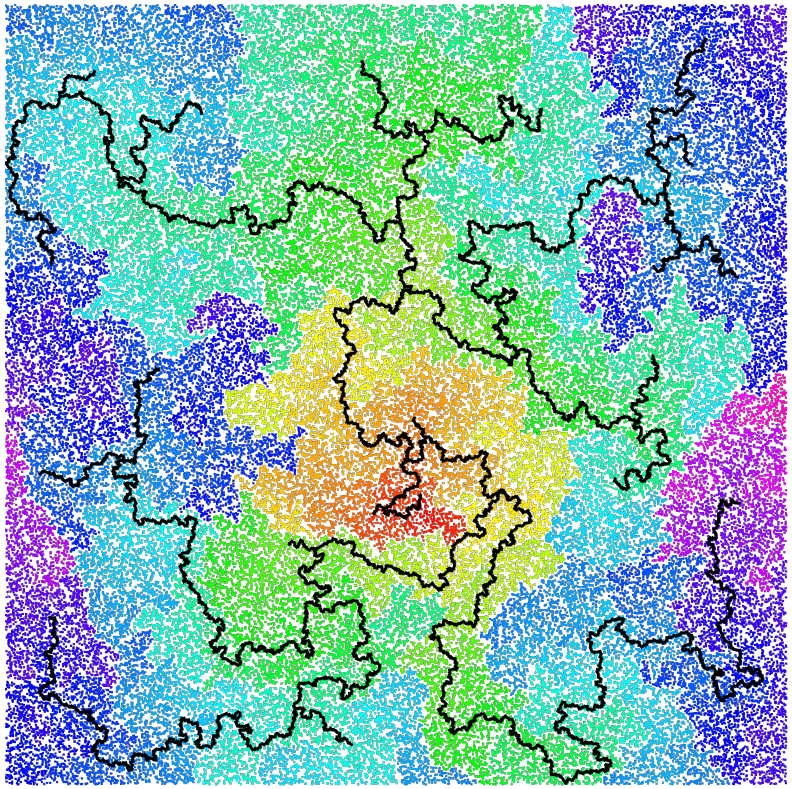
The critical random graph



L. Addario-Berry

Random Structures and Dynamics

Oxford

April 11-14, 2011

Parts are joint with Nicolas Broutin, Luc Devroye, Christina Goldschmidt, Svante Janson, Grégory Miermont

Limit for a component of $G_{n,1/n}$

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Now have an asymptotic description of the structure of each "macroscopic" component of the critical random graph.

Brownian motion with drift?

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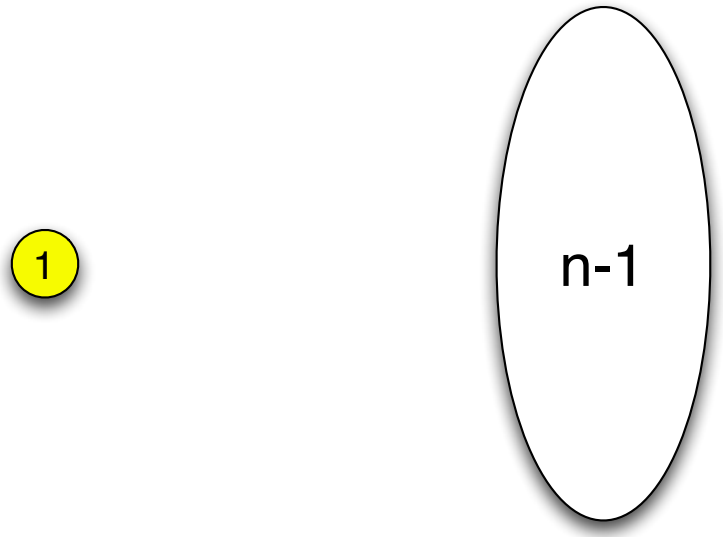

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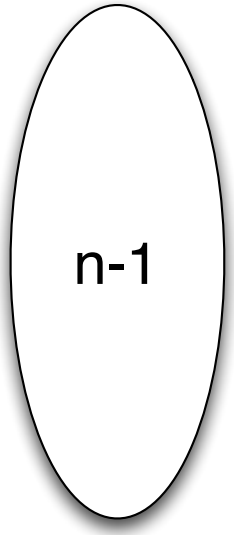

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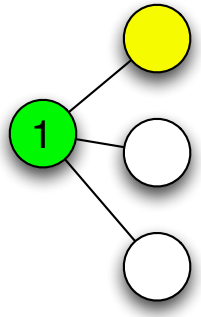

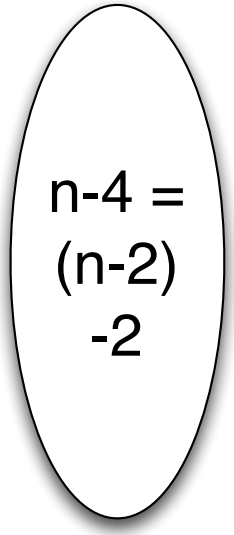

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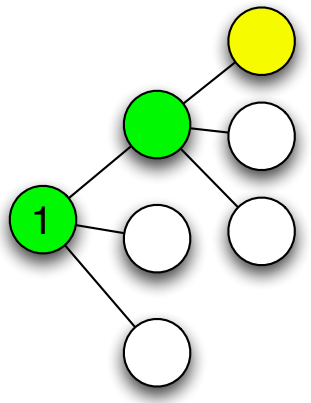

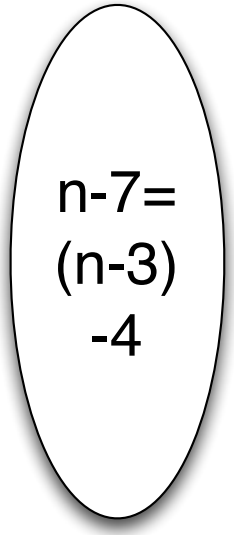

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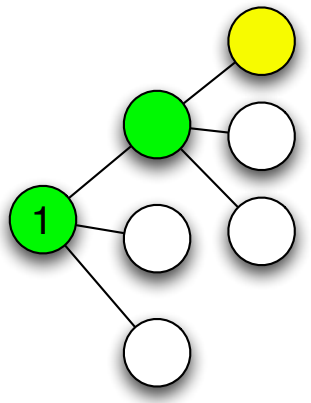

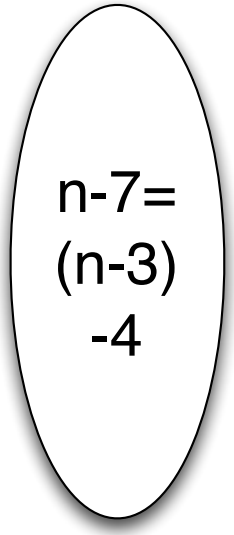

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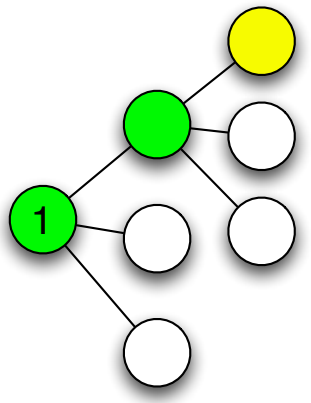

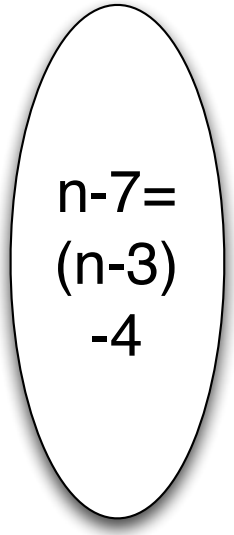

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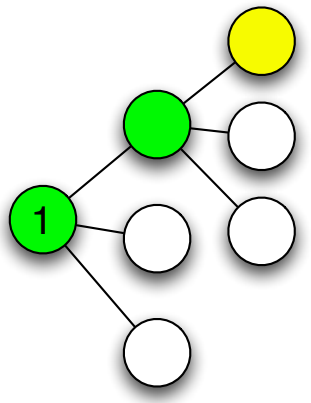

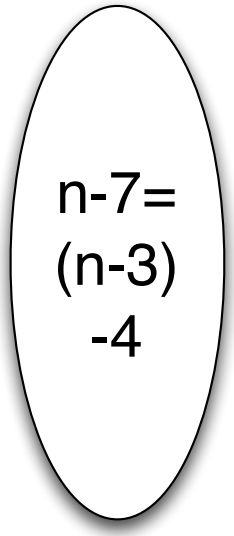

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- The first component is explored when discrete walk hits -1; start again from 0 and explore another component! This is why reflection at running infimum.

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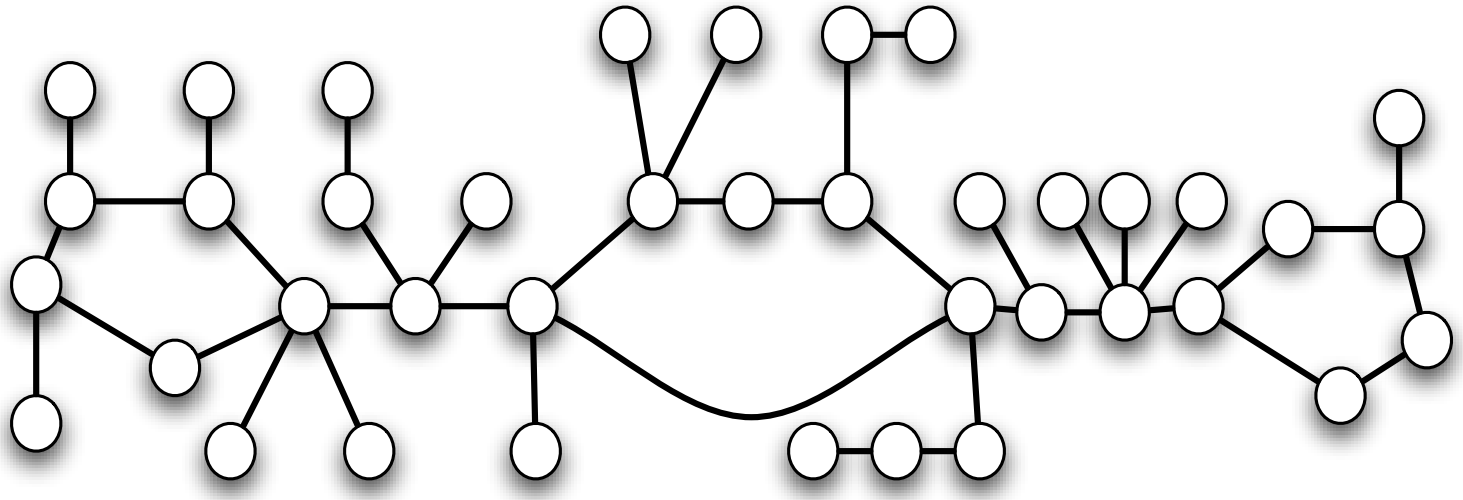
- First need to explain another description of a component of $G_{n,1/n}$, so we can better see the geometry (cycle structure).

The cycle structure of a connected graph.



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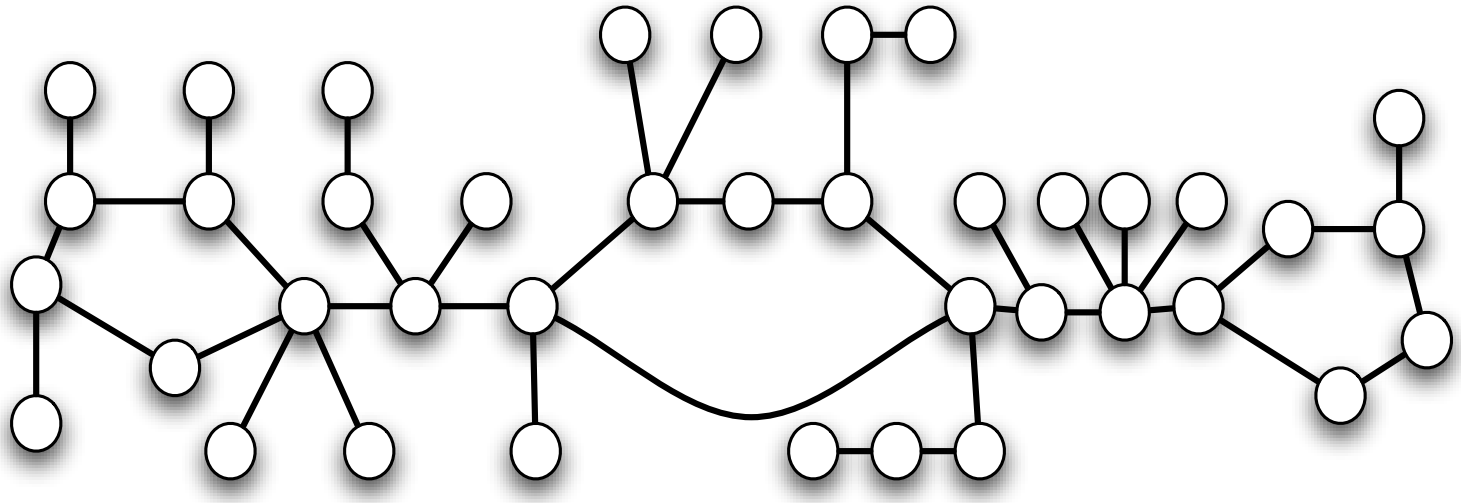

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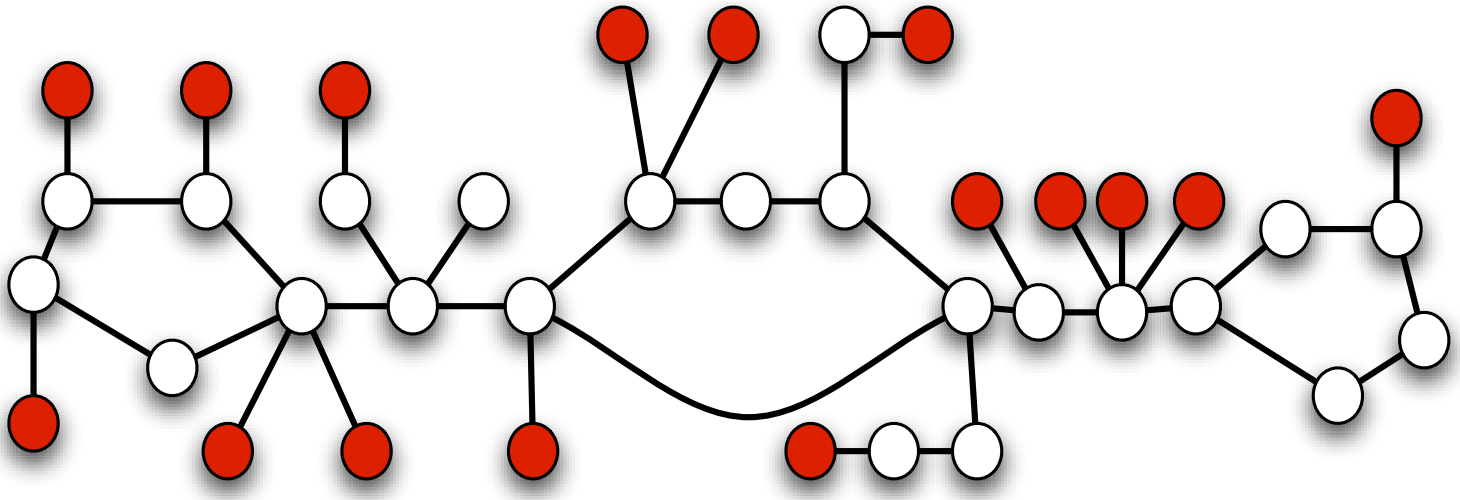

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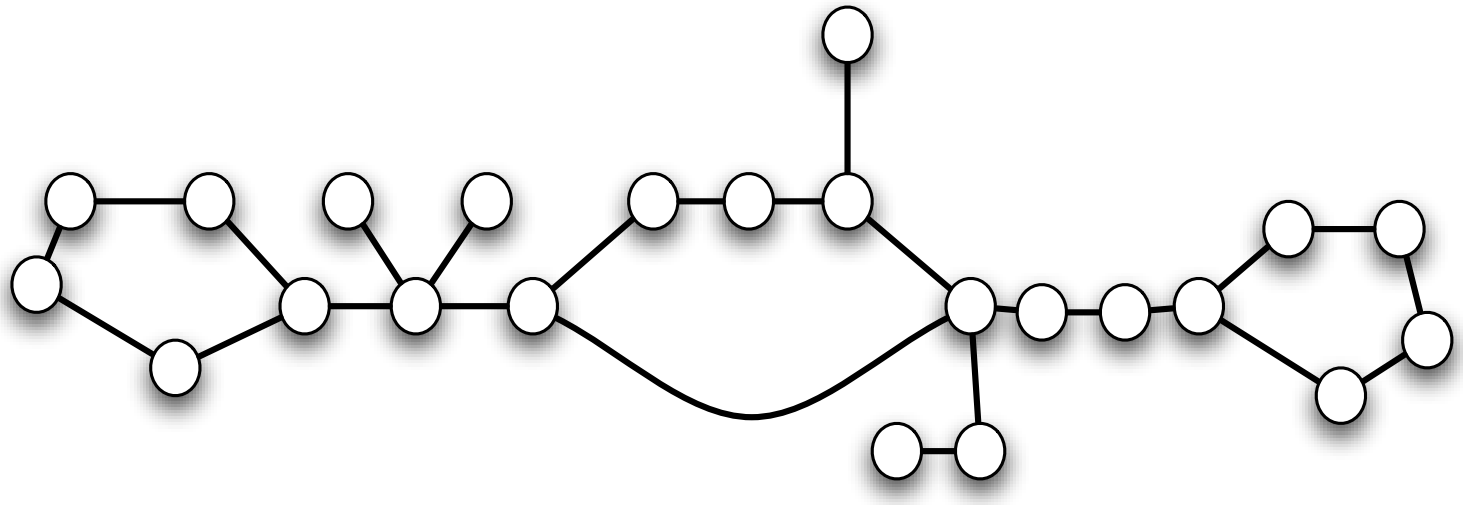

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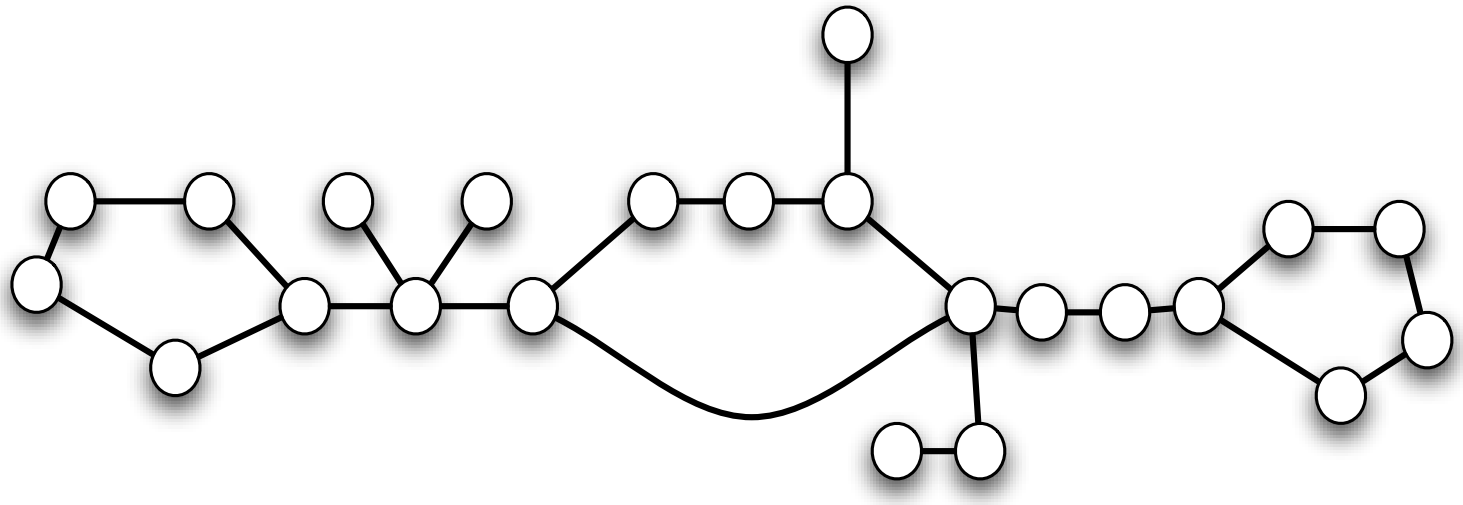

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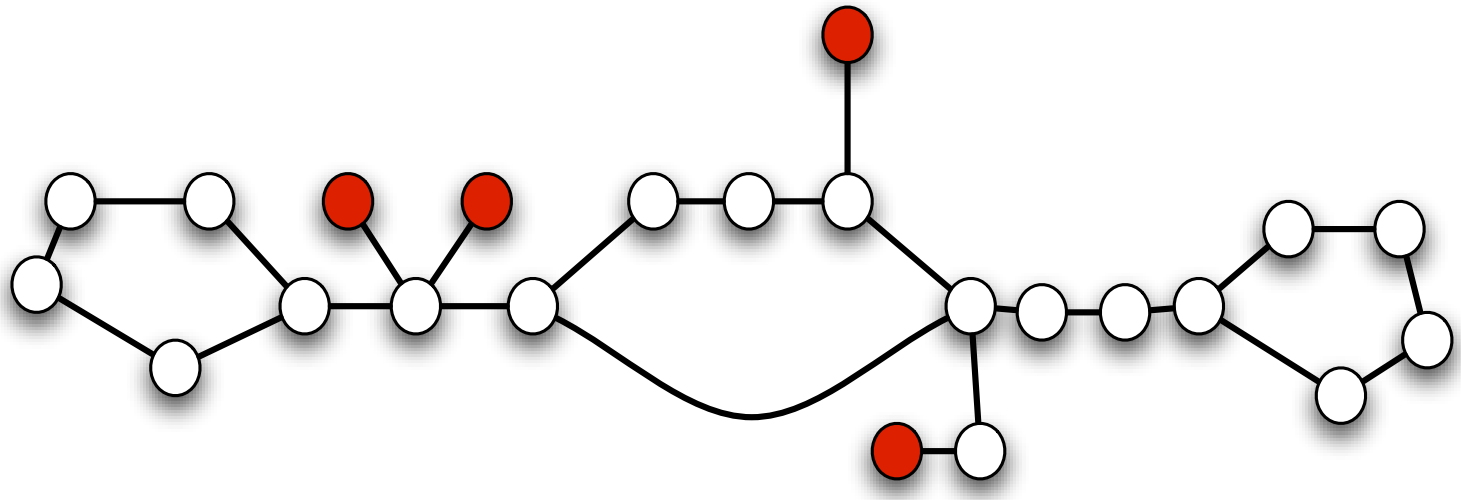

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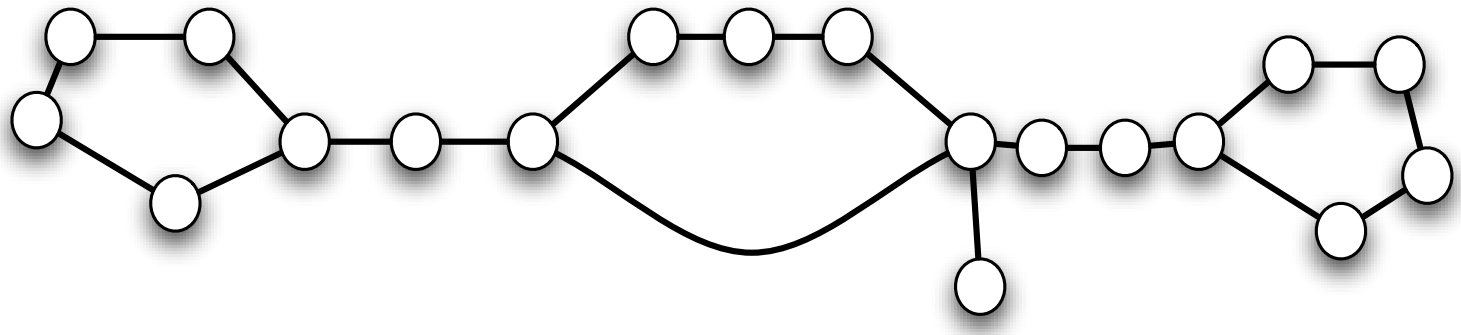

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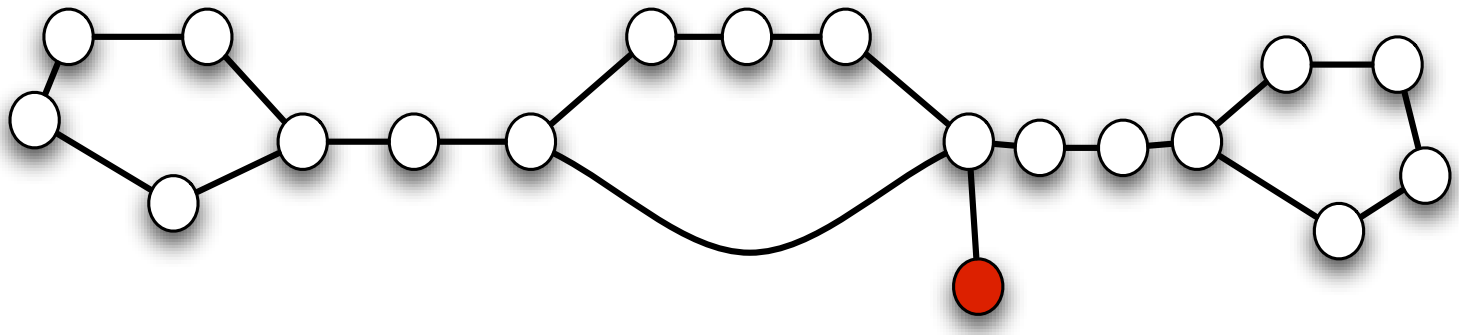

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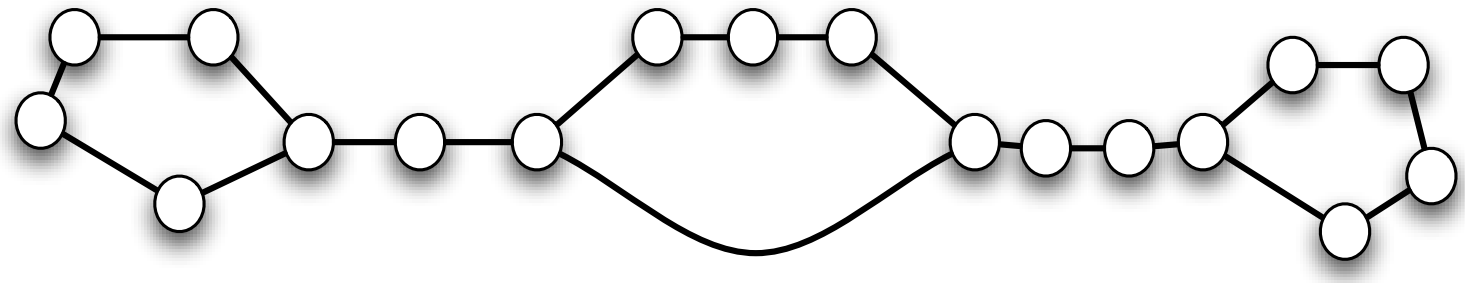

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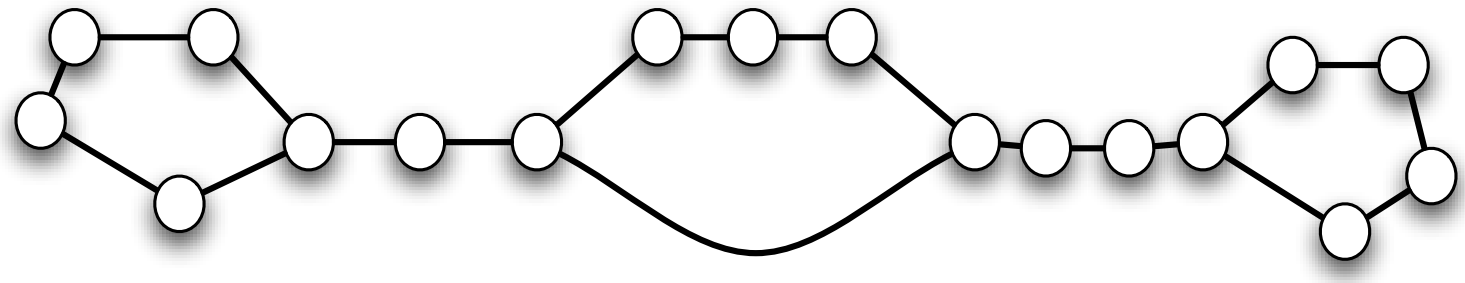

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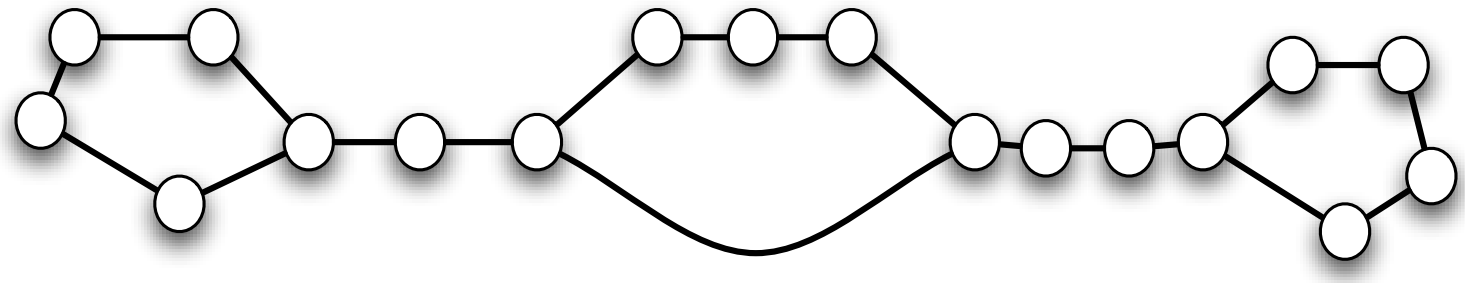

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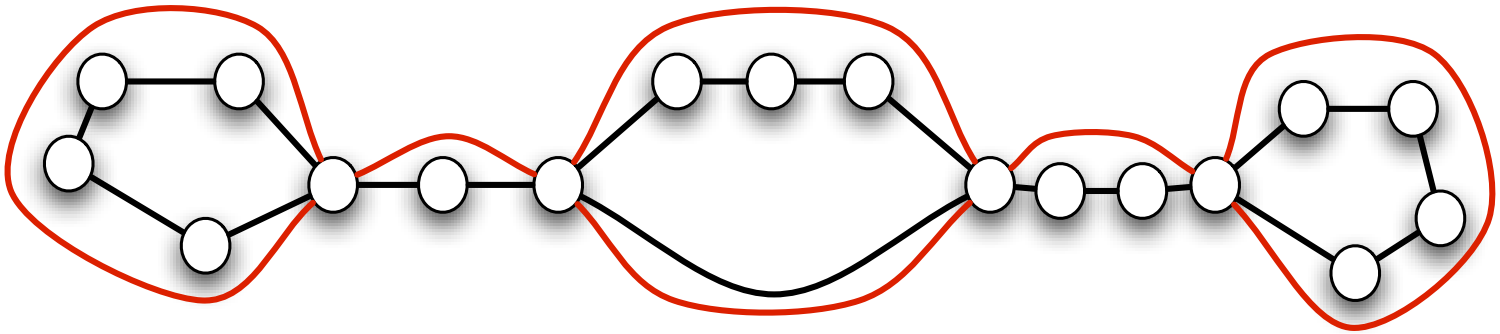

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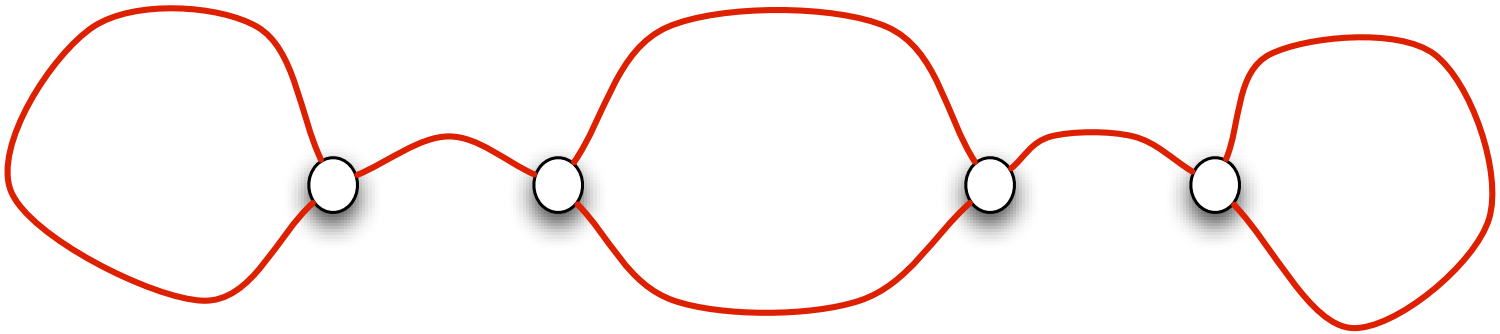

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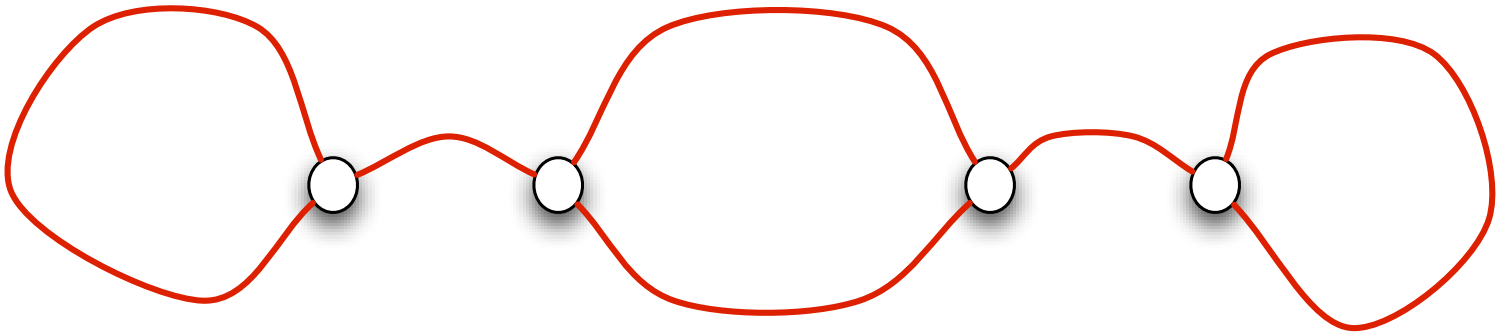

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- Result after suppressing degree-2 vertices is called the *kernel*.

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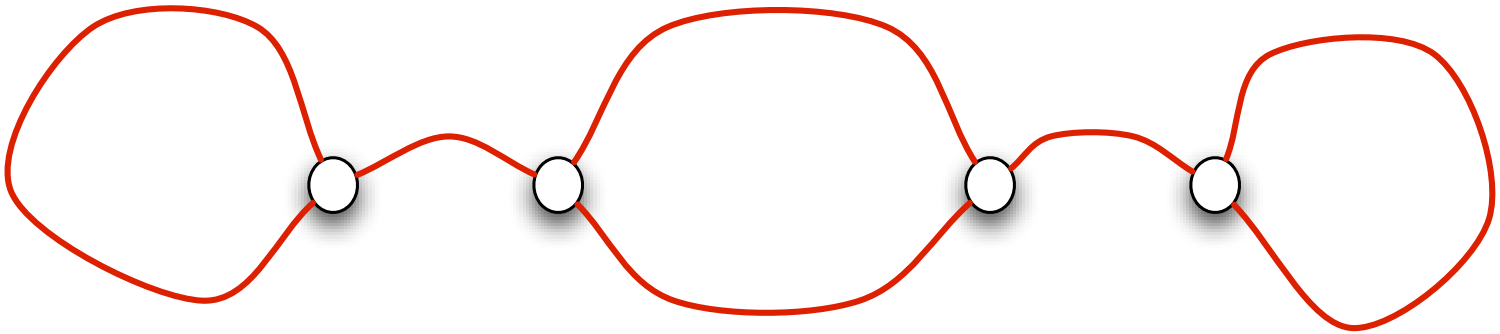

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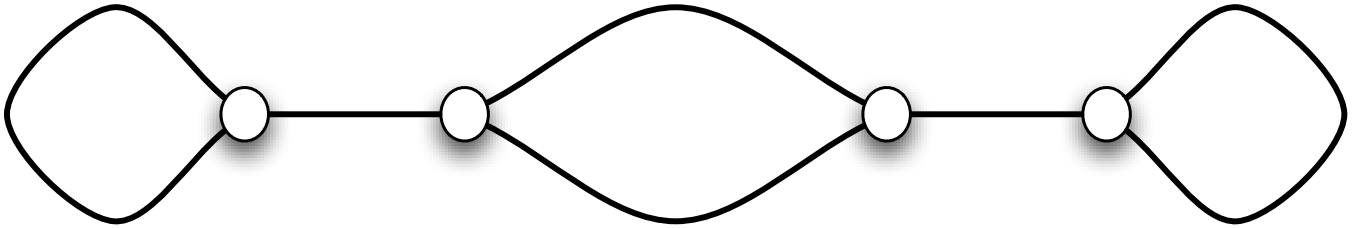
- Kernel has same surplus as the original graph.

The kernel



- The kernel is a connected, unlabelled multigraph with minimum degree 3.

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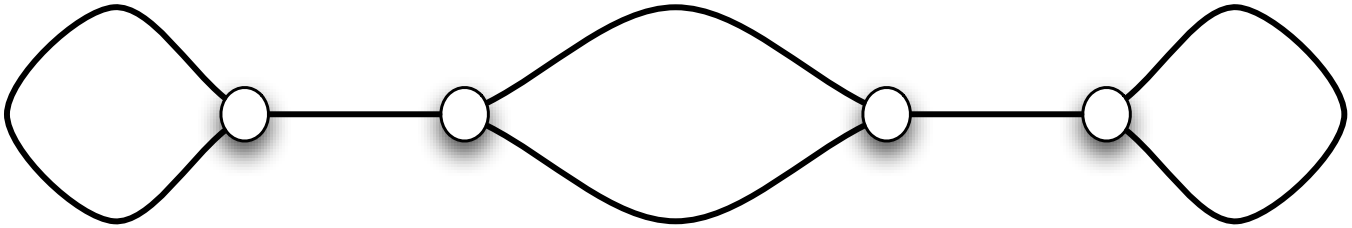

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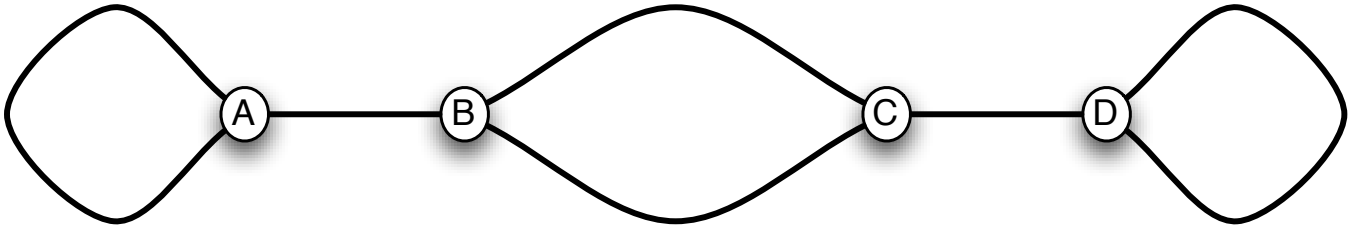

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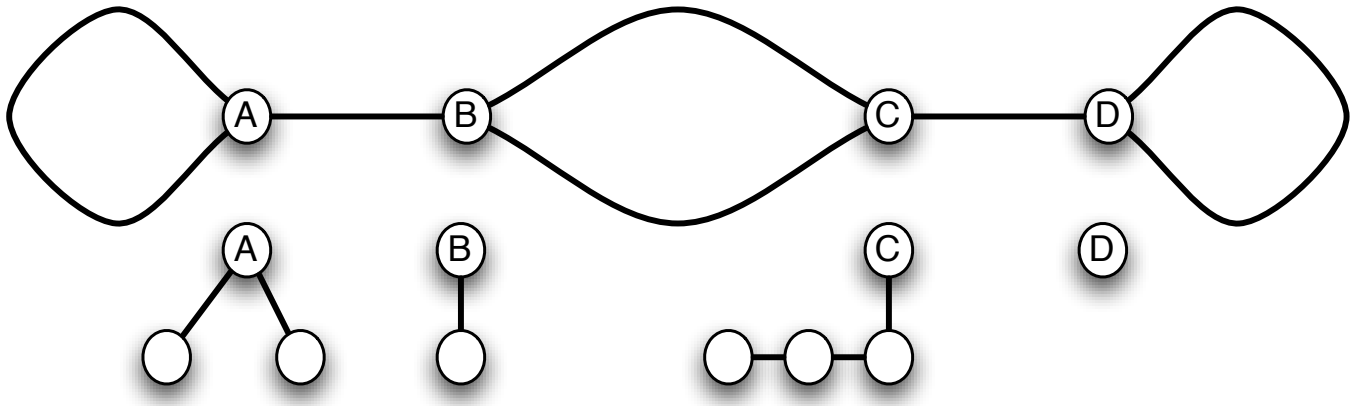

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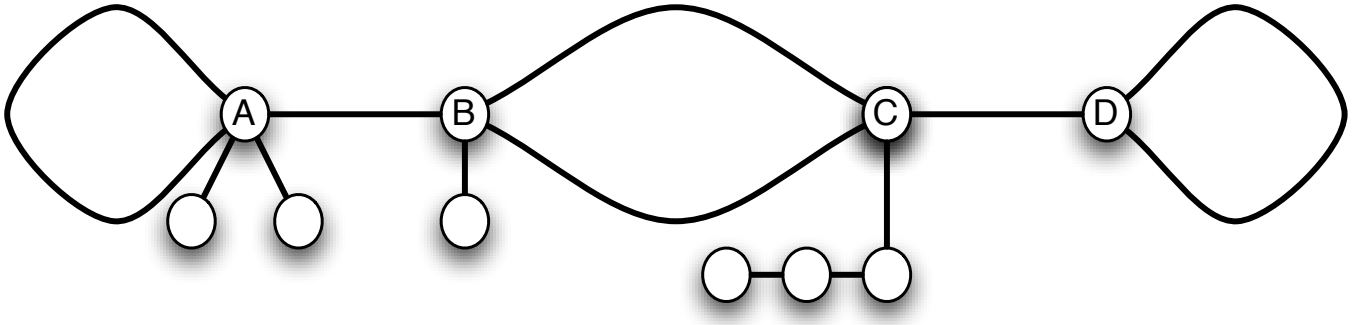

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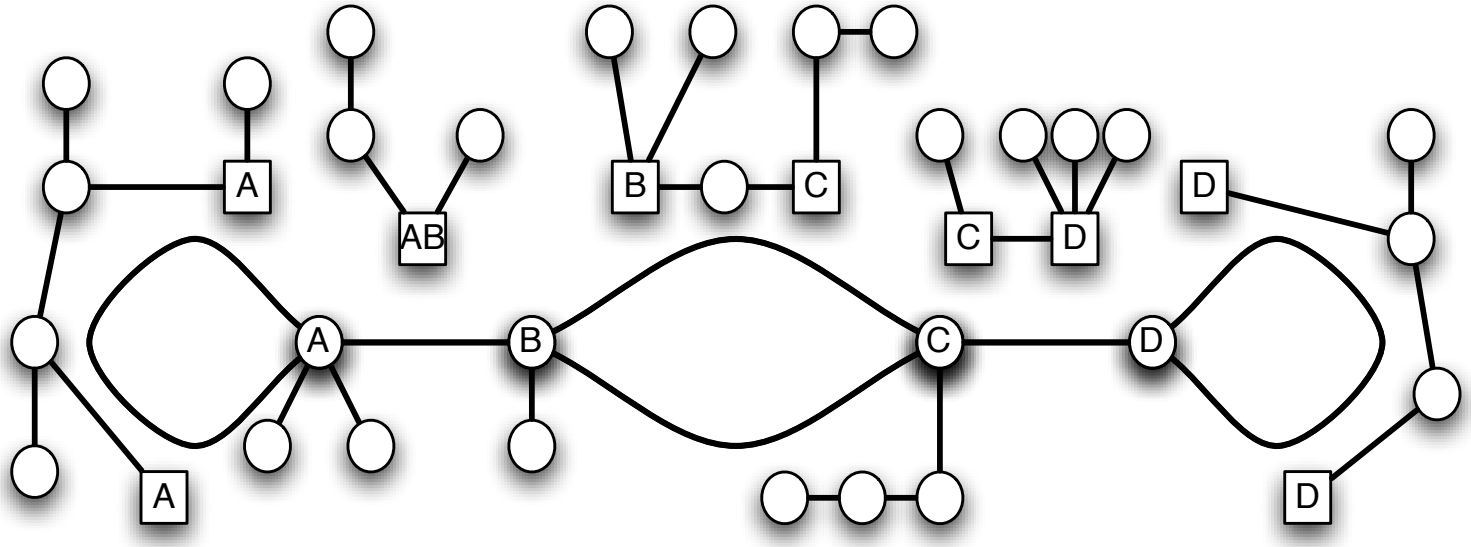

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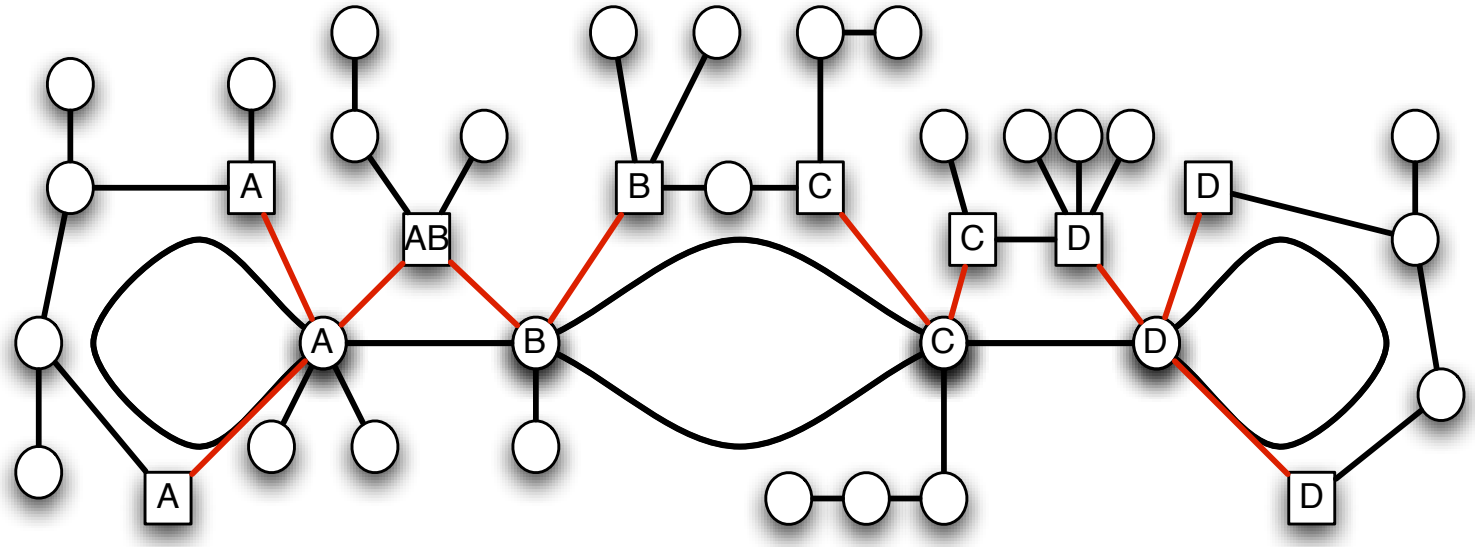

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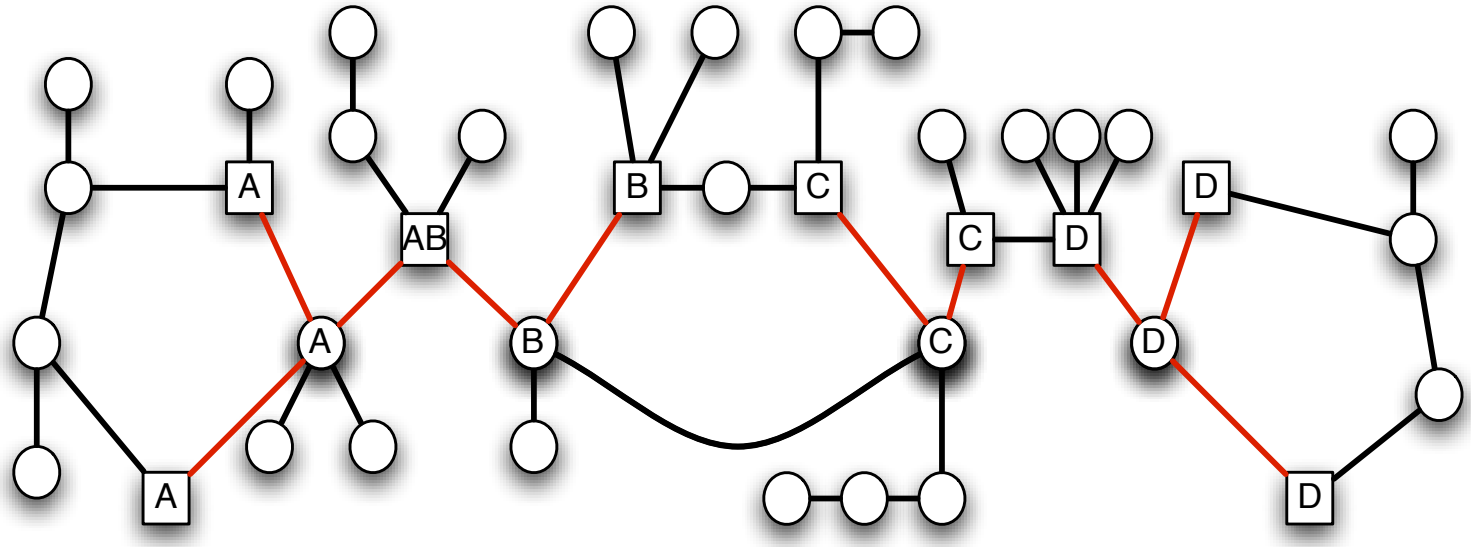

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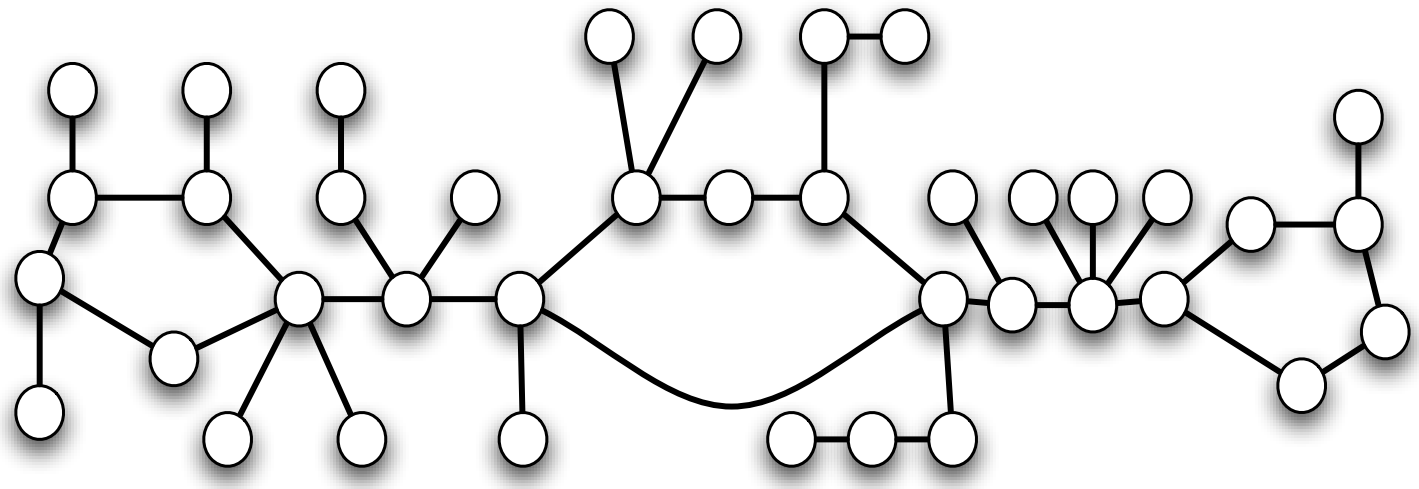

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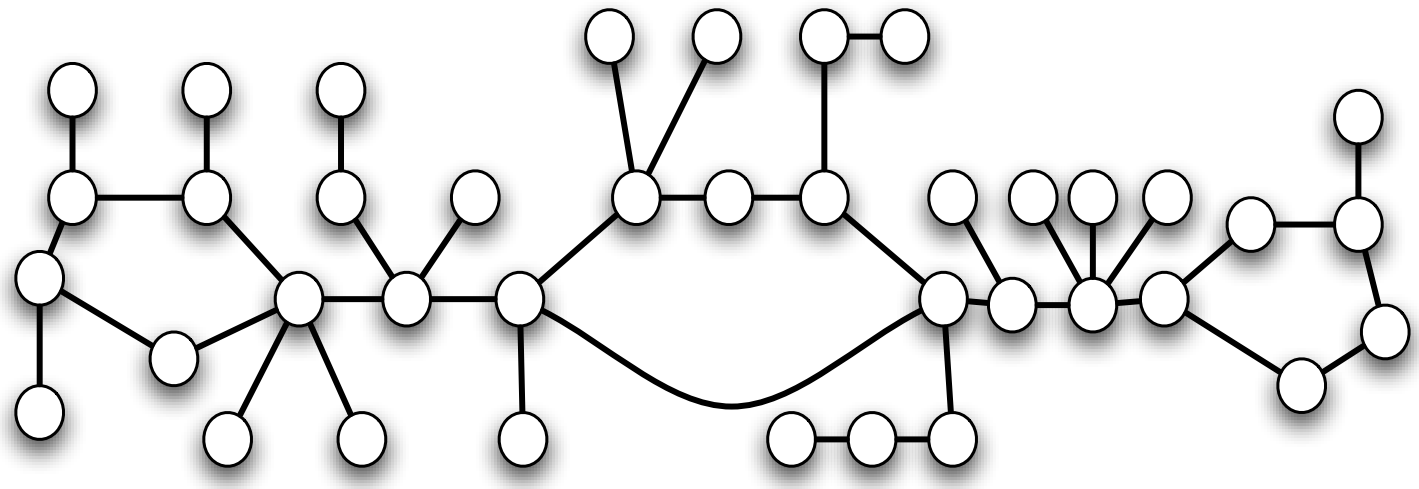

- The kernel is a connected, unlabelled multigraph with minimum degree 3.
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- Symmetry factors: $1/2$ for loops, $1/k!$ for each k-multiedge.

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- The vector $(|T_1|/m, \ldots, |T_{3(s-1)}|/m)$ has distributional limit Dirichlet$(1/2, \ldots, 1/2)$.

The limiting picture

In the limit, then, a random component can be built by gluing *randomly rescaled Brownian CRTs* along the edges of a random kernel.

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- In other words, core path lengths are just given by splittings of an interval by uniforms.

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Repeat until all cycles are broken.

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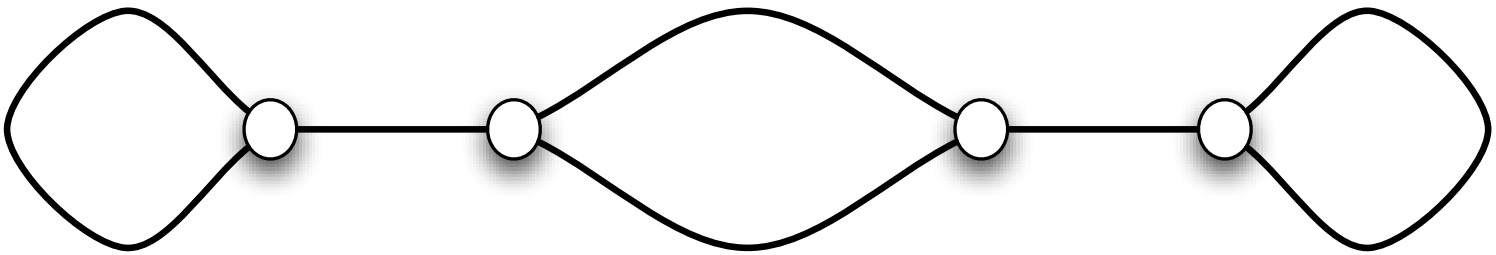

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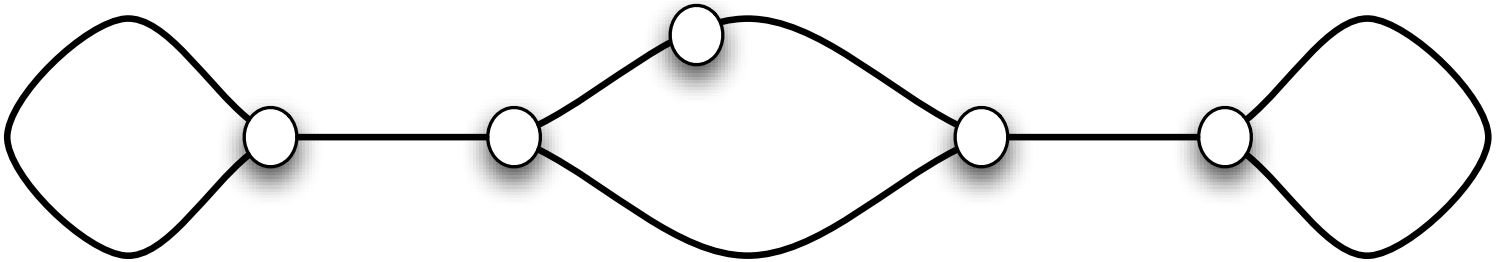

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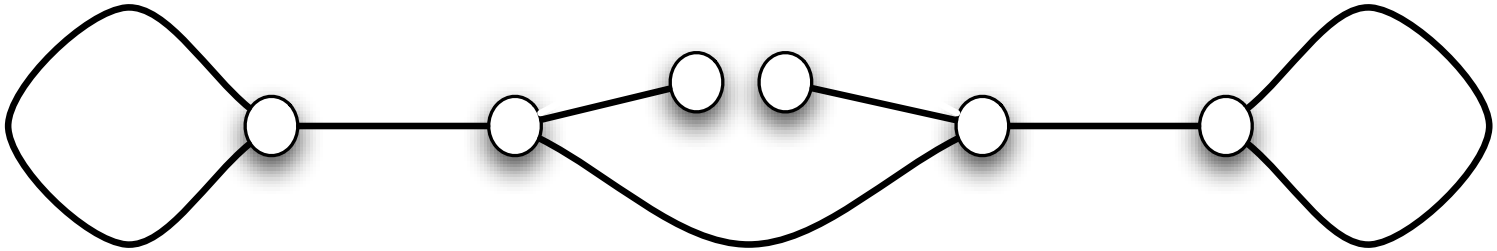

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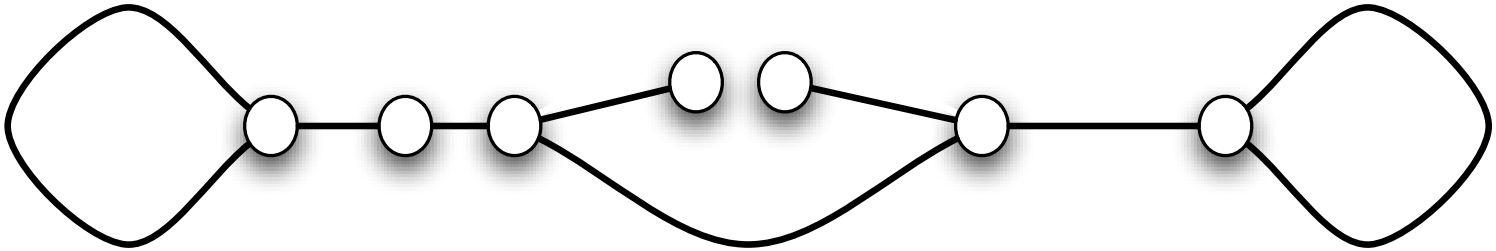

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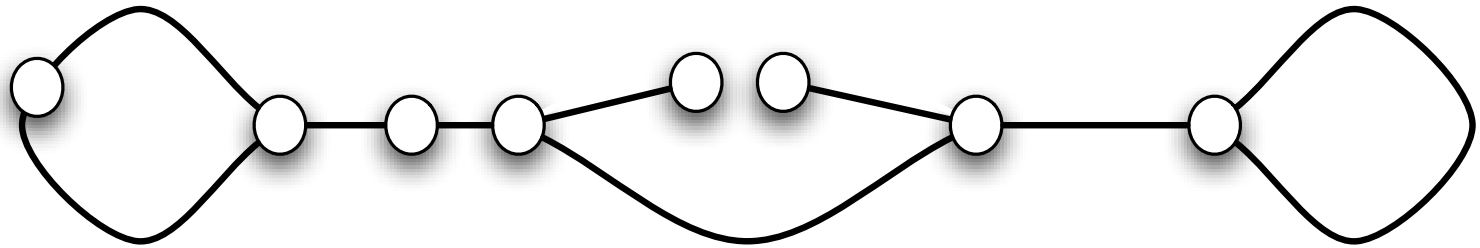

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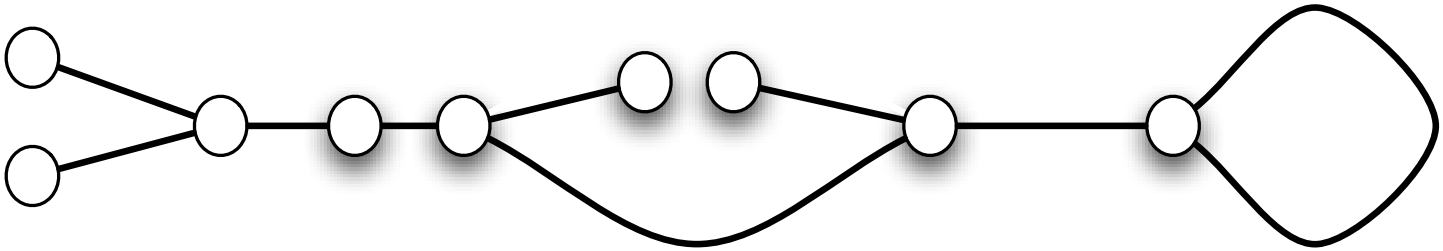

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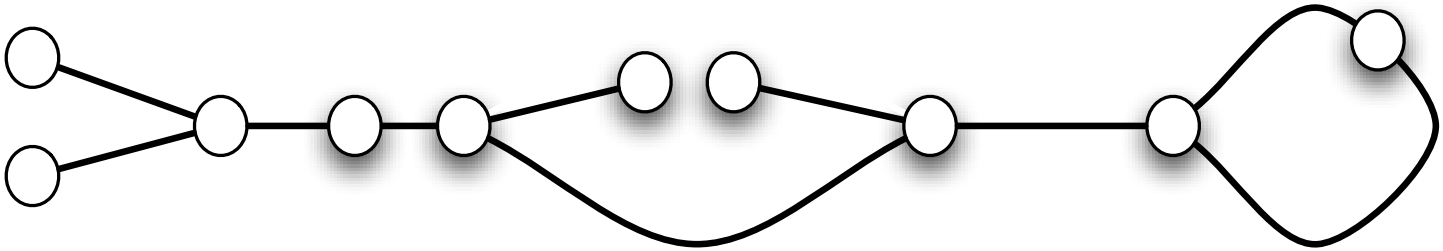

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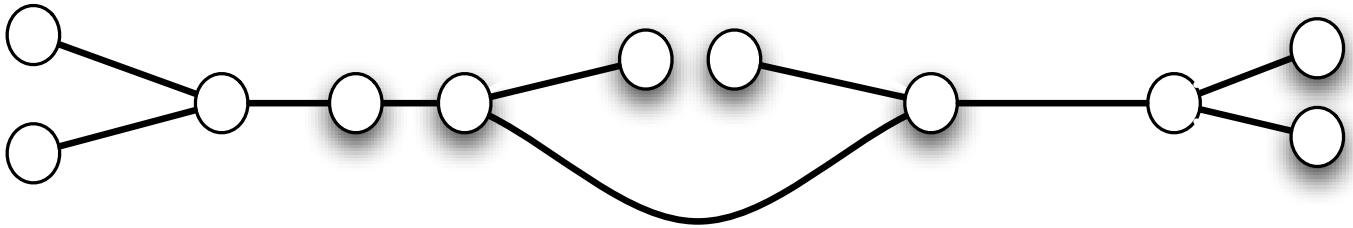

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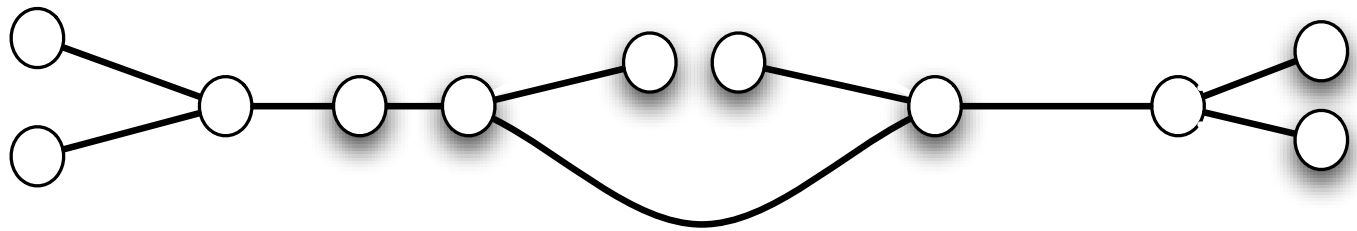

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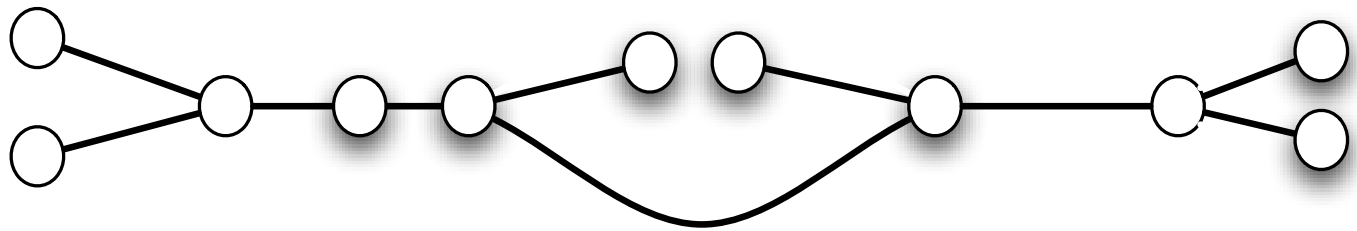

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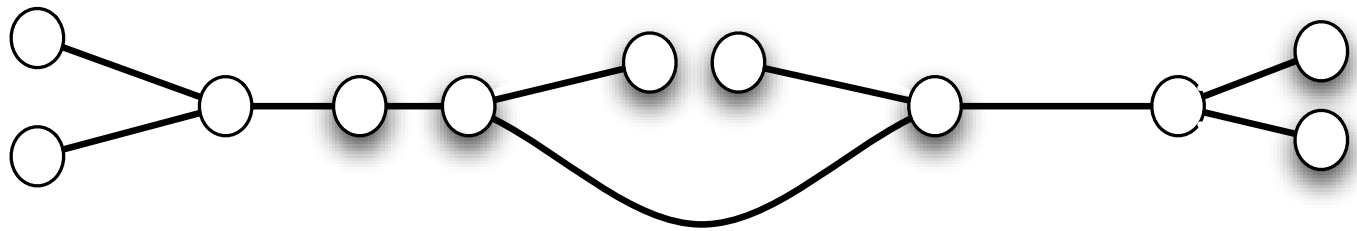

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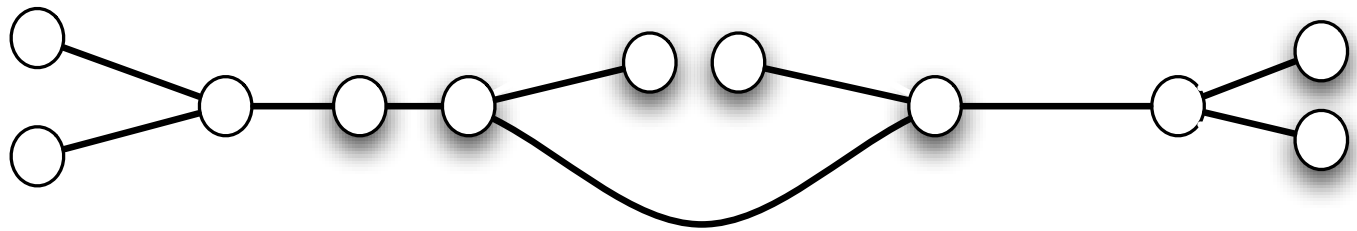

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In the $s \to \infty$ limit, it must have height of order $s^{1/3}$, because in the $s \to \infty$ limit it encodes the structure of the MST of K_n, and this is known to have order $n^{1/3}$.

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- The largest has mass tending to infinity but its diameter stays bounded.
- The limiting object is the distributional GH limit of the MST of K_n (with distances rescaled by $n^{1/3}$). (*)


COMPLETELY
DIFFERENT

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- One model we think we can handle: random graphs with a given degree sequence.

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- How to generate such a graph?

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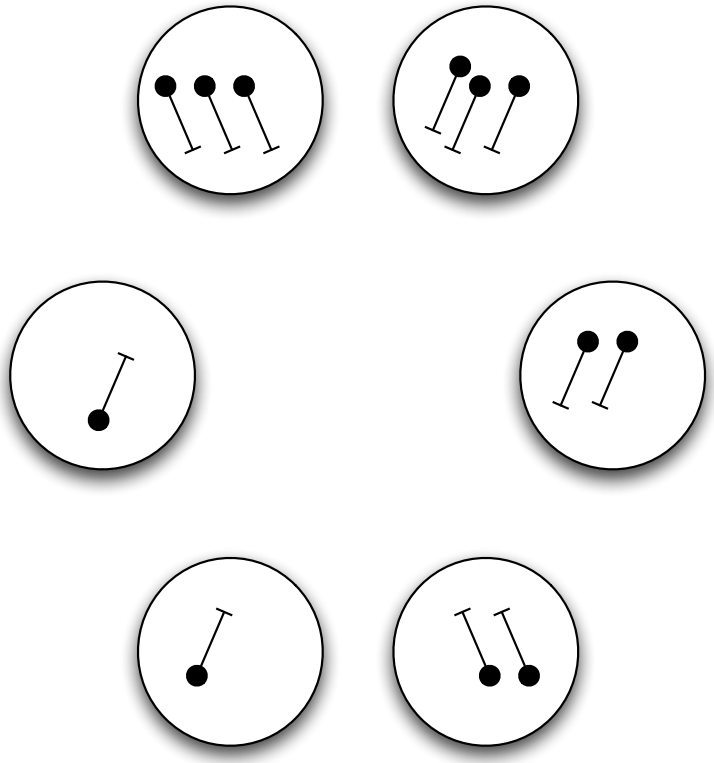

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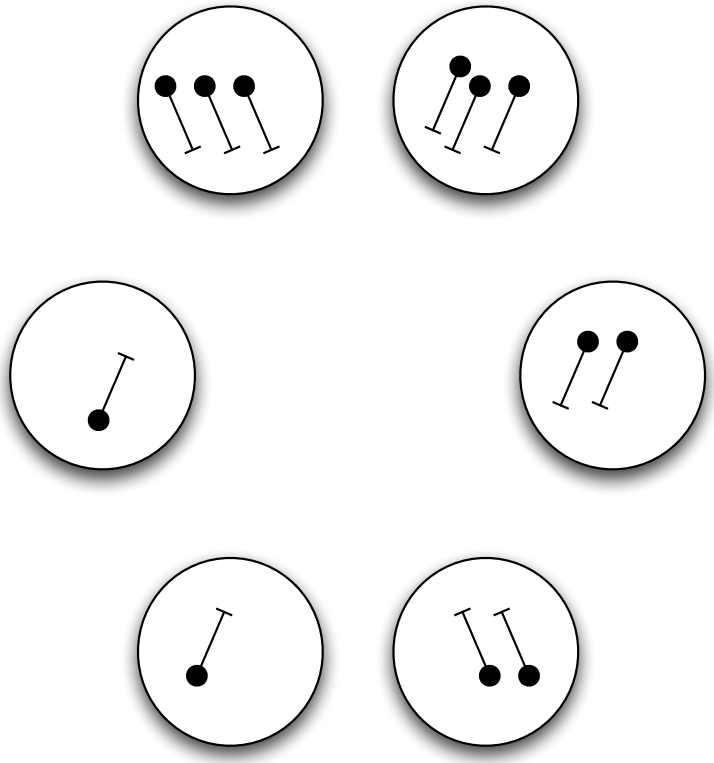

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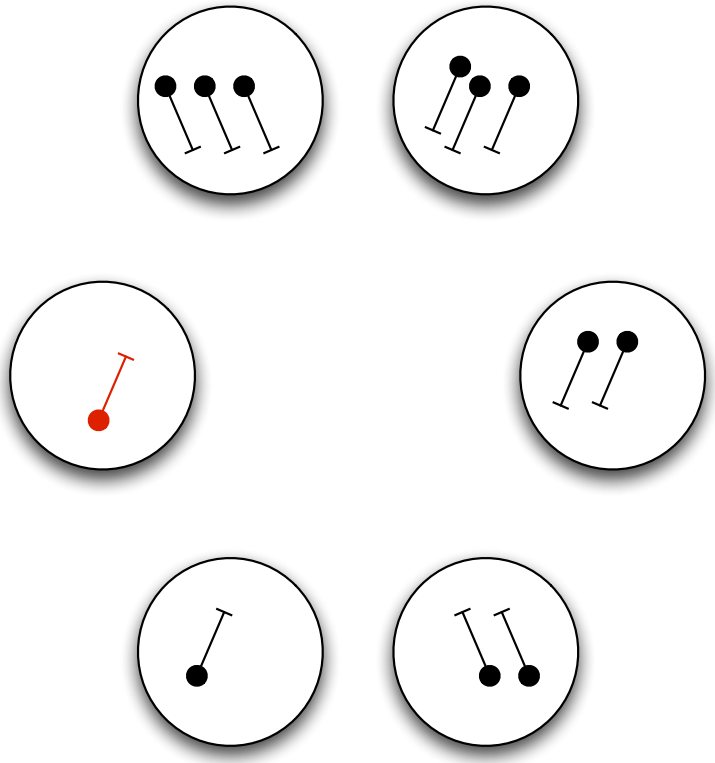

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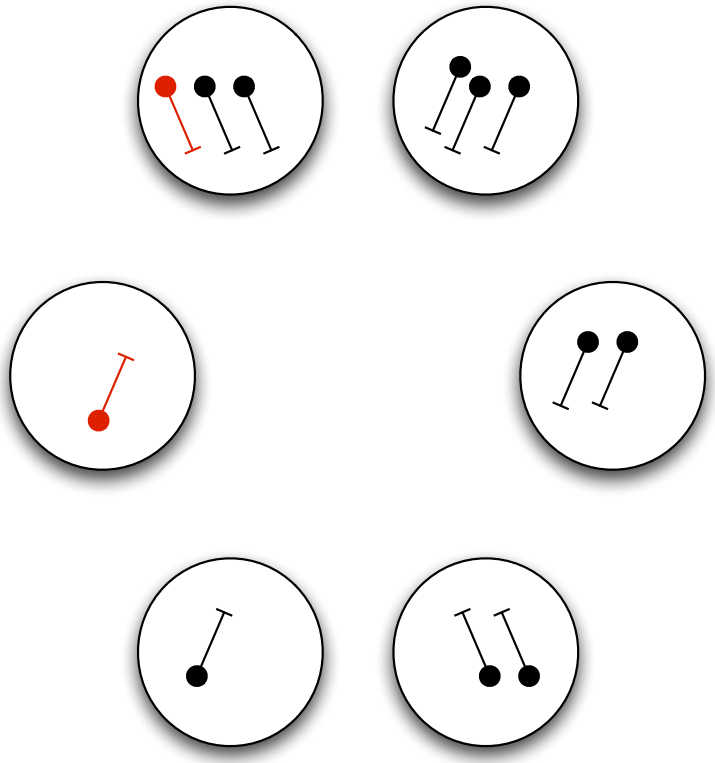

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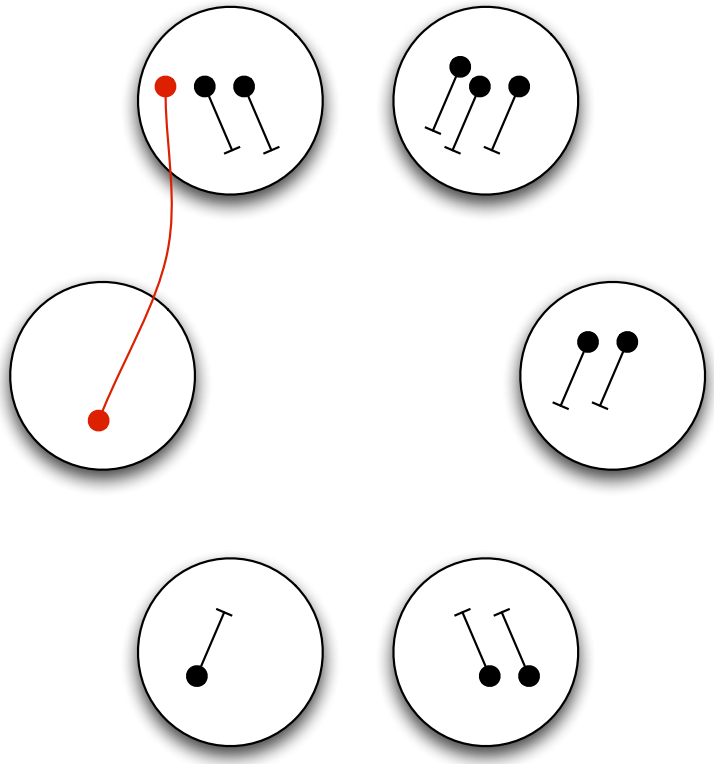

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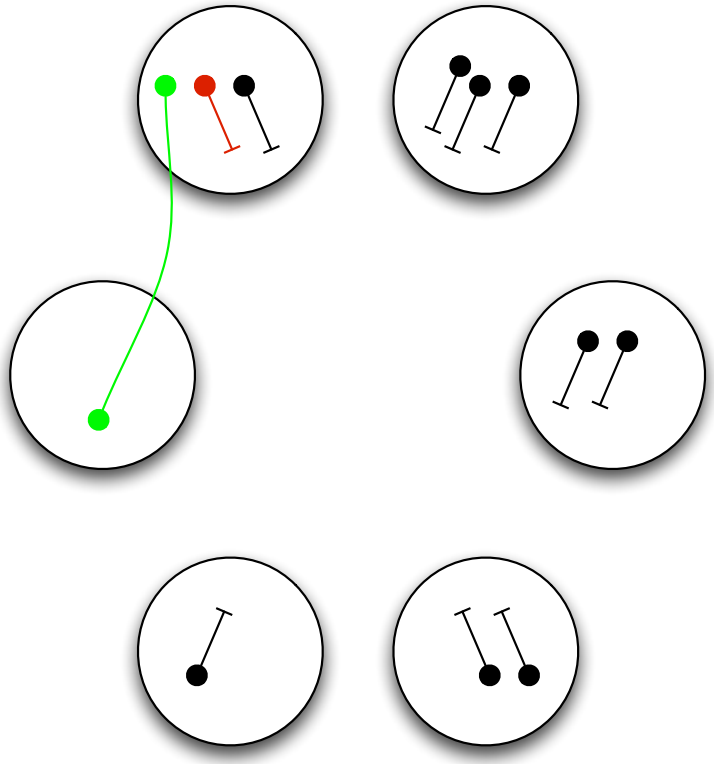

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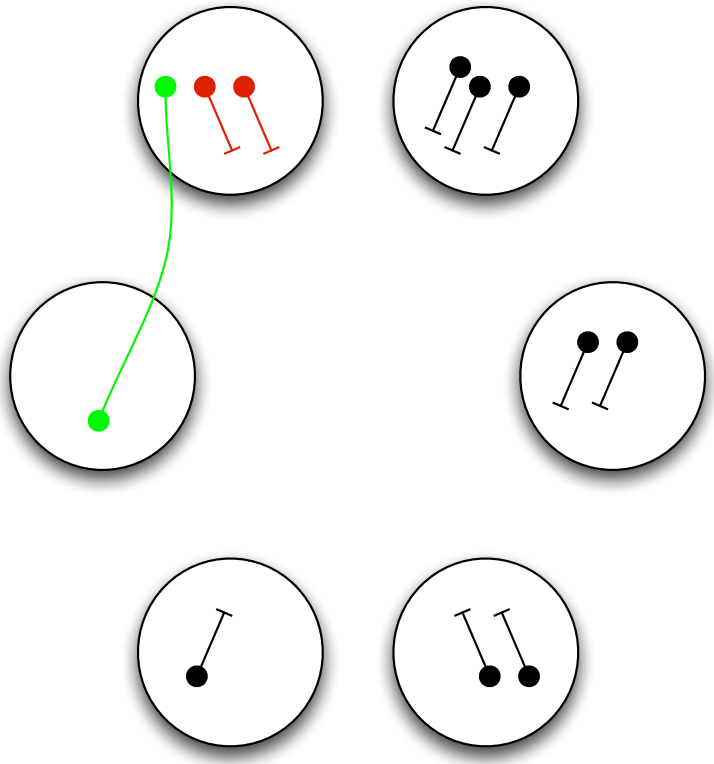

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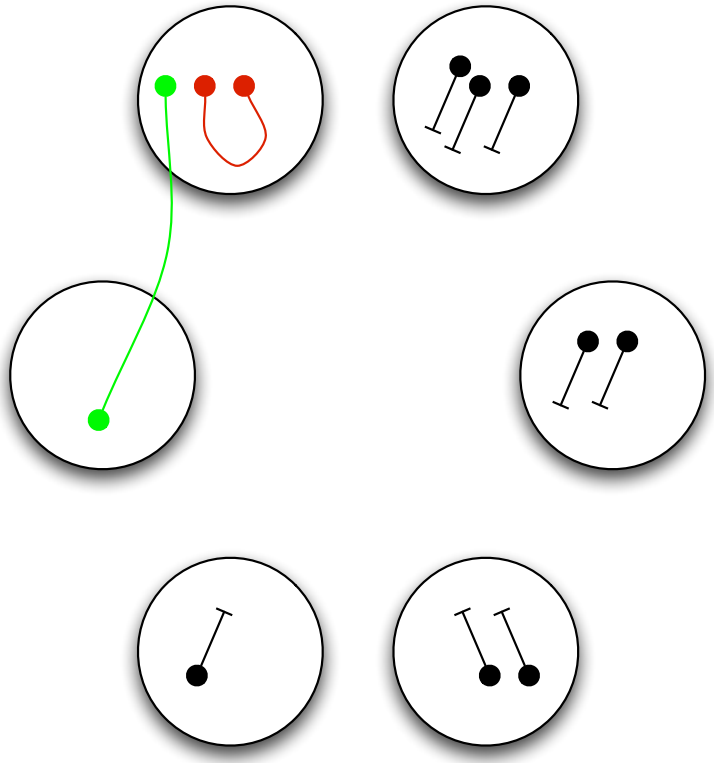

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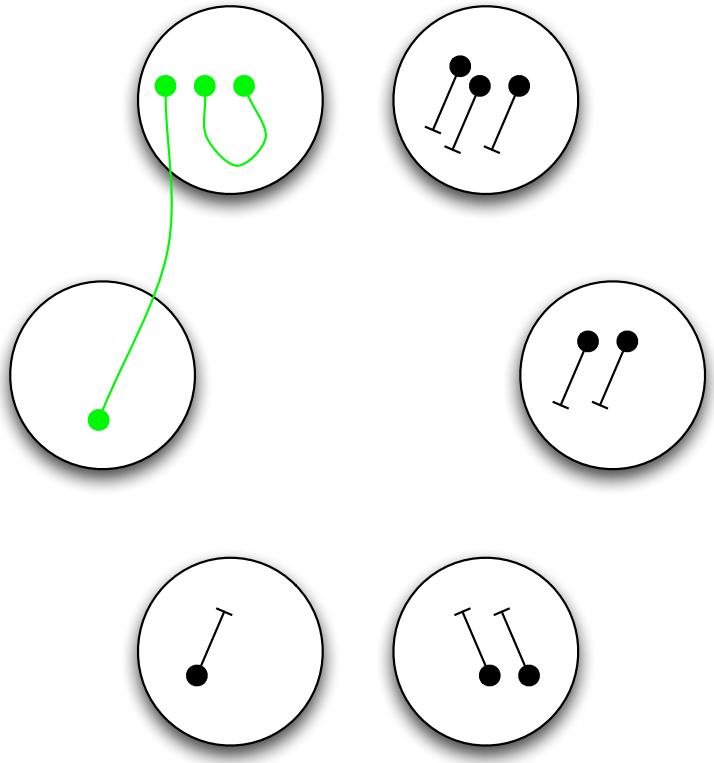

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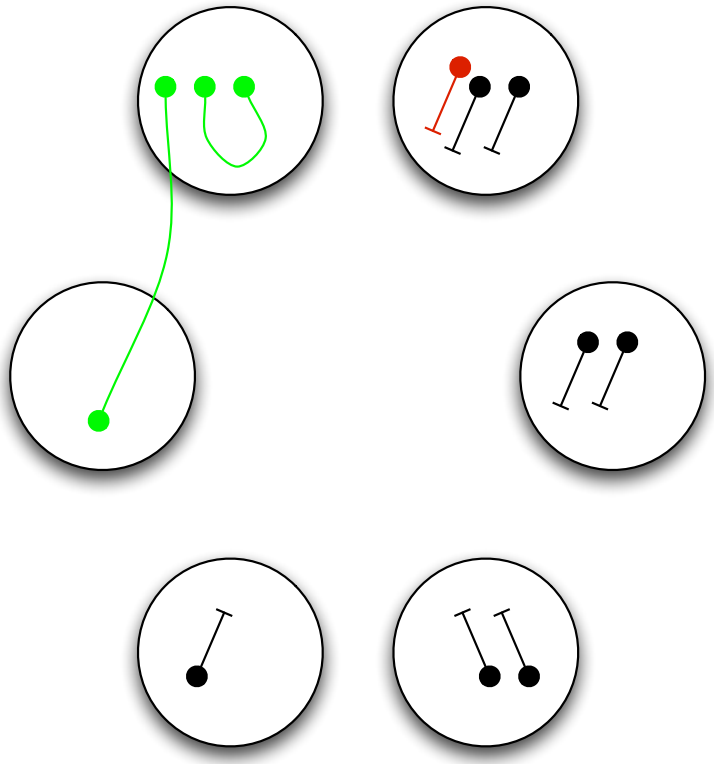

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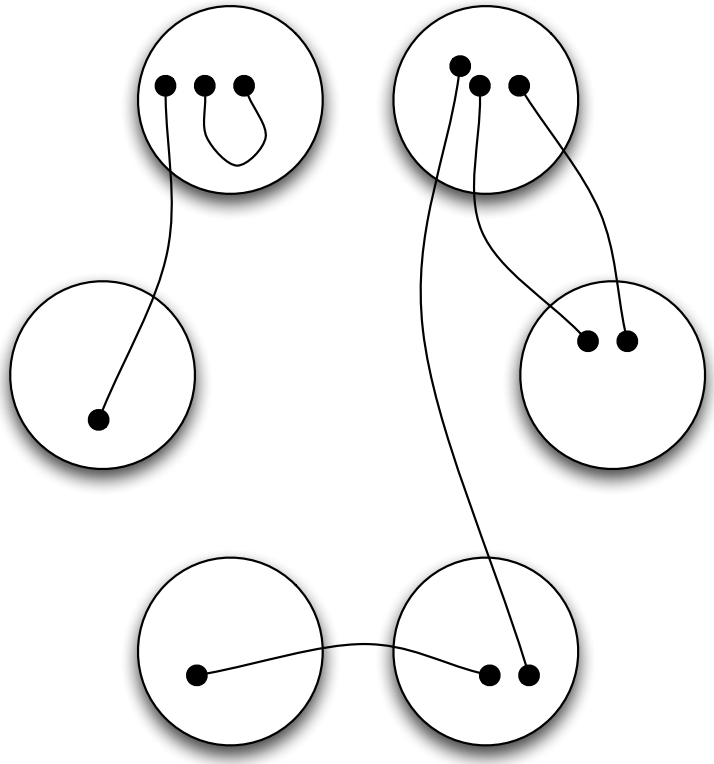

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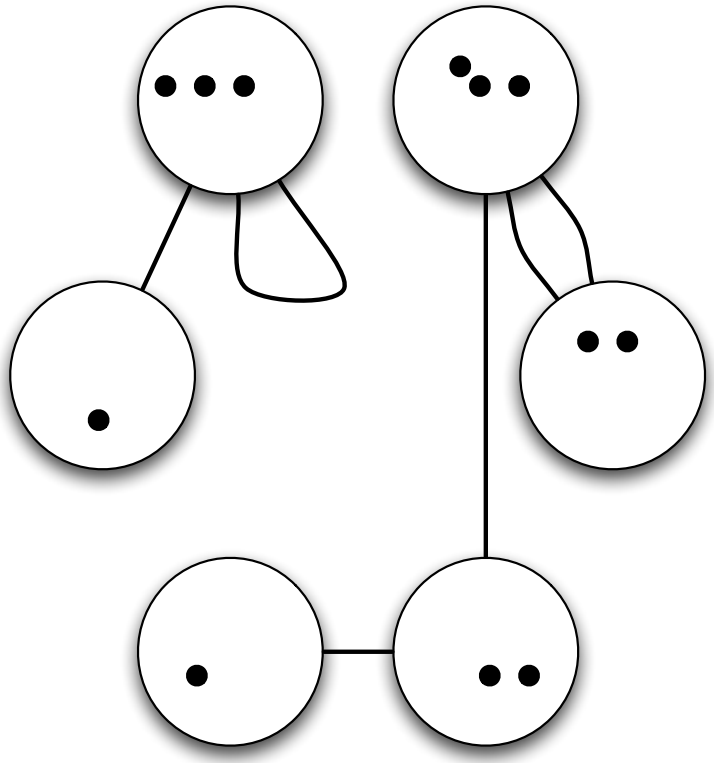

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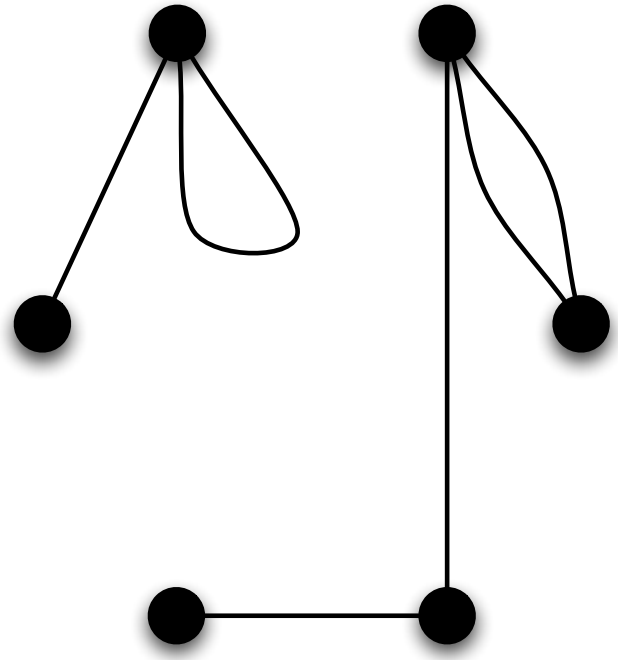

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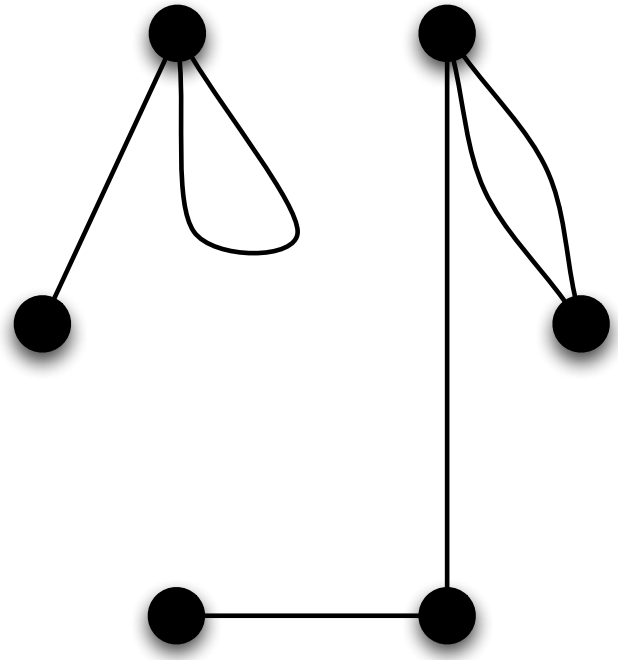

Conditional on obtaining a simple graph, result has distribution $G_{\mathbf{d}}$.

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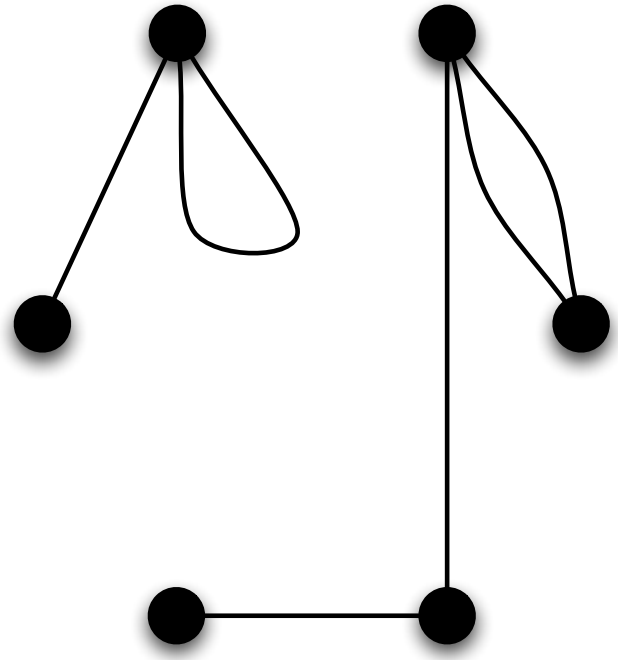
- Collapse bags down to points.



Conditional on obtaining a simple graph, result has distribution $G_{\mathbf{d}}$.
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- Without finite third moments, the limit is genuinely different. Reflected Brownian motion with drift becomes a thinned Lévy process. (Bhamidi, van der Hofstad, van Leeuwaarden, 2010.)

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- A random tree with a given degree sequence and finite second moments, has sub-Gaussian tails for the height (Addario-Berry, Devroye, Janson 2010 and in preparation).

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- At the critical k, the vacant set precisely has a critical degree sequence, and so will have the same scaling limit as $G_{n,1/n}$.

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Are two uniformly random lattice points at expected distance $n^{d/3}$ in T?

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Thanks again to Ben, James, and Matthias, and to all of you.