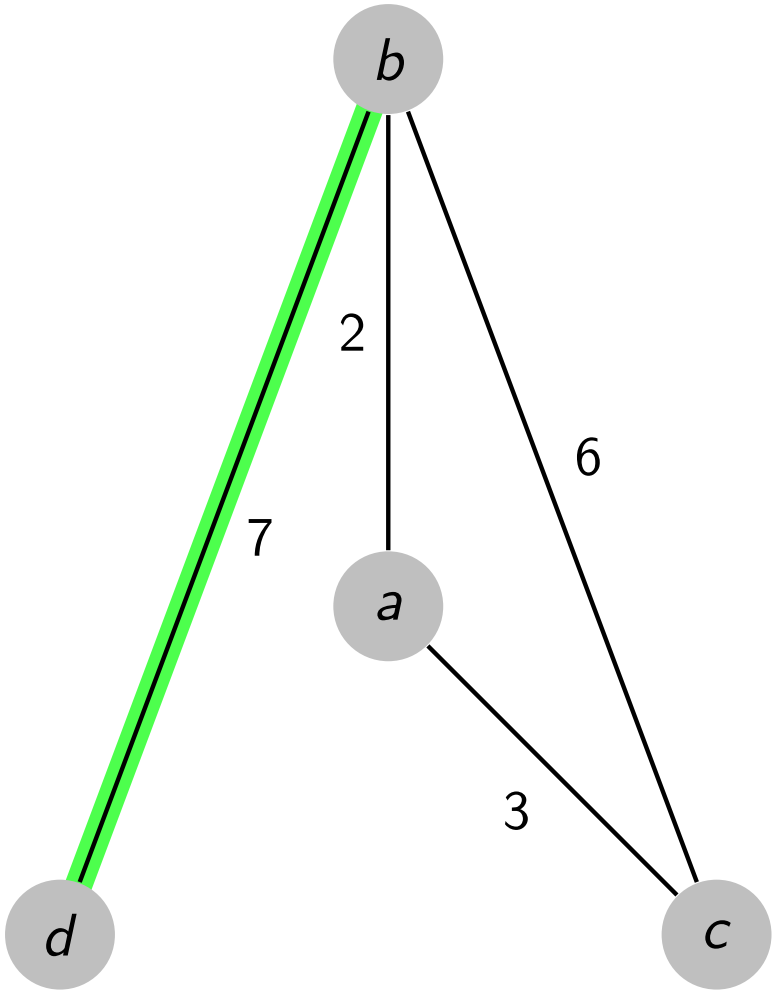

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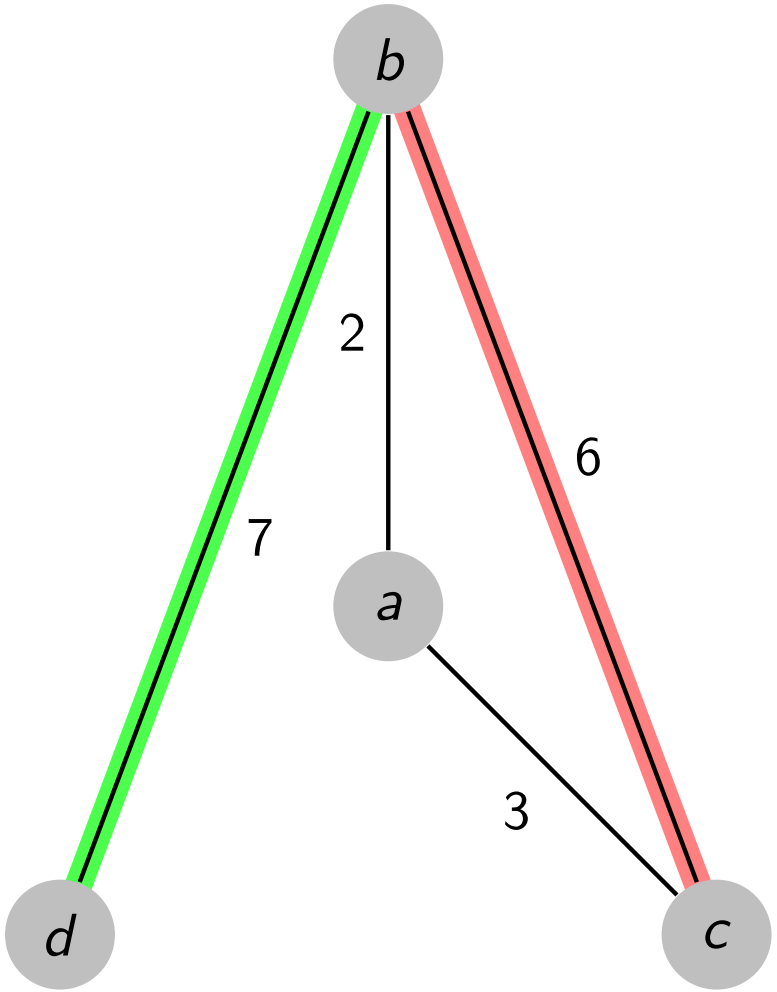

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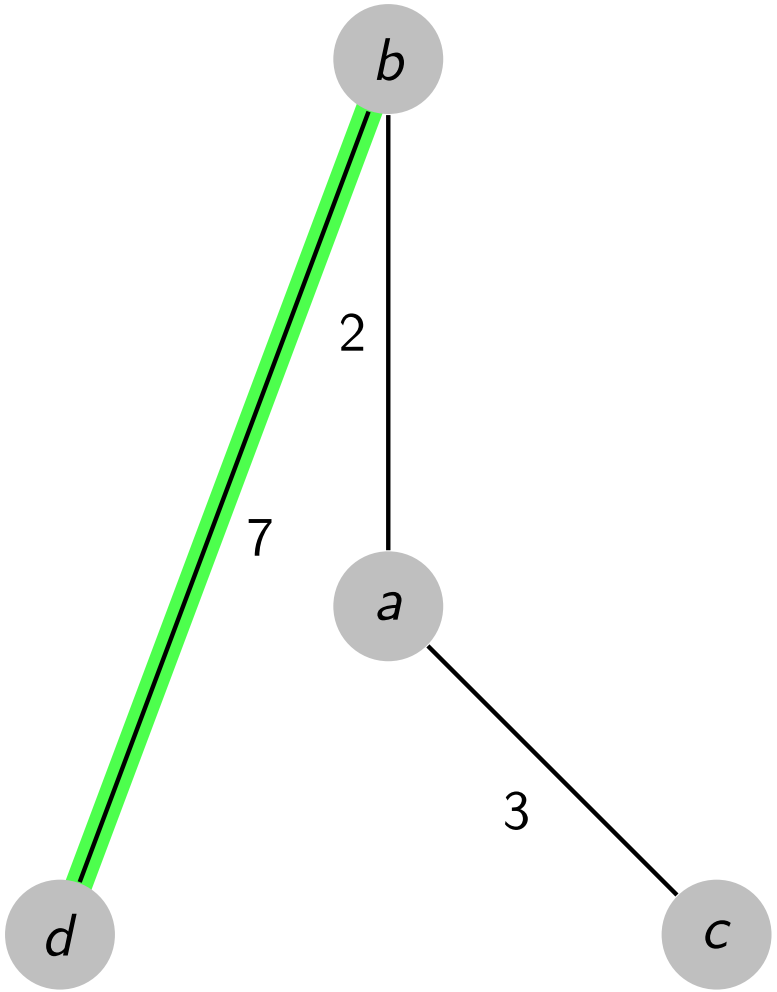

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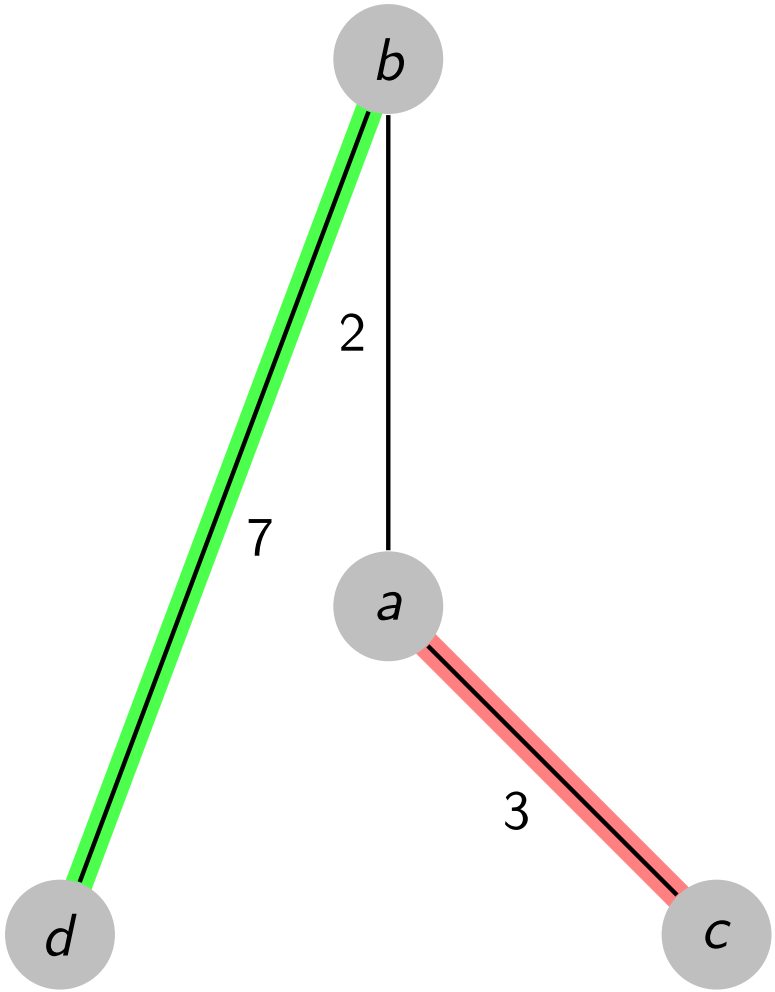

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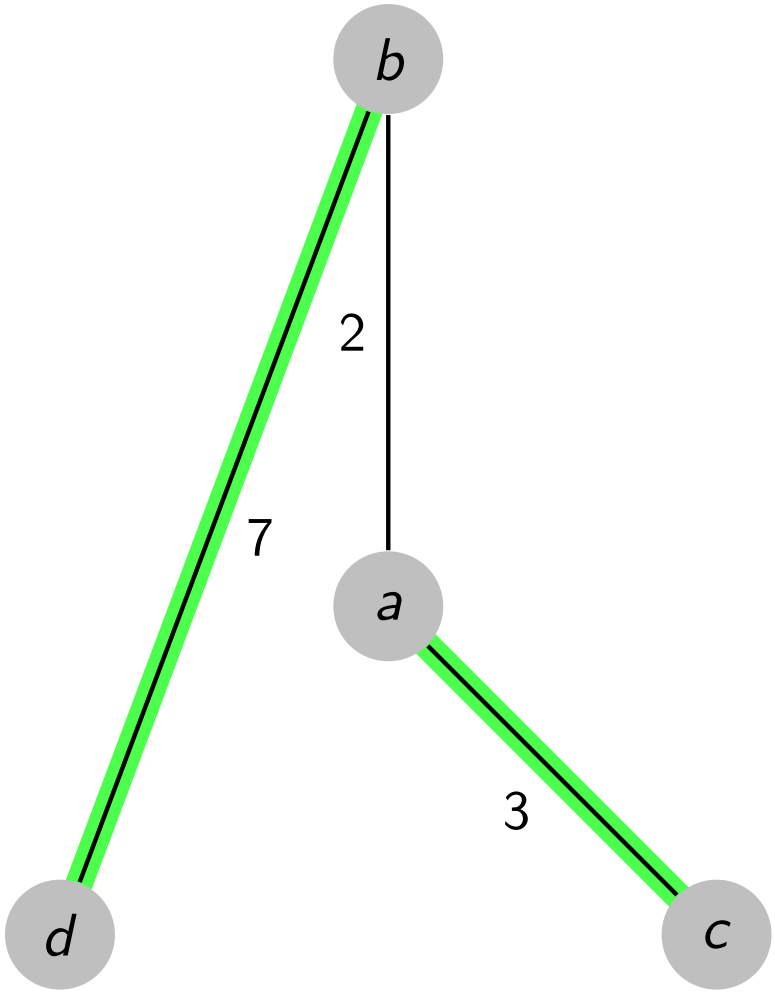

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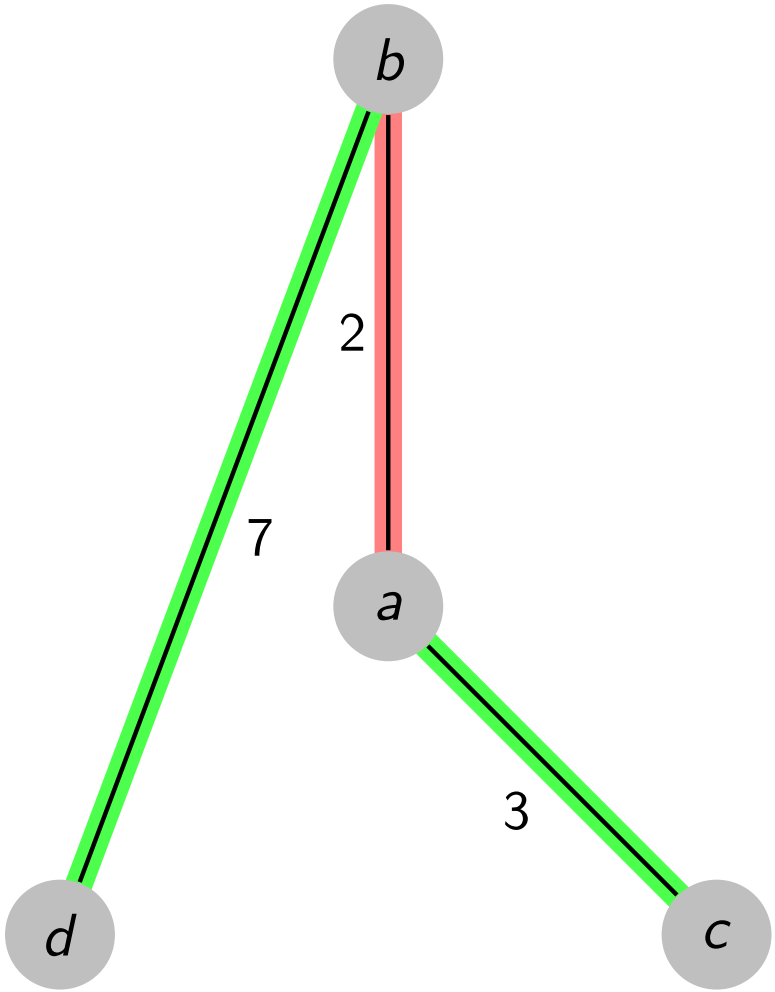

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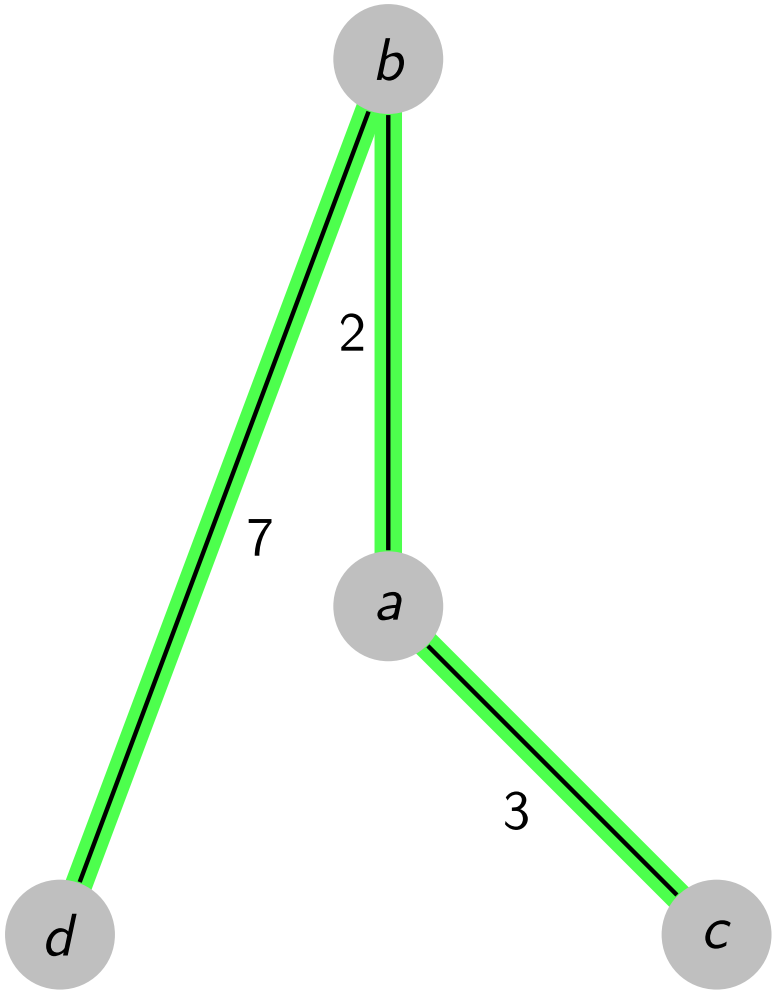

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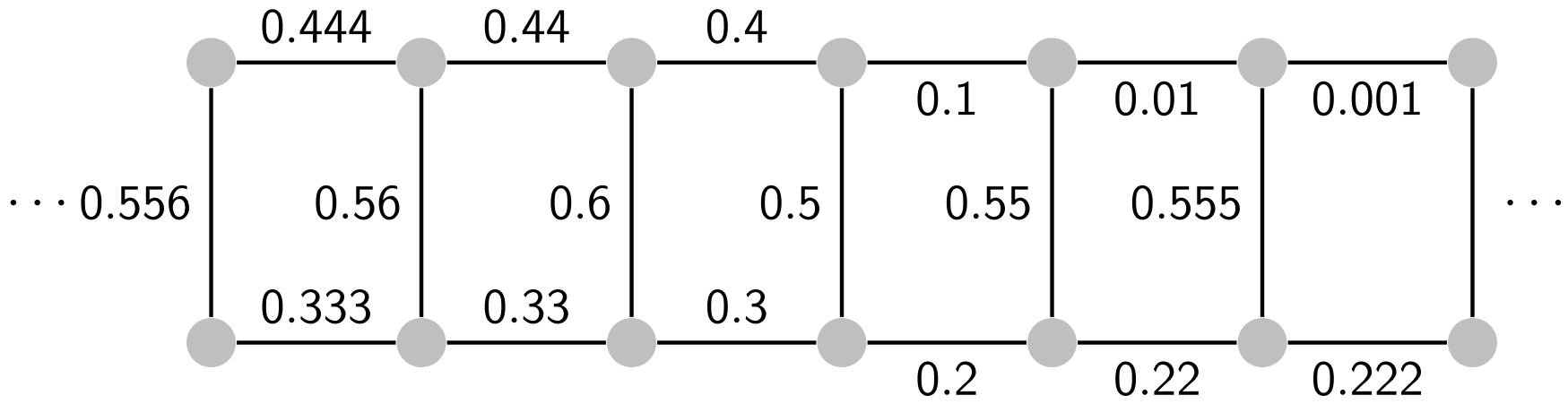

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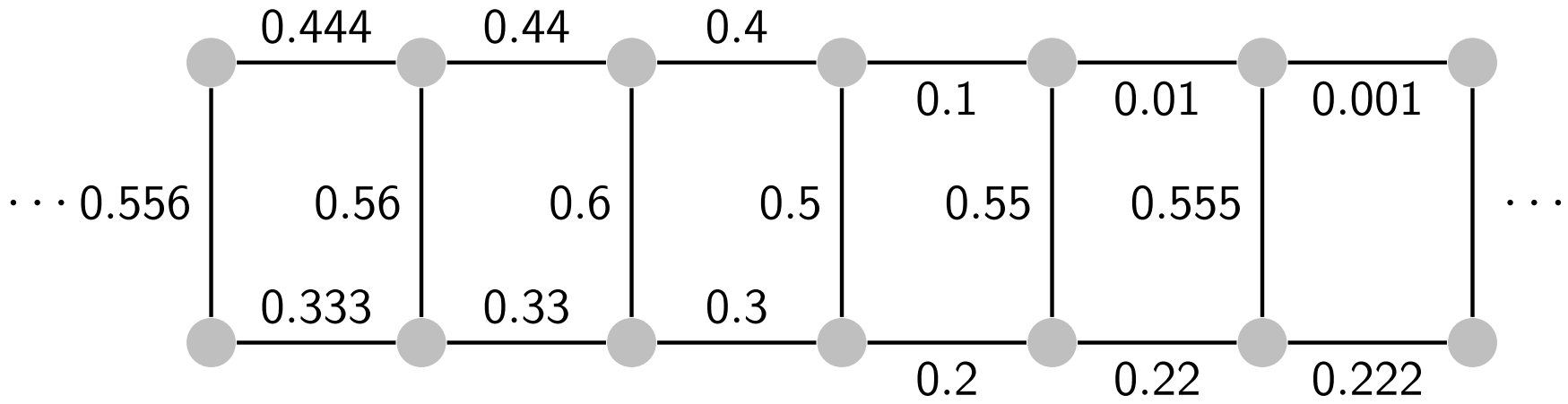

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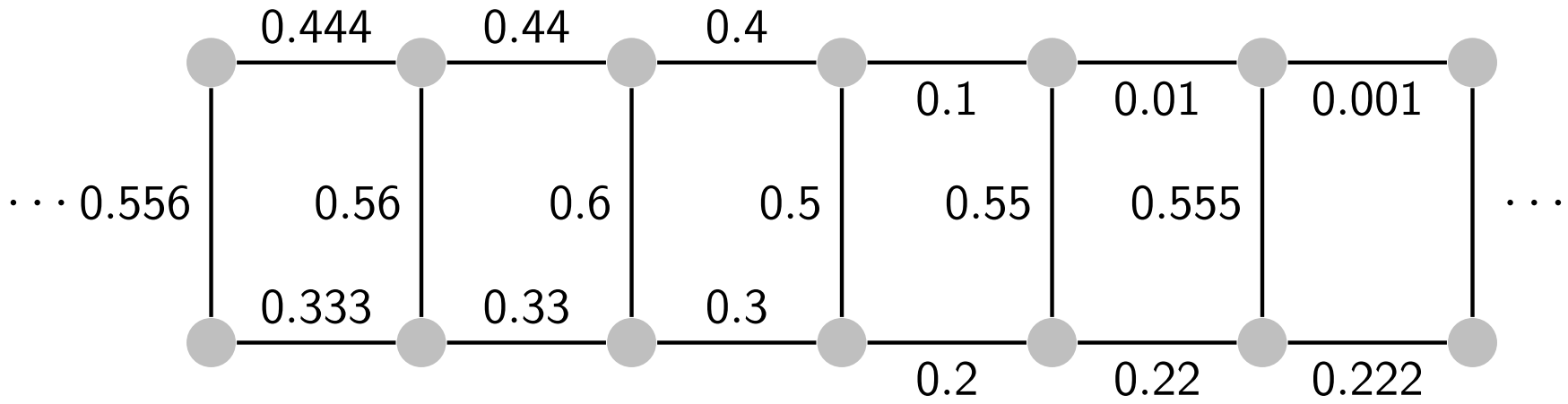

Kruskal's algorithm: There is no smallest edge!

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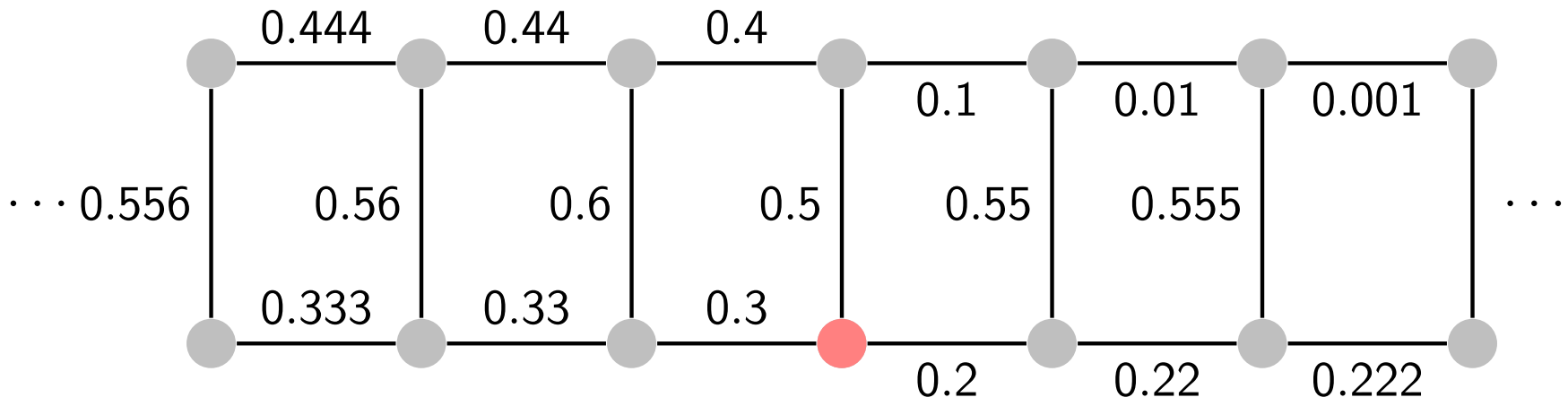

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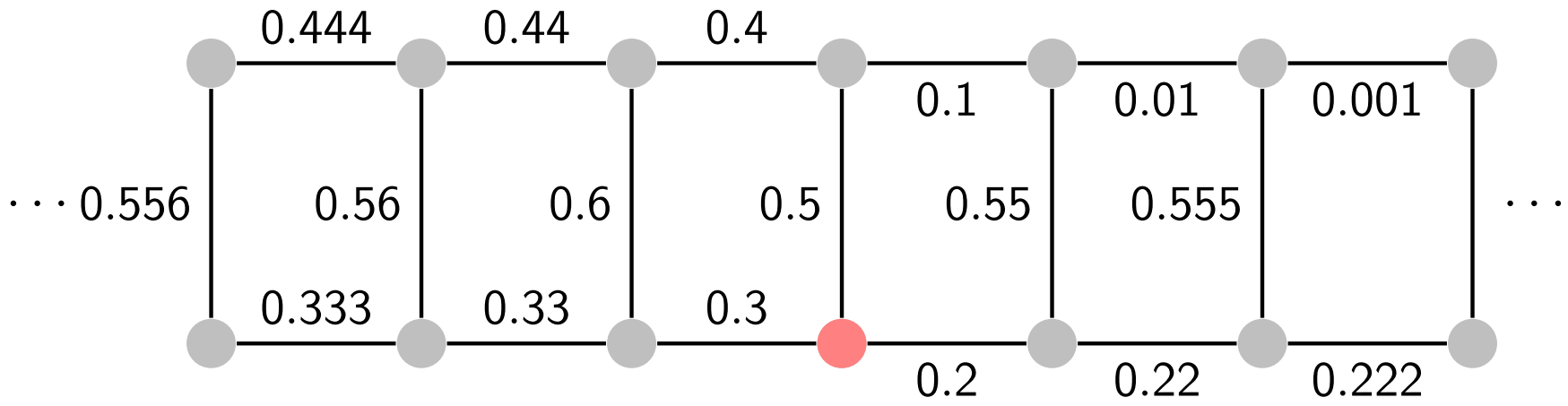

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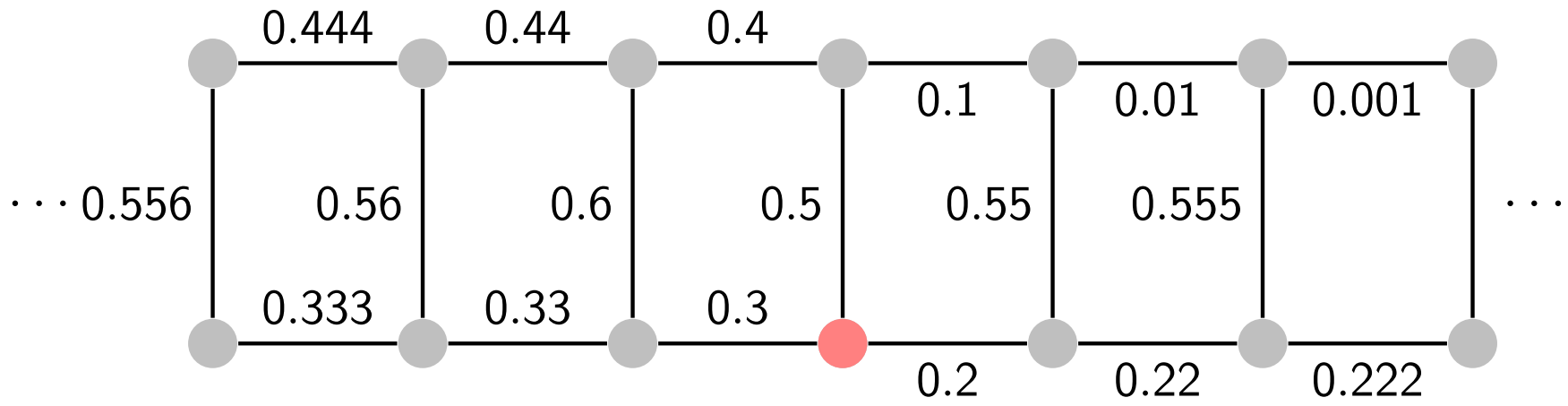

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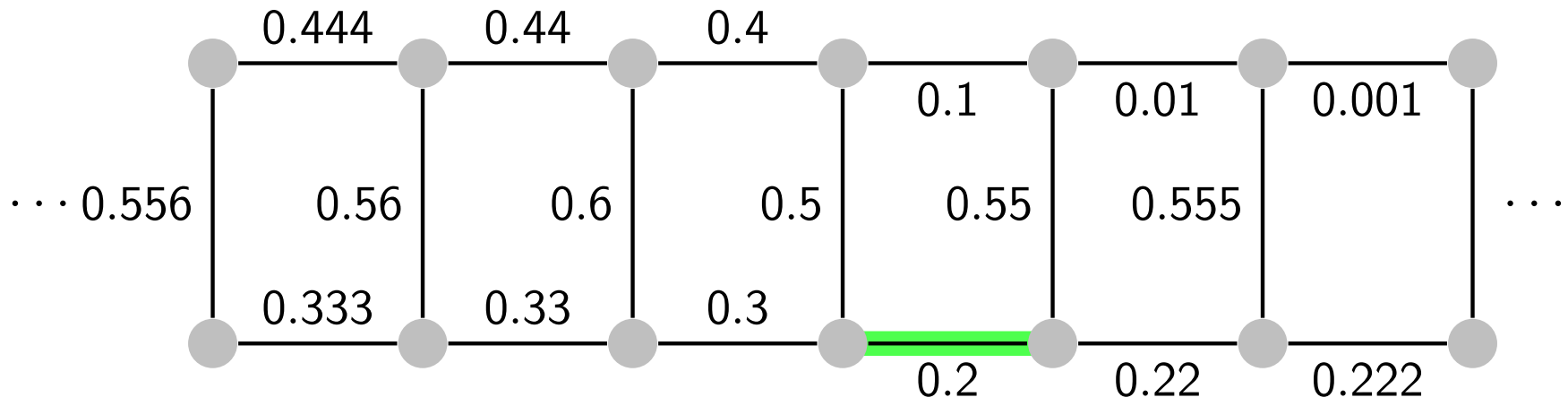

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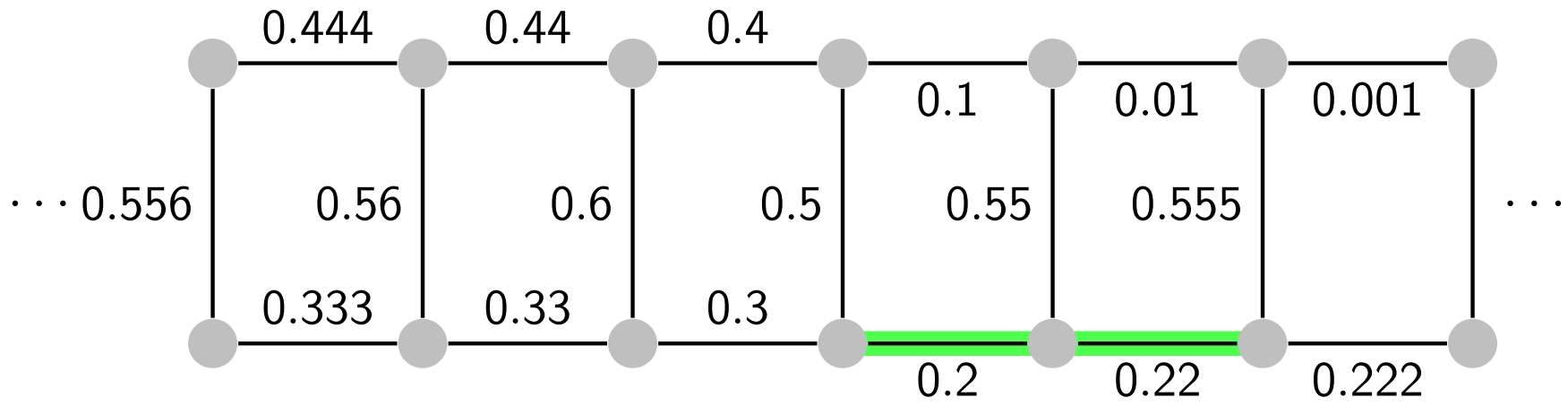

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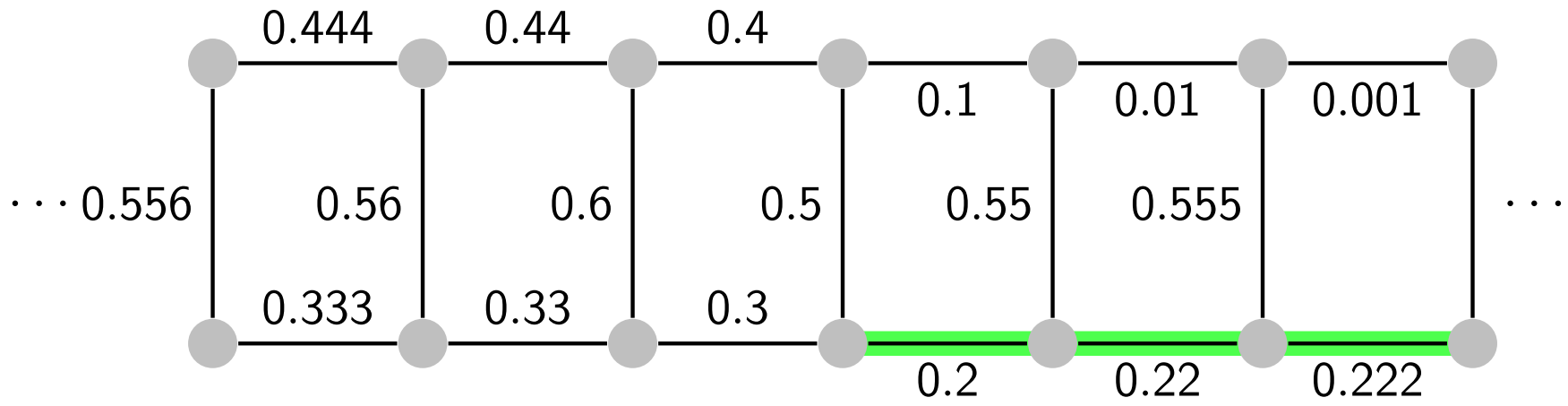

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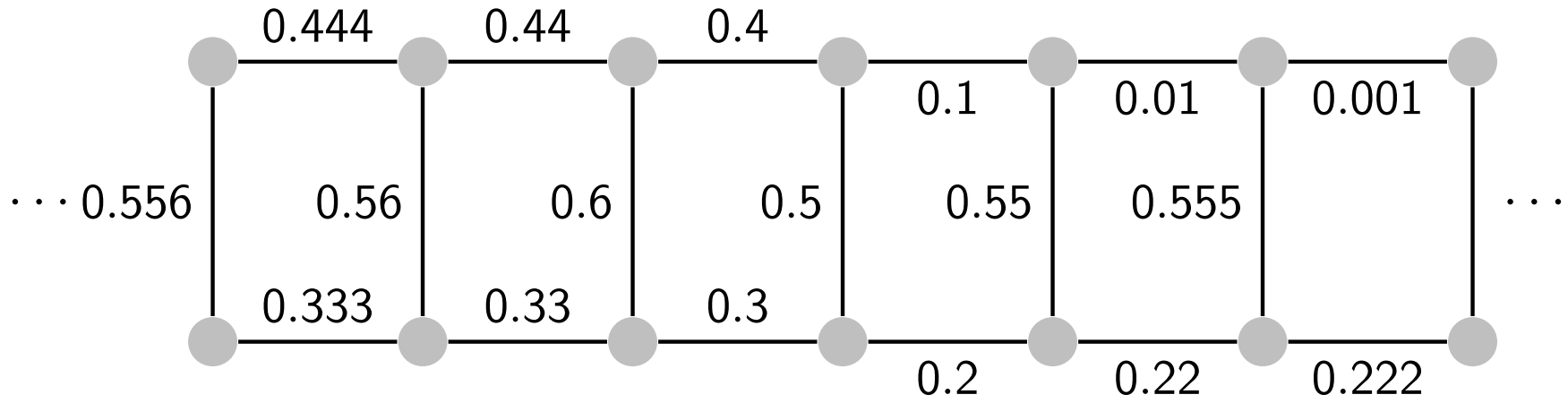

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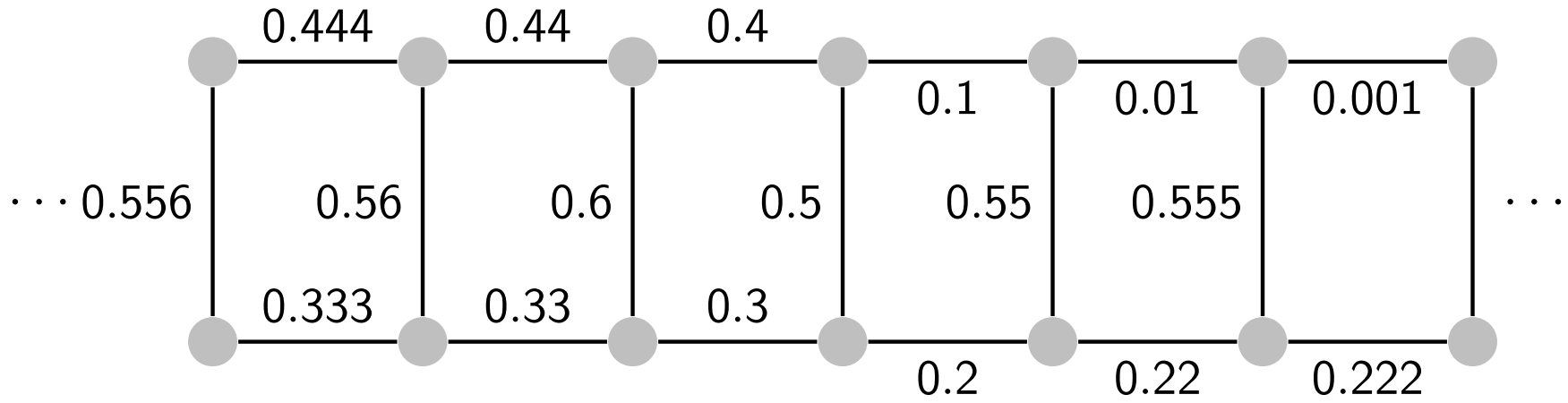

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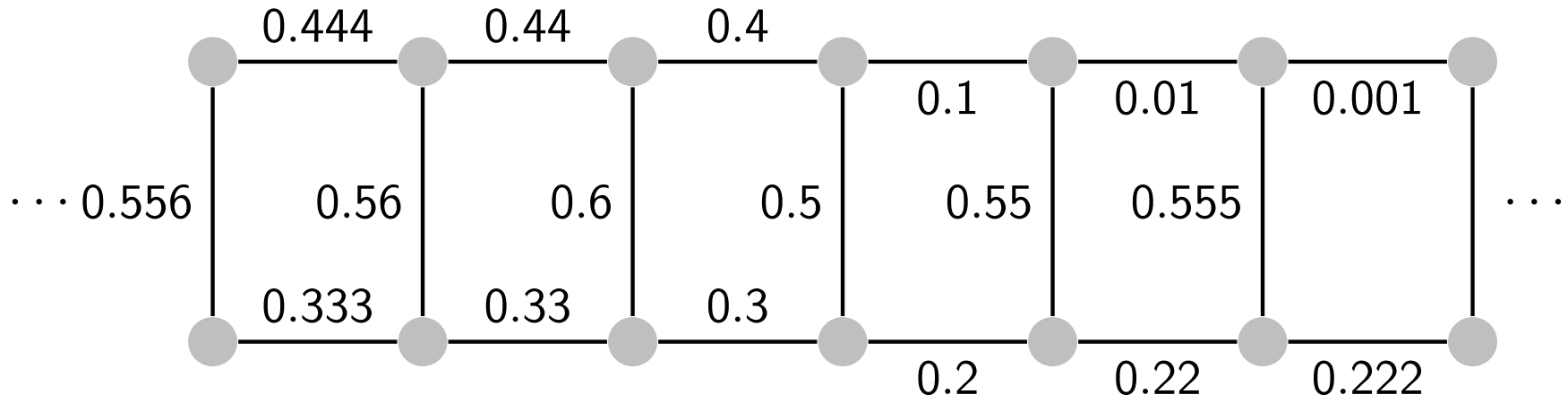

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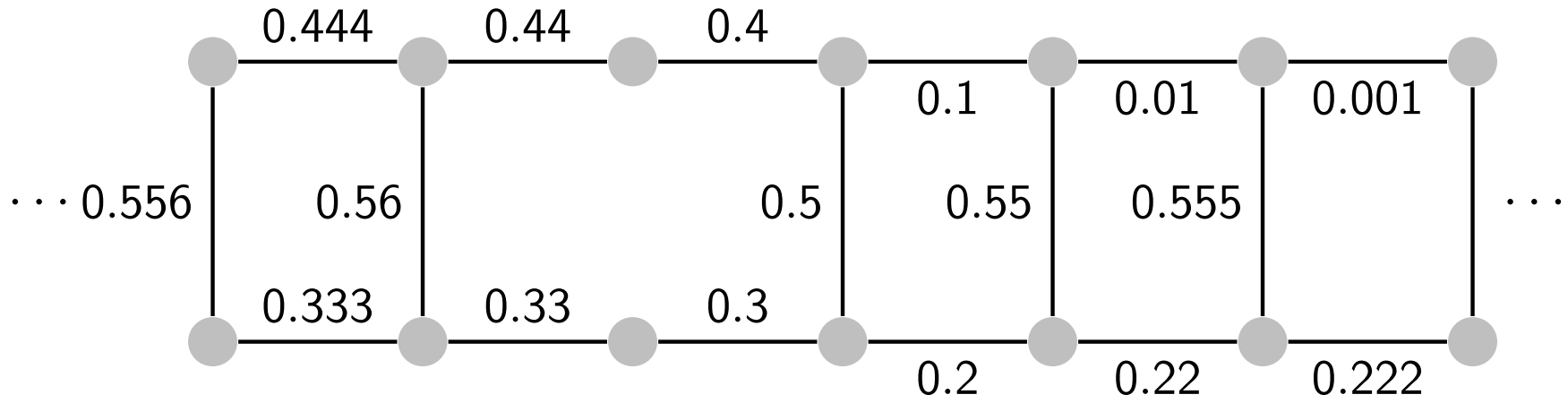

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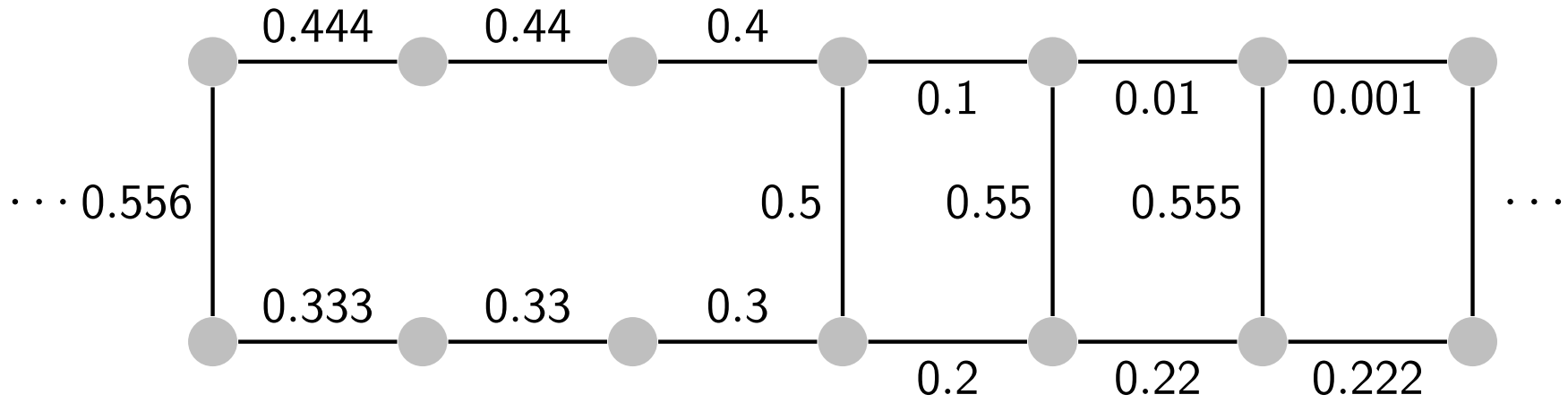

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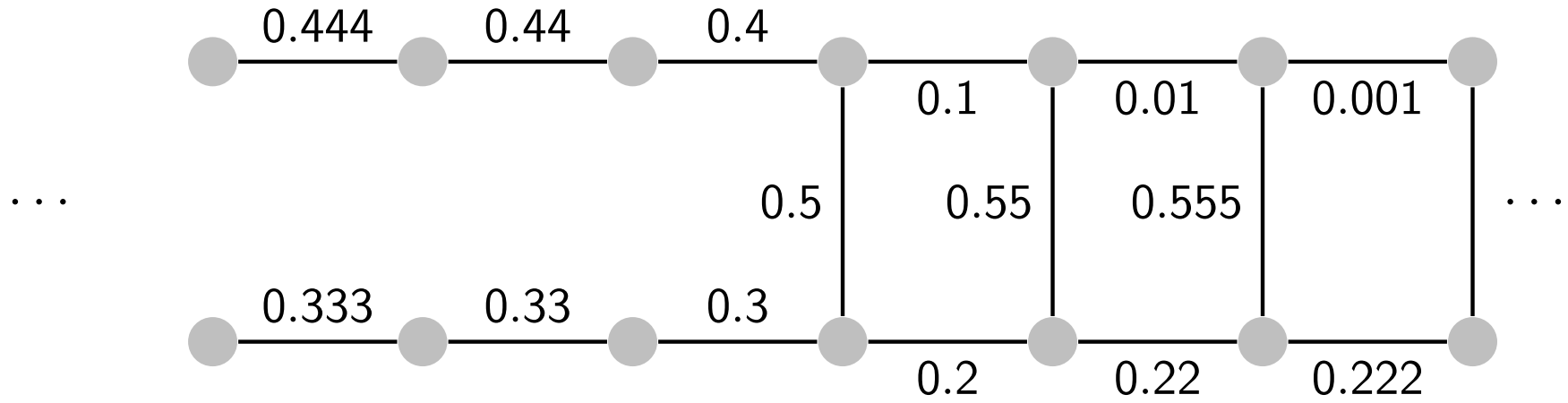

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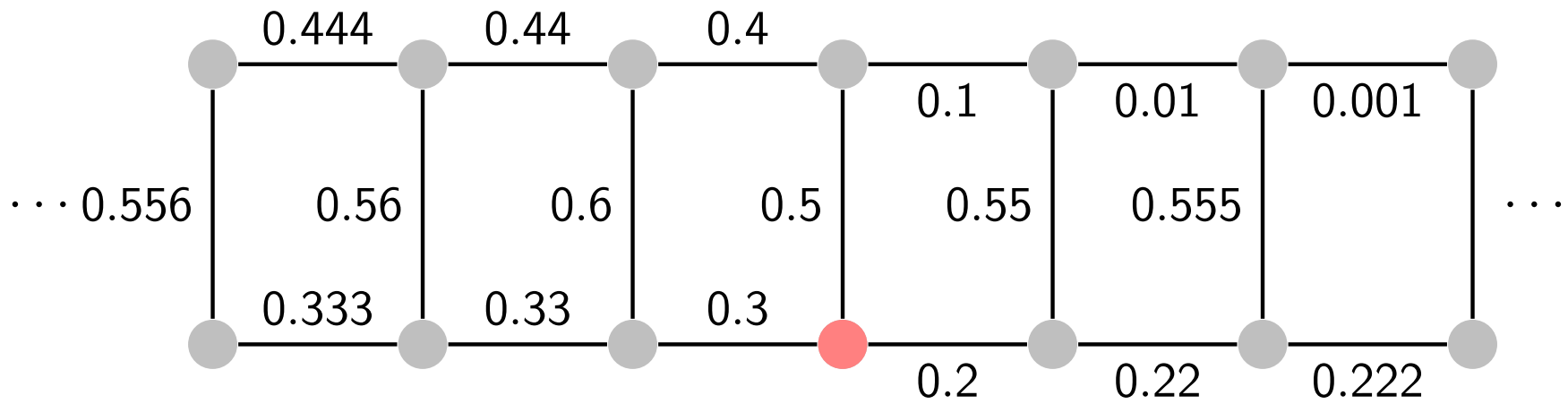

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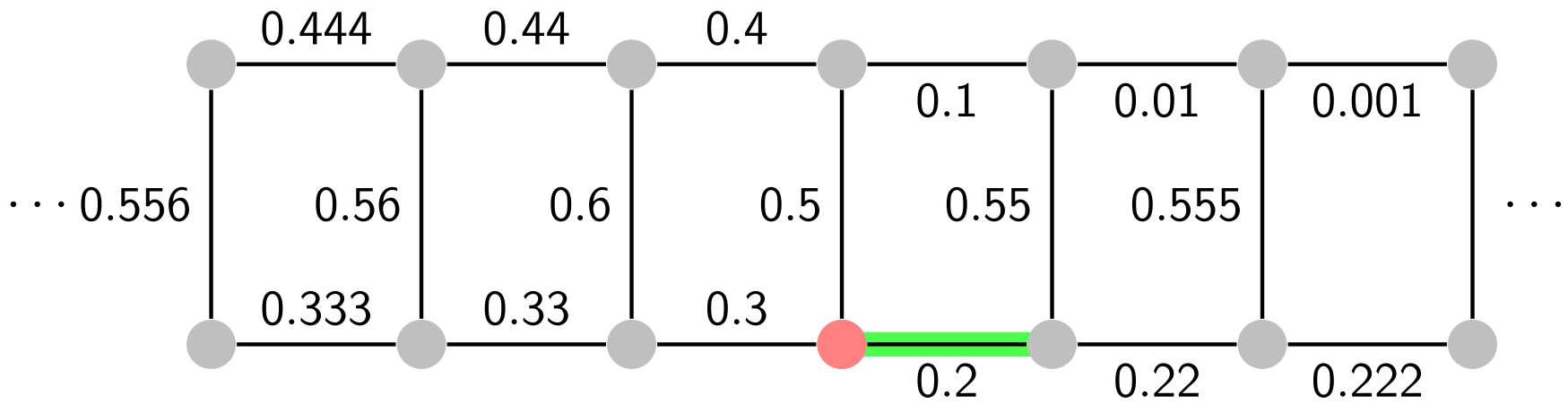

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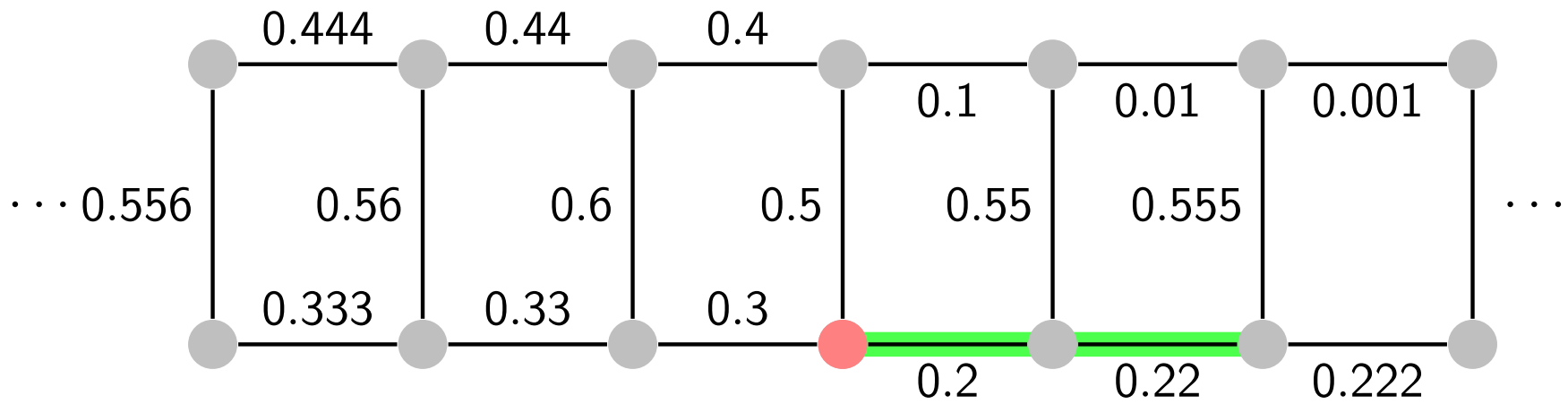

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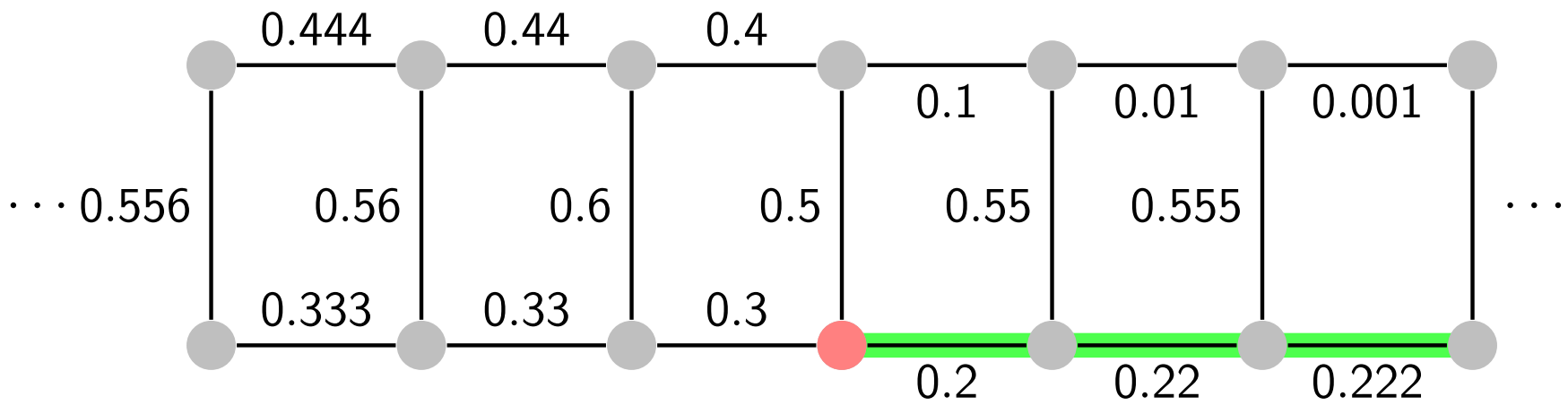

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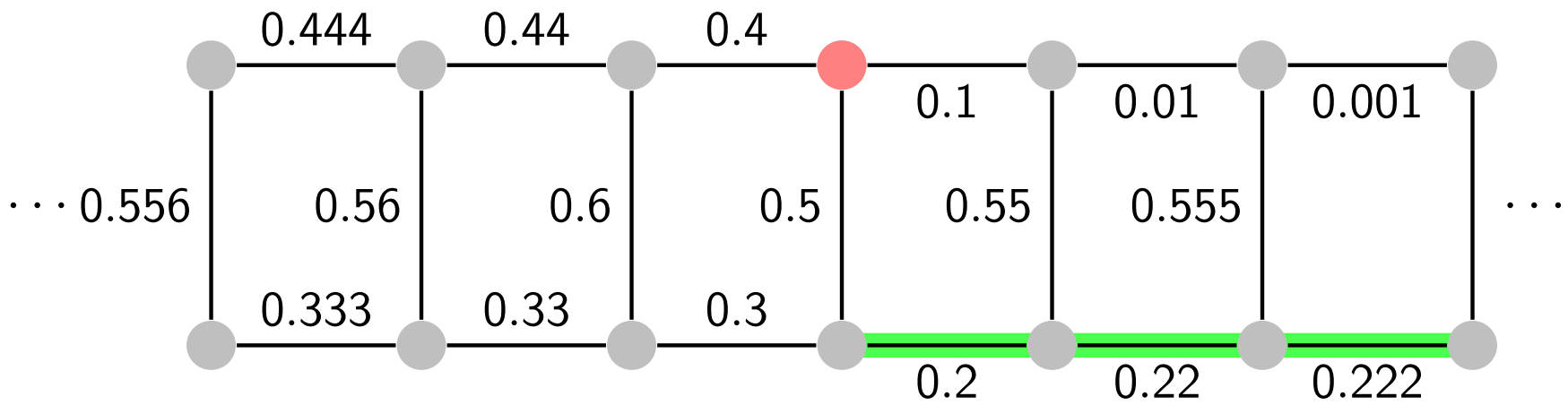

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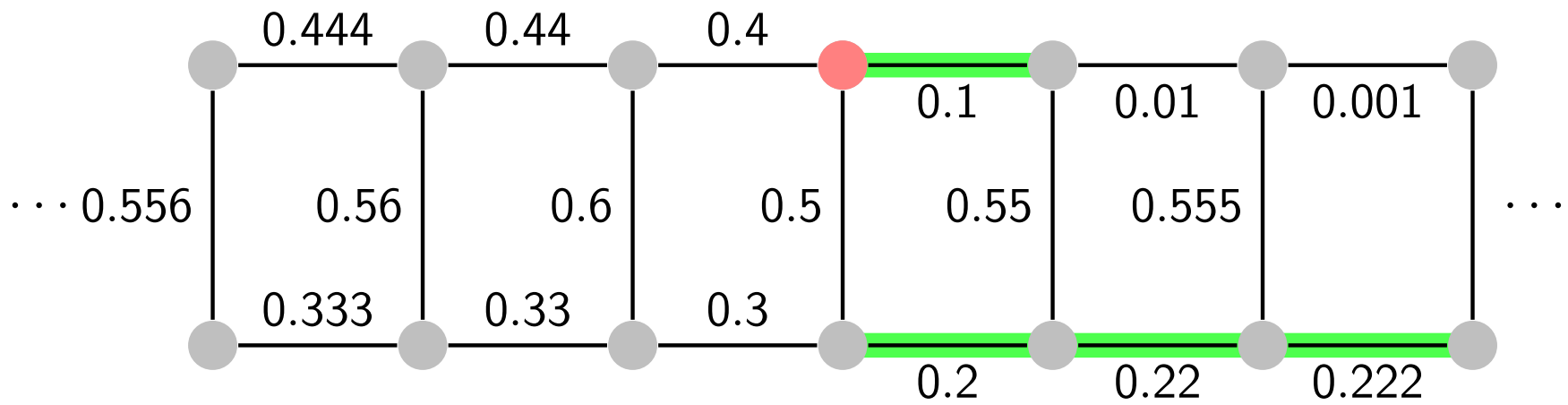

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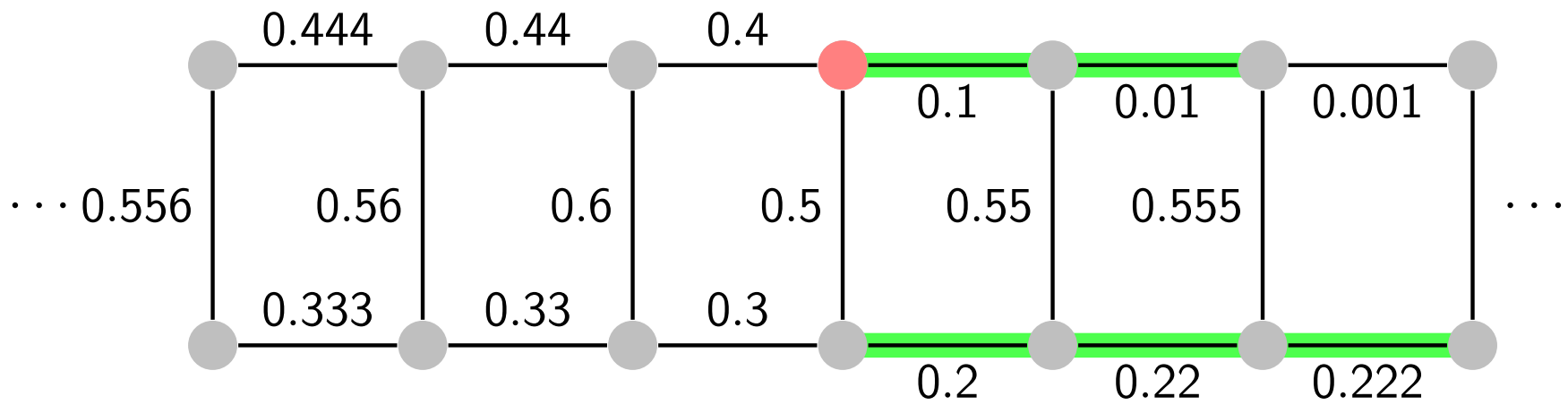

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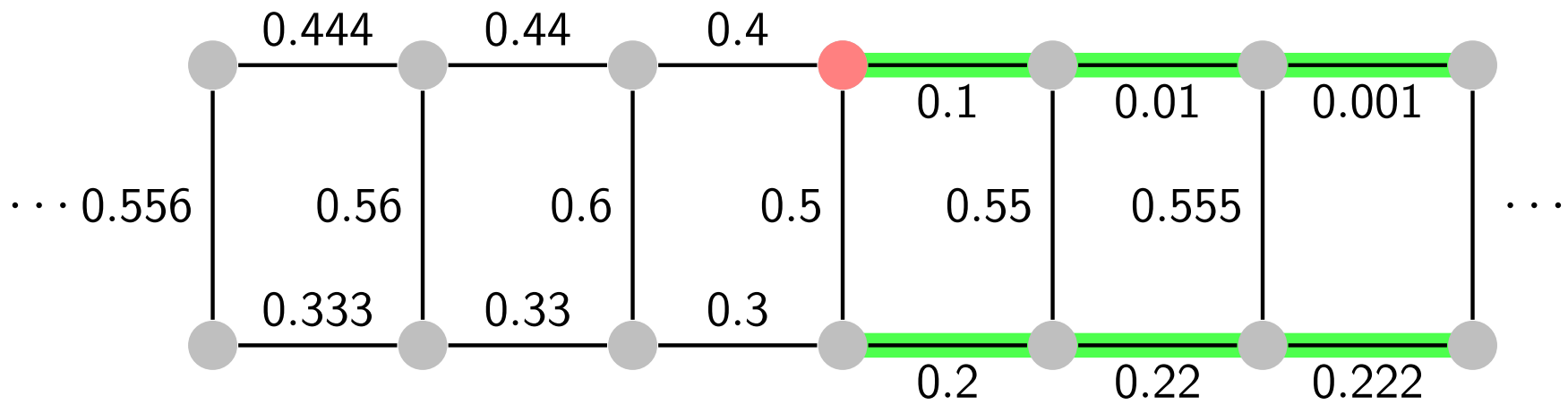

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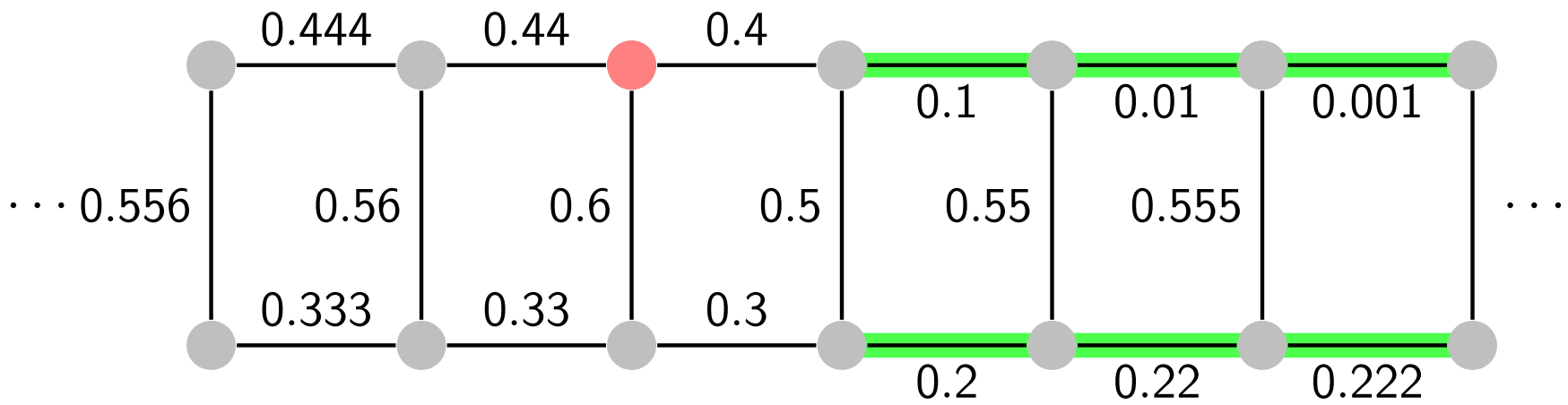

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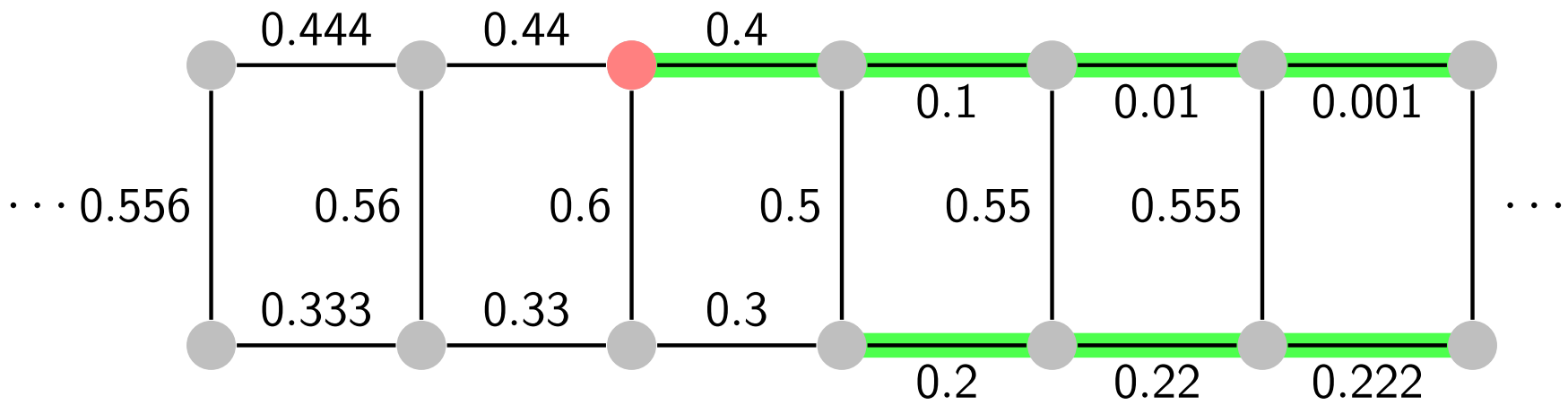

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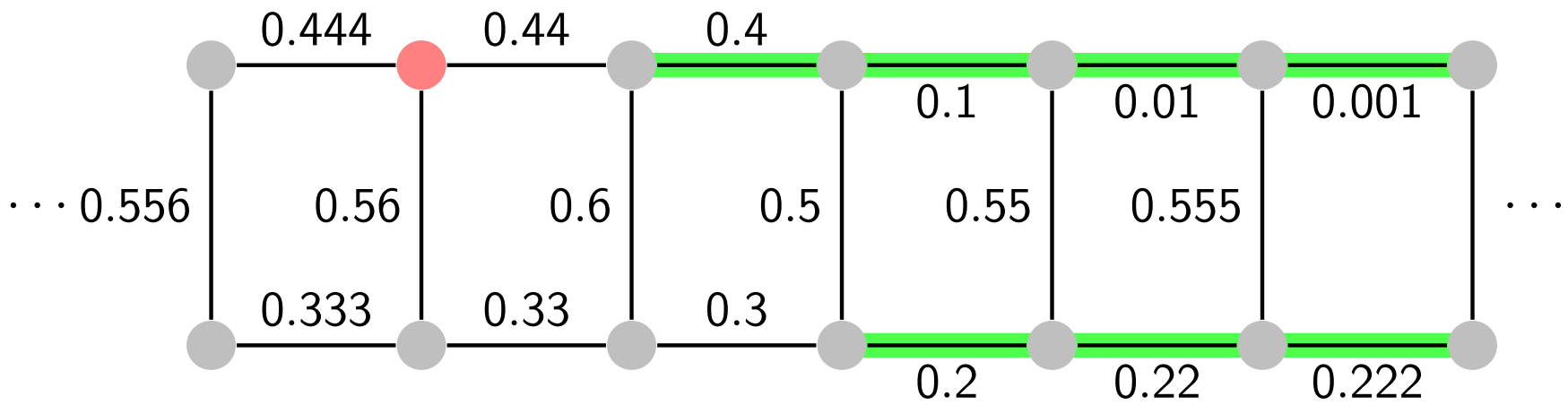

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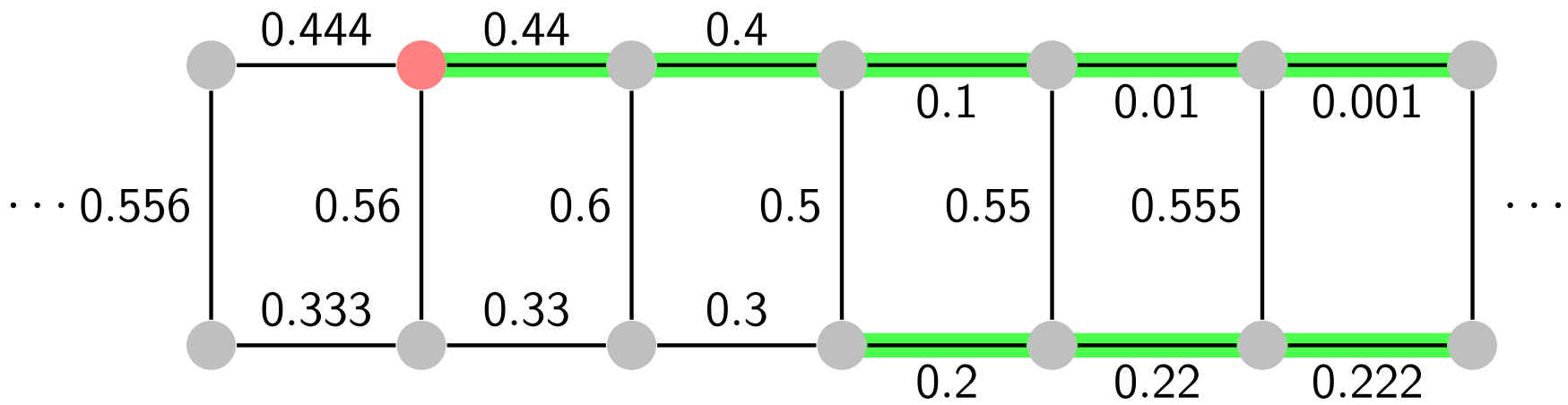

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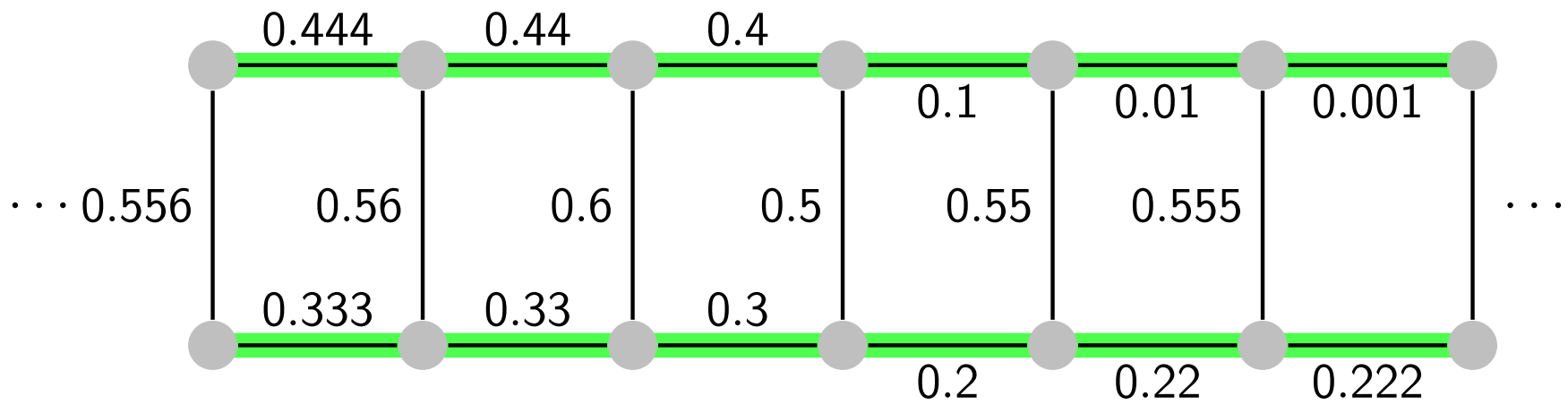

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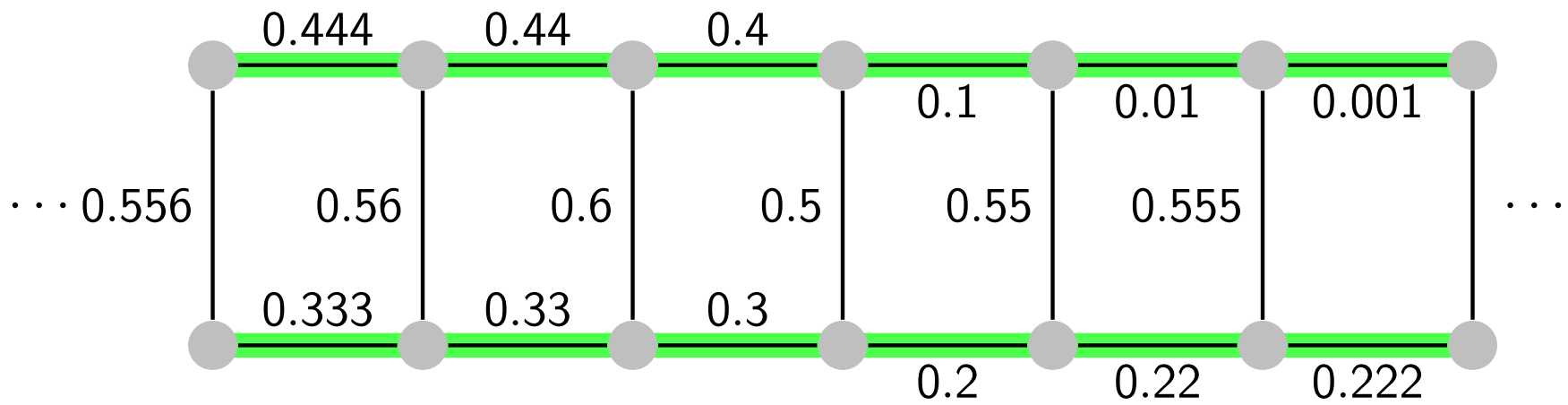

The result is called the Minimum Spanning Forest.

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nice: infinite, locally finite, all edge weights distinct.

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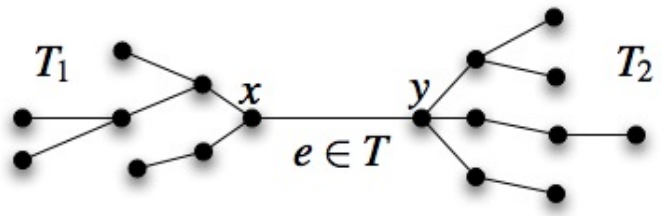

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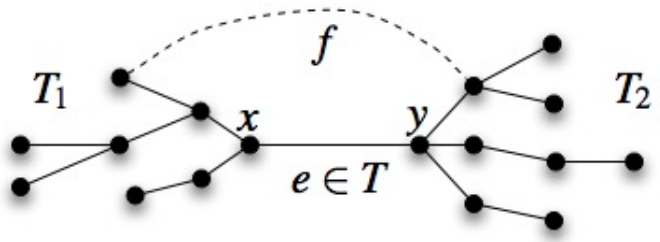

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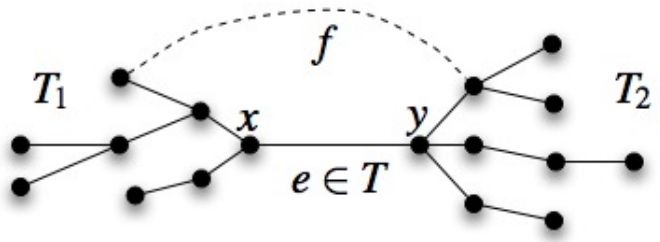
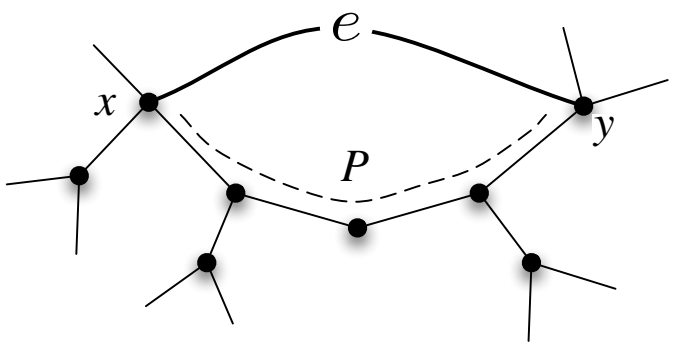


Every edge f of P satisfies $w_f < w_e$



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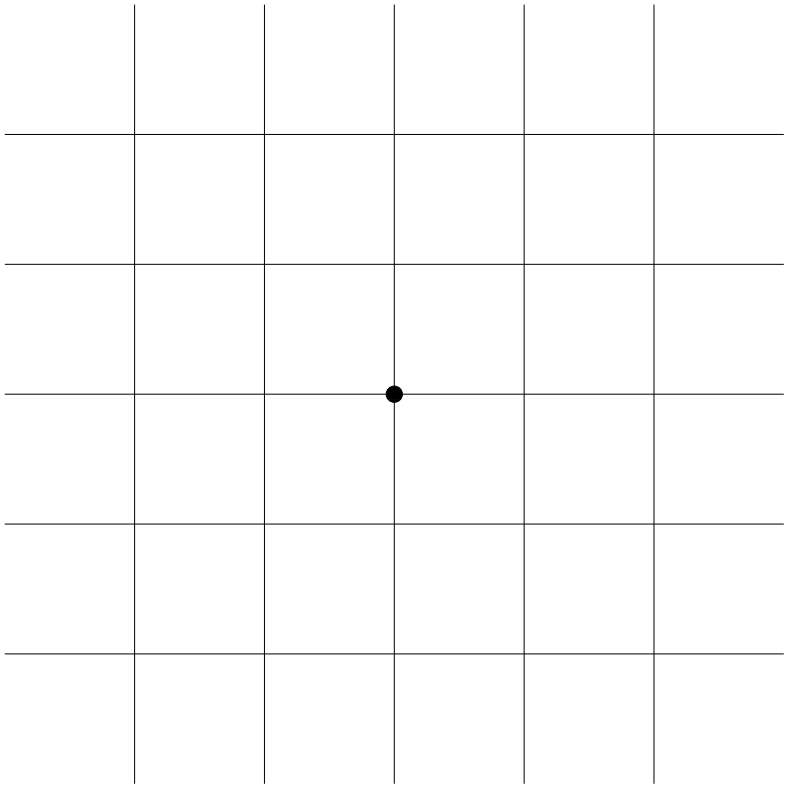
Note: in the percolation process, there are never two distinct infinite components!

Next: a very quick refresher on percolation.

Bernoulli bond percolation

- Example: Nearest neighbour graph on $\mathbb{Z}^2$.



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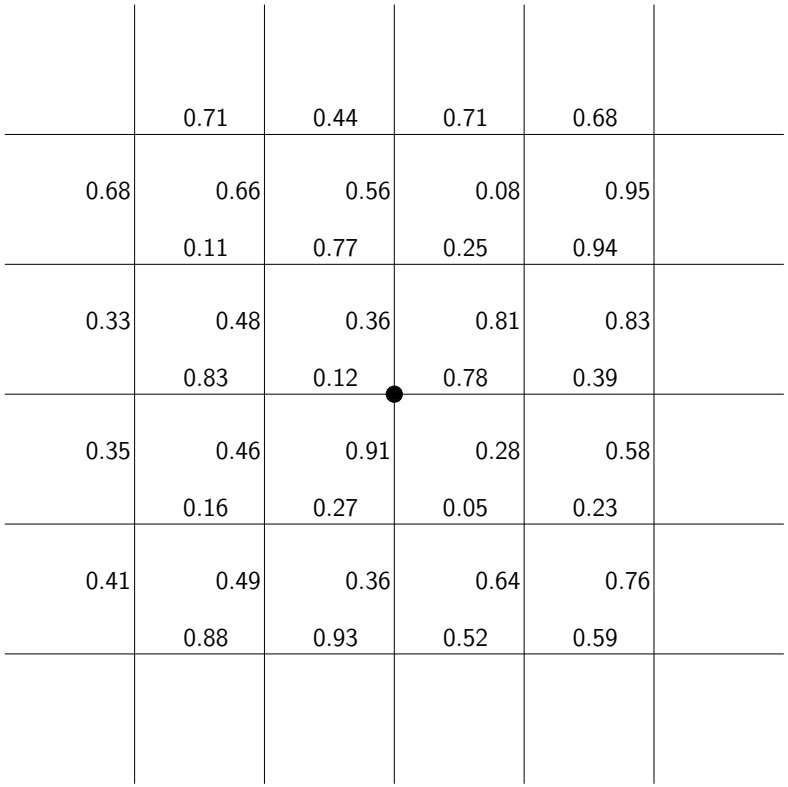

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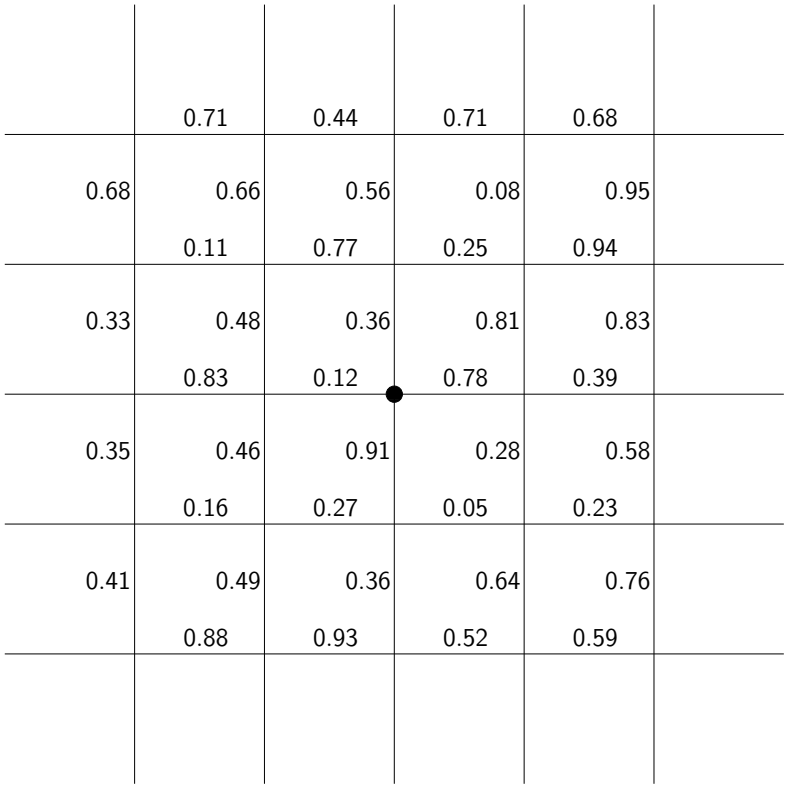

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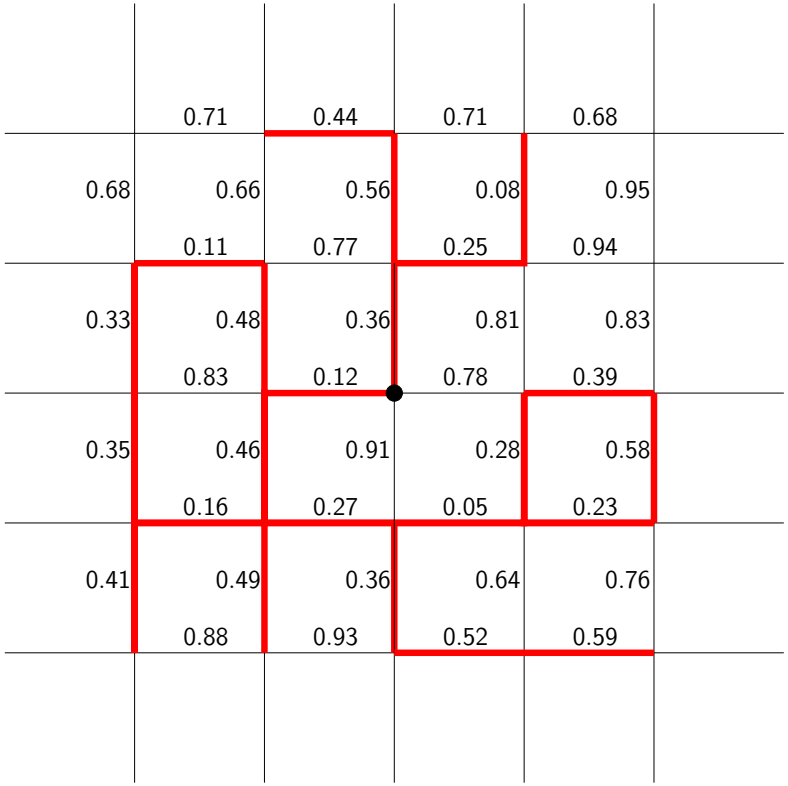

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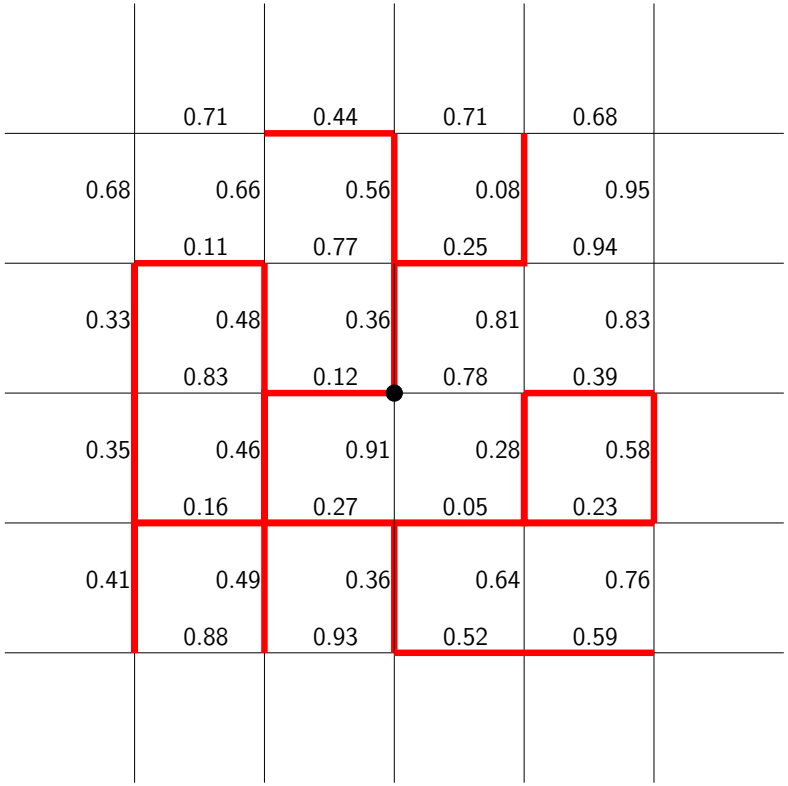

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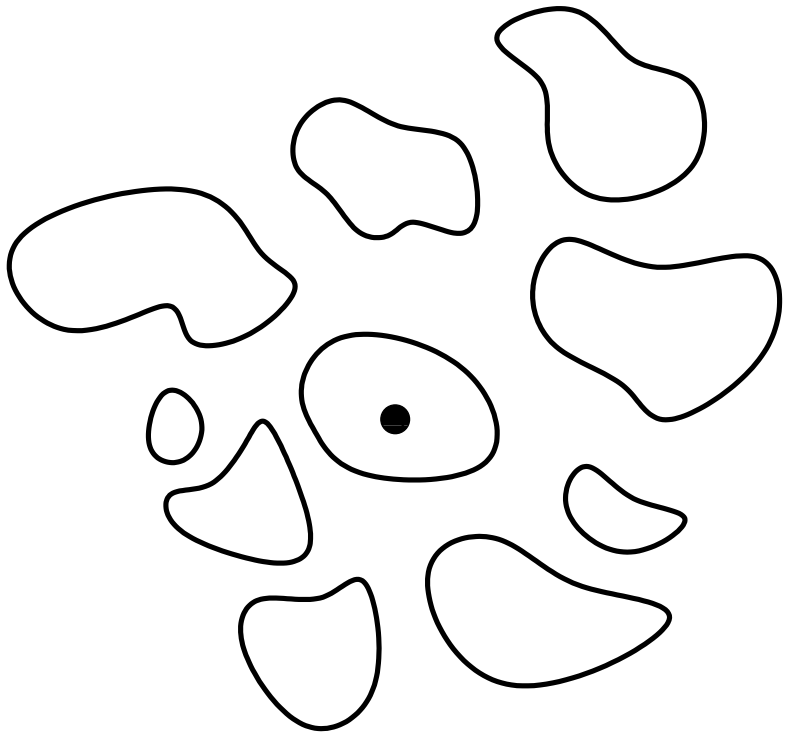

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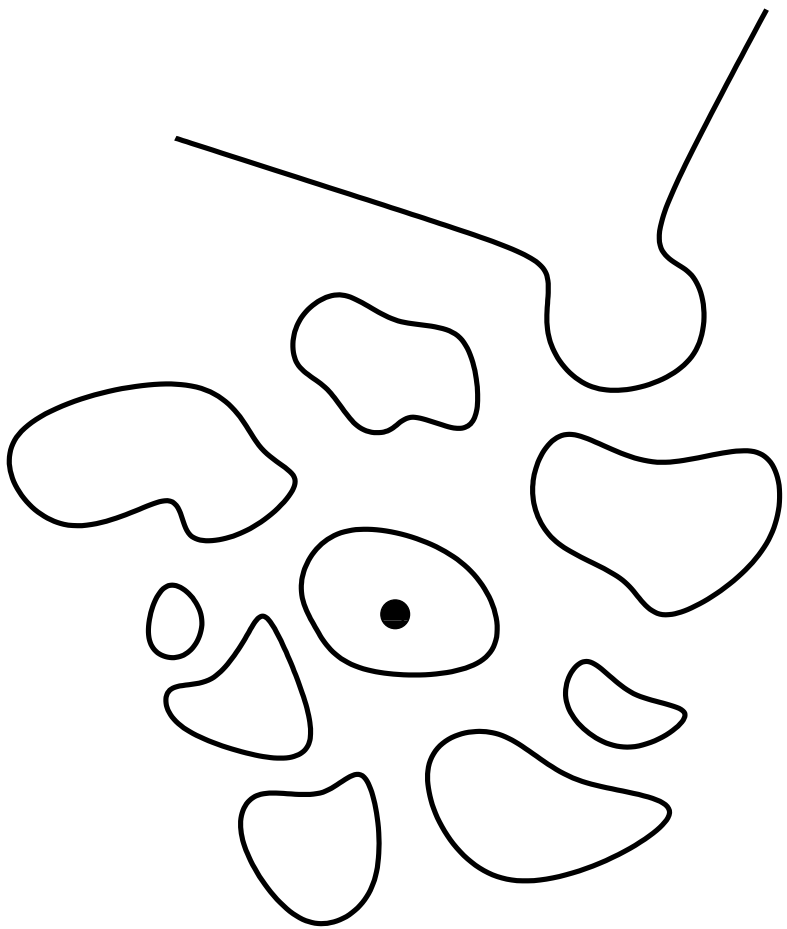

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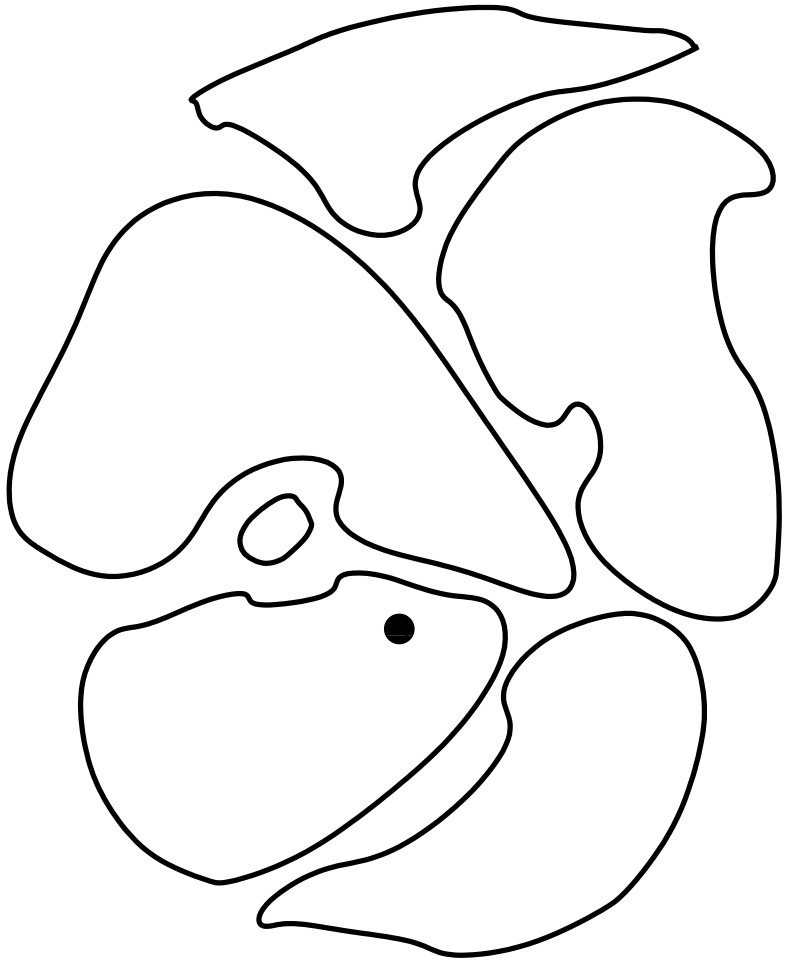
  (Critical case)



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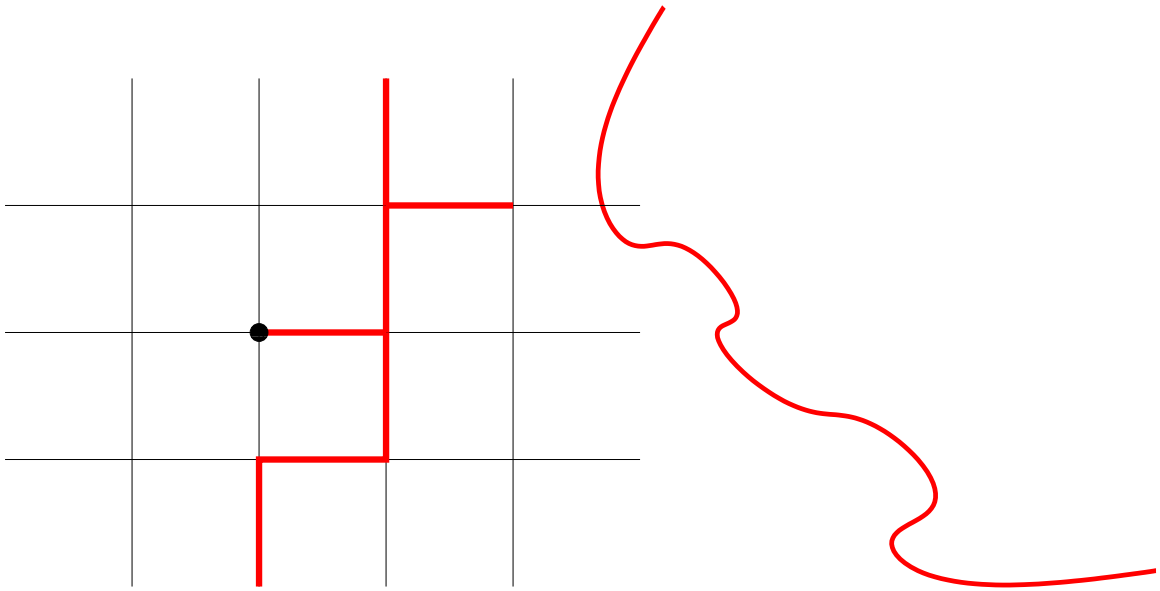
Self-organized criticality



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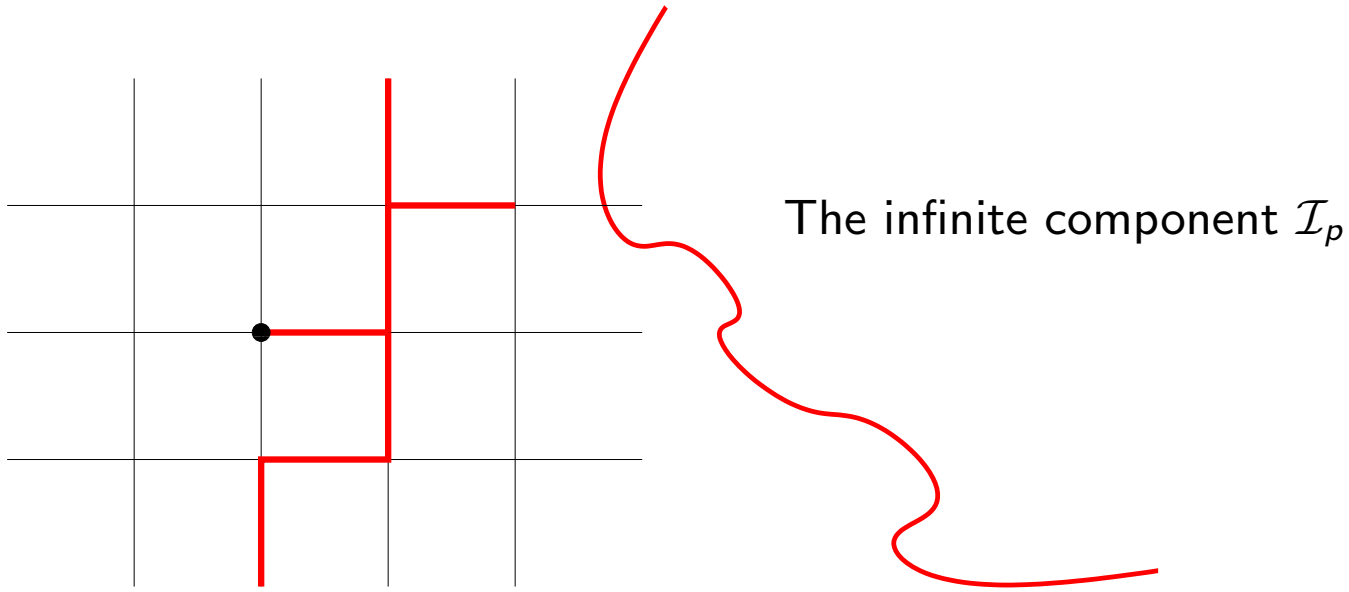

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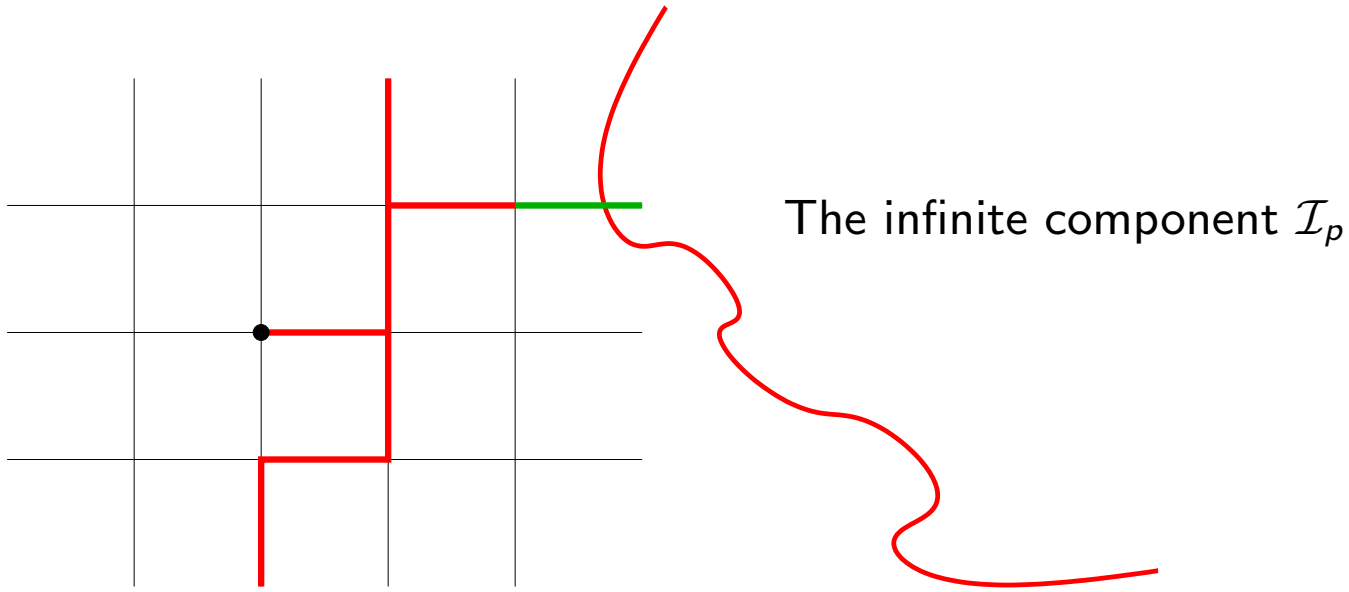

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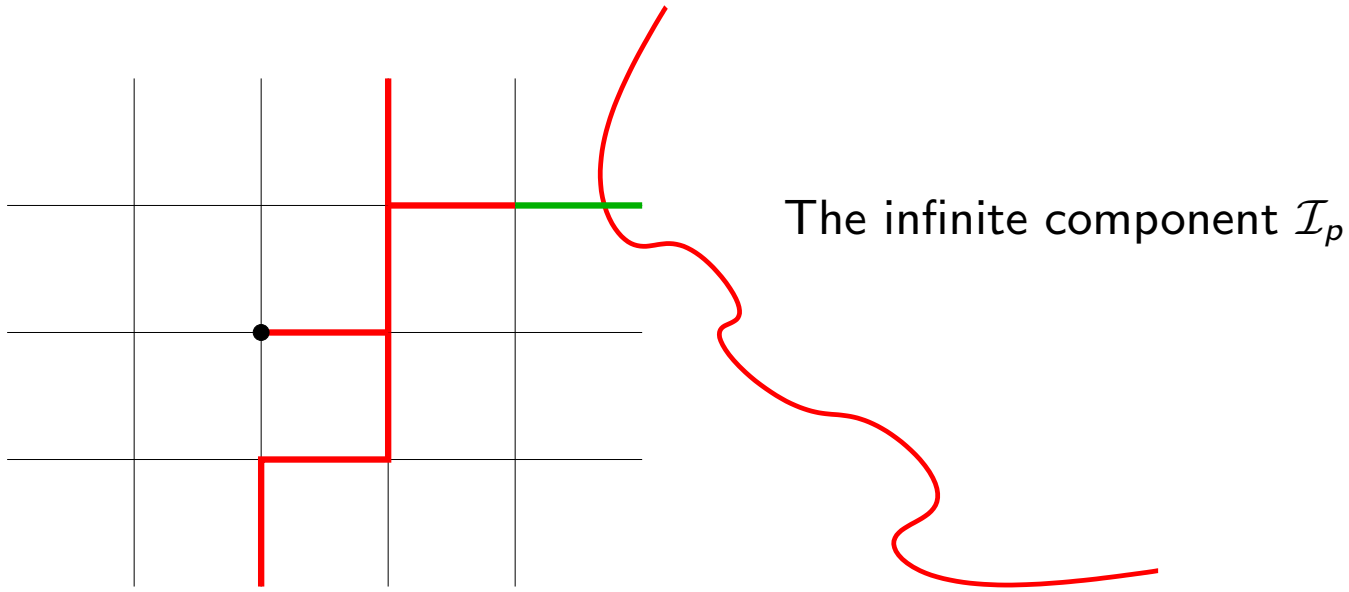

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- After this point, no edge outside of $\mathcal{I}_p$ will ever be explored.

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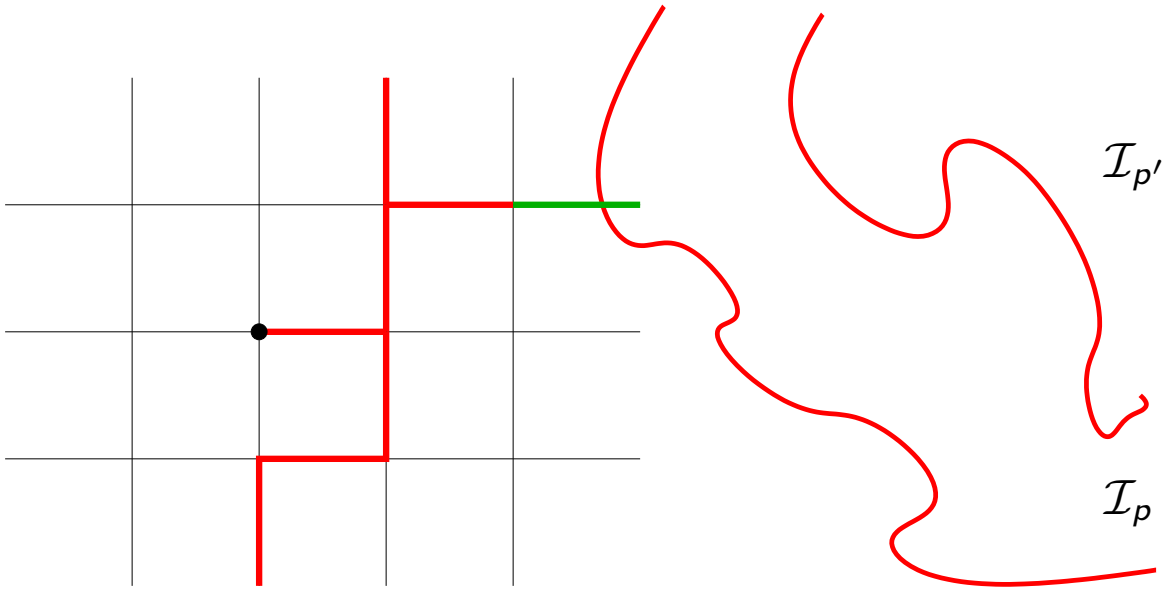

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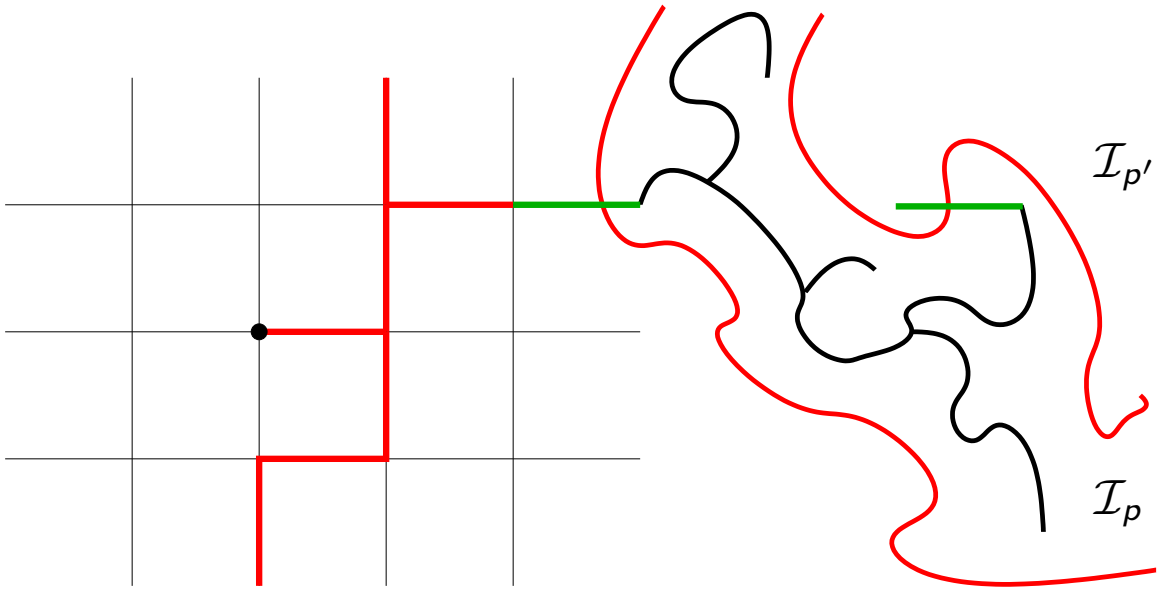

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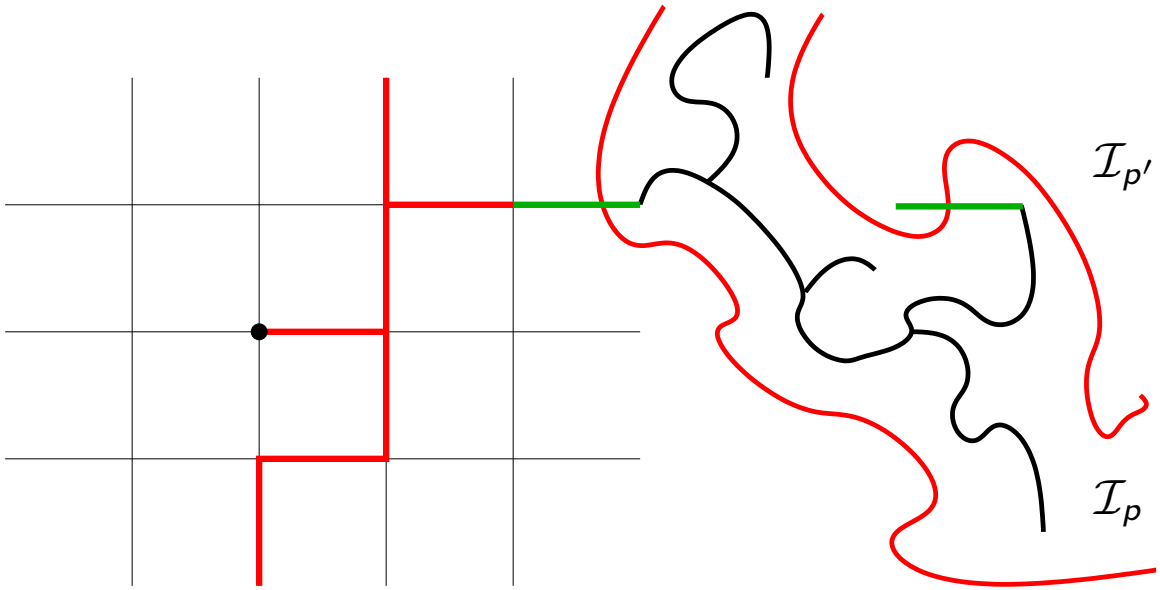

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- Invasion percolation constructs a tree with a unique infinite path that "zooms in on" the near-critical infinite Bernoulli percolation cluster. ("Self-organized criticality")

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Gives one possible approach to proving that there is no percolation at criticality.

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- Thus, for $d > 4$ the effects of "self-interaction" between first and second halves of an n-step self-avoiding walk are negligible.

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- Theorem (Angel et. al. 2006): Invasion percolation on the $2d$-ary tree is 4-dimensional.

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- For $d > 6$, the invasion percolation tree should take its mean-field behaviour and so be a four-dimensional object.
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- Thus, the minimum spanning forest (MSF) should be a tree in dimensions $d < 8$, but have infinitely many components in dimensions $d > 8$.

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- No one can prove anything at all about this, except when $d = 2$.

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- For $d = 2$, Alexander & Molchanov (1994) used duality to show that the MSF is indeed a tree on the square, triangular, and hexagonal lattices.

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- These facts allow Alexander & Molchanov's proof to be extended to two-dimensional Euclidean MSF.
- (In fact, in two dimensions the Euclidean MSF was already known to be a tree (Alexander 1995).)

Another connection with critical percolation

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This gives another possible approach to (re-)proving that $\theta(p_c) = 0$ for $d > 8$.

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Heuristic: structure of critical percolation on $\mathbb{Z}^d$ well-modelled by a critical Galton-Watson tree with branching distribution $\mathrm{Bin}(2d, 1/2d)$.

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- Understanding the geometry of the cycles is key to understanding the structure of the MST.
- Another mean-field candidate: the critical random graph $G_{n,1/n}$.
- This is locally well-modelled by a critical Binomial Galton-Watson tree, but has sparse additional long-range connections (cycles).

Next lecture

Explain the global limiting structure of the critical random graph $G_{n,1/n}$.

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If you want to work on this with me, find me at tea!