

# Some recent results in probability

Louigi  
Addario-Berry



McGill

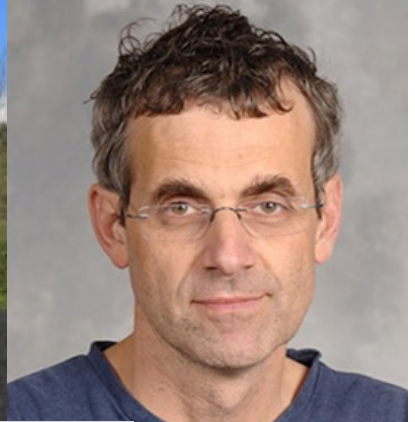
July 24

SPA

2023



# Schramm Lecture- Past speakers



I am the first speaker in this series who never met Oded Schramm.

I know him only through his work and through the words of those who knew and loved him.

Many people have written and spoken of his his gentleness, his curiosity and his kindness.

Taking inspiration from that (and from his math):

This talk:

I aim to tell you about some results I like, and that I hope Oded Schramm would have liked, by researchers who I think would be great candidates for future Schramm Lectures.

(Consider this talk a nomination!)

I didn't choose these results for their "importance" (whatever that means) but because a) I find them beautiful and b) I feel competent to explain their statements (though not all aspects of their proofs).

# 1. Cramér's theorem is atypical

Nina Gantert



Kavita Ramanan



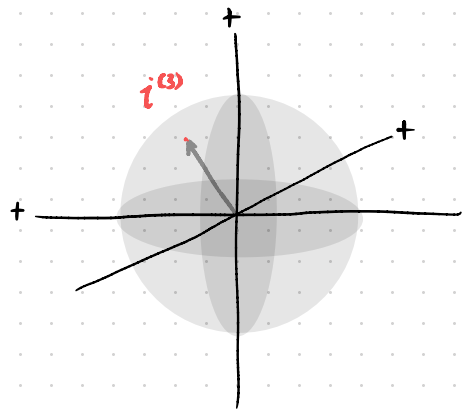
# Cramér's Theorem is atypical

Assume  $(X_i, i \geq 1)$  IID, s.t.  $\mathbb{E}X_1 = 0$ ,  $\Lambda(t) := \log \mathbb{E} e^{tX_1} < \infty$  for  $|t| < t_0$ . ↖  $t_0 > 0$

Let  $\Lambda^*(x) := \sup_{t \in \mathbb{R}} (tx - \Lambda(t))$  be the Legendre transform.

Write  $X^{(n)} := (X_1, \dots, X_n) \in \mathbb{R}^n$

$\hat{i}^{(n)} := \frac{1}{\sqrt{n}} (1, \dots, 1) \in S^{n-1}$  ← "Diagonal" Unit vector



So  $W_n := \frac{1}{n}(X_1 + \dots + X_n) = \langle \frac{1}{\sqrt{n}} X^{(n)}, \hat{i}^{(n)} \rangle$

**Cramér's Theorem** (1937)  $\mathbb{P}(W_n \geq x) = \exp(-(1+o(1))n\Lambda^*(x))$ ,  $0 < x < x_0$  ↖ suff. small

"Story": To have  $W_n \approx x$ , the r.v.s  $X_1, \dots, X_n$  must behave like they have mean  $x$ .

Cheapest way to do this: exponential tilting of  $\mathcal{L}_{X_1}$  to  $\hat{\mathcal{L}}$  with mean  $x$ .

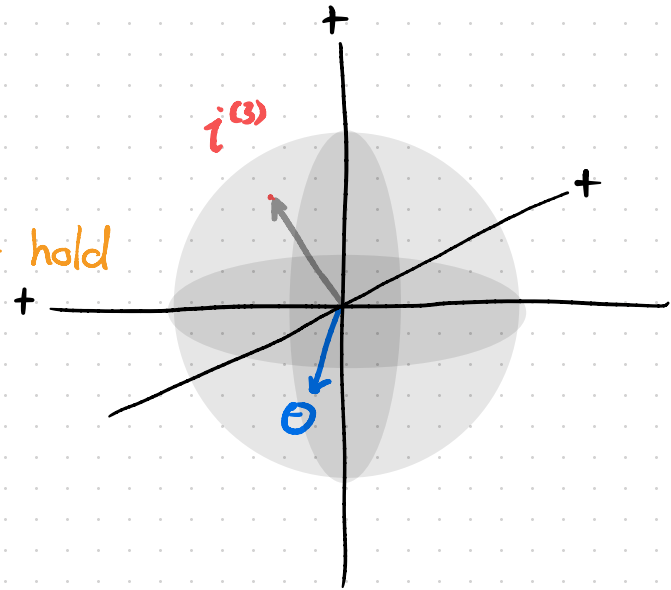
Cost of tilting: relative entropy  $D(\hat{\mathcal{L}} | \mathcal{L}_{X_1}) = \Lambda^*(x)$ .

Cramér's Theorem is atypical

Q: What happens if we replace  $i^{(n)} = n^{-1/2}(1, \dots, 1)$  by  $\Theta^{(n)} \in_n \mathbb{S}^{n-1}$ ?

Now studying deviations of  $W_n(\Theta^{(n)}) := \frac{1}{\sqrt{n}} \langle X^{(n)}, \Theta^{(n)} \rangle$ .

A: Most  $\Theta^{(n)}$  are far less balanced than  $i^{(n)}$ . Some coordinates play a bigger role than others  $\Rightarrow$  Cramér's Thm may no longer hold



However, an LDP still exists!

Let  $\Psi(t) := \mathbb{E}[\Lambda(tN)]$  where  $N \sim \mathcal{N}(0, 1)$ .

$$\Psi^*(x) := \sup_{t \in \mathbb{R}} (tx - \Psi(t))$$

(If  $\Lambda(\sqrt{t})$  is sym., conc., non-linear then  $\Lambda(t) = \Psi(t) < \infty$  iff  $t = 0$ .)

Theorem (Gantert, Kim, Ramanan 2016) Let  $\Theta = (\Theta^{(n)}, n \in \mathbb{N})$ . Then under suitable moment conditions, almost surely  $\mathbb{P}(W_n(\Theta^{(n)}) \geq x | \Theta) = \exp(-(1+o(1))n\Psi^*(x))$

## 2. Random maps under the Cardy embedding

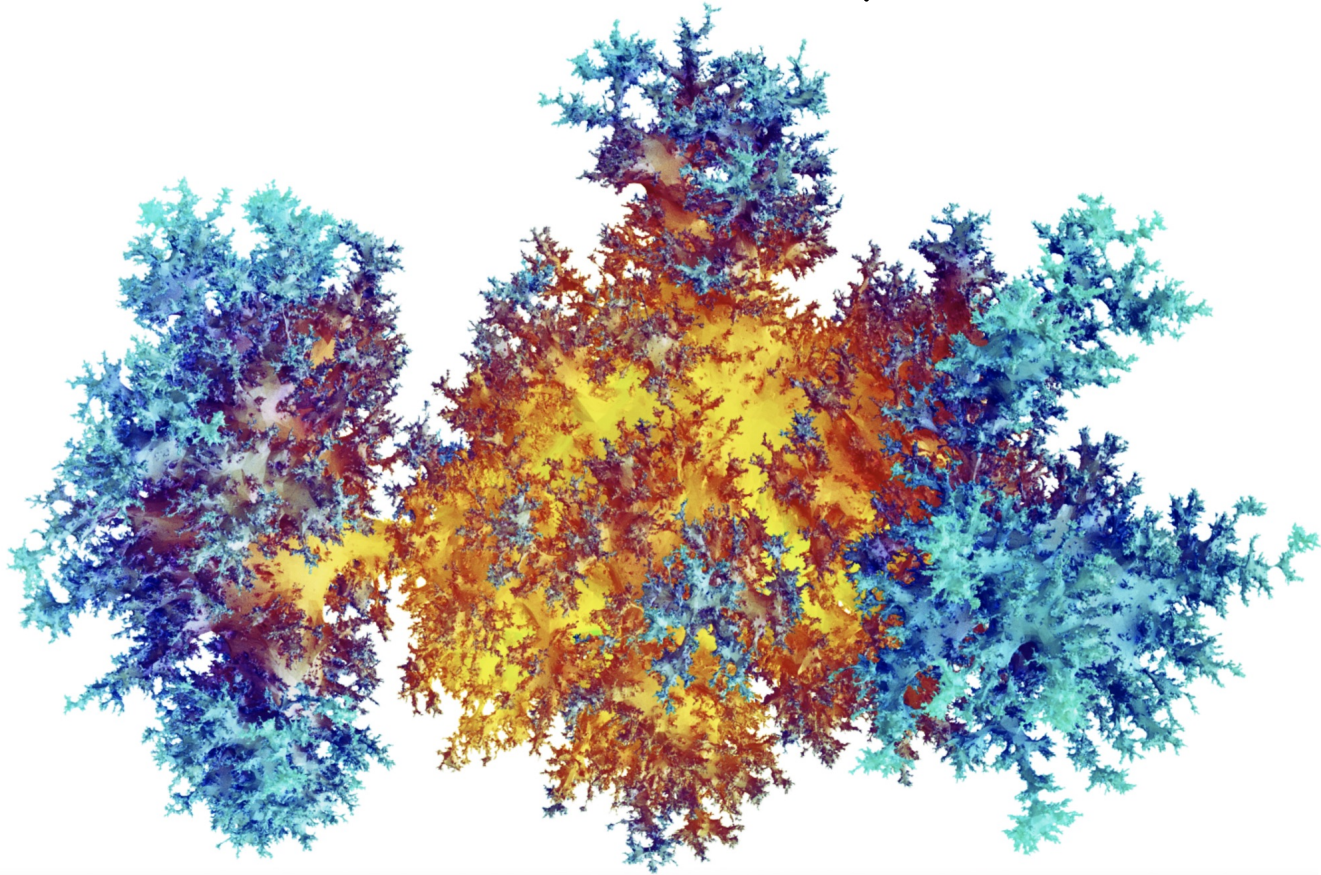
Nina Holden



# Brownian Map



Image by  
Benedikt  
Stufler



Scaling limit of many random map ensembles (Le Gall, Miermont, ...)

# Brownian Disk:

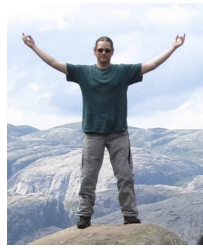
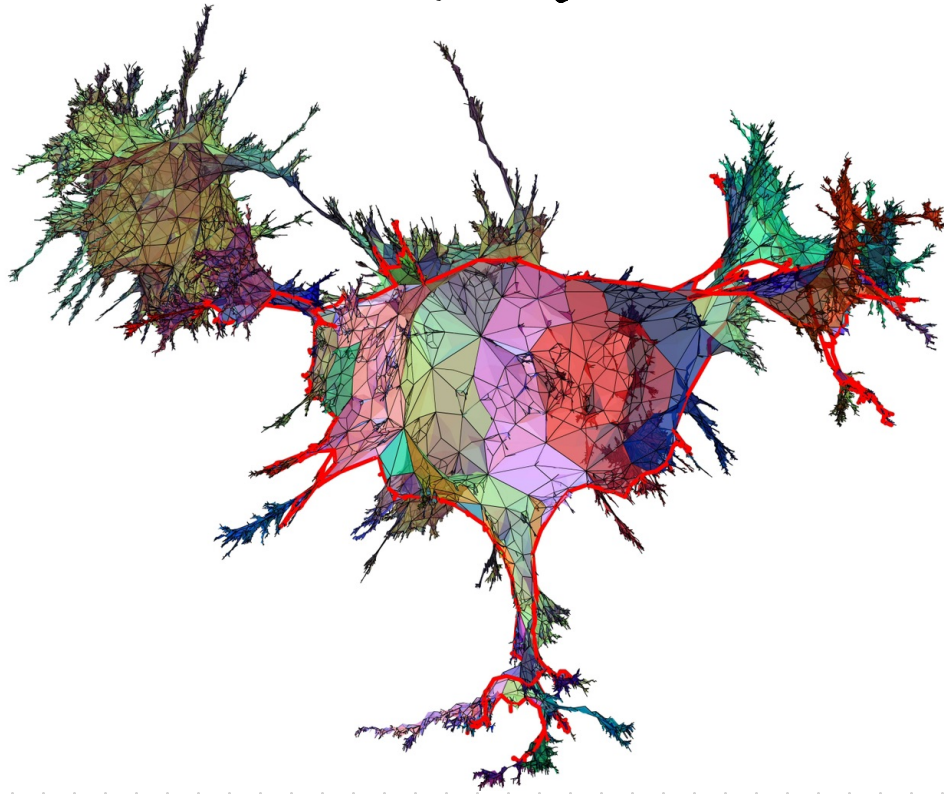


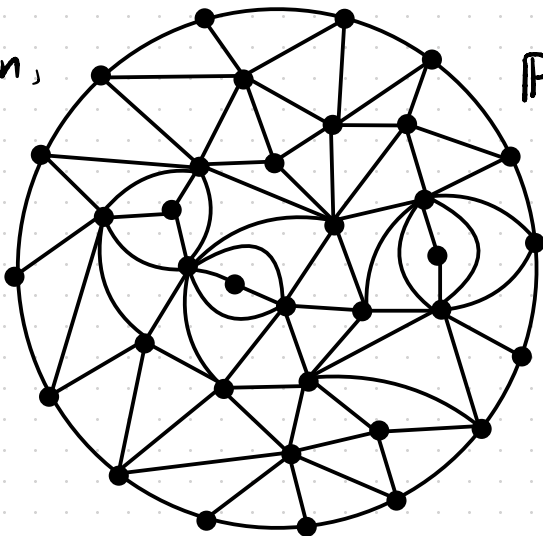
Image by  
J. Bettinelli

Analogous limit for random maps with a boundary  
(Bettinelli, Miermont, ...)

# Example of a convergence result: Boltzmann triangulations

Let  $\mathcal{M}_n = \{\text{loopless triangulations with boundary length } n\}$

Choose  $M_n \in \mathcal{M}_n$ ,



$$\mathbb{P}(M_n = M) \propto \left(\frac{2}{27}\right)^{|M_n|} \quad \leftarrow \# \text{ vertices.}$$

$\mathcal{D} = \sqrt{3/2}$ -Liouville quantum gravity disk (obtained from  $e^{\gamma h} dz$   $h = \text{GFF on disk}$ ,  $\gamma = \sqrt{3/2}$ ).

Measured metric space structure of  $\mathcal{D}$  is called

$\text{BD}_1 = \text{Free Brownian disk with perimeter 1.}$

vertices of  $M_n$

Graph dist. on  $V(M_n)$

Counting meas. on bdy. vert.s

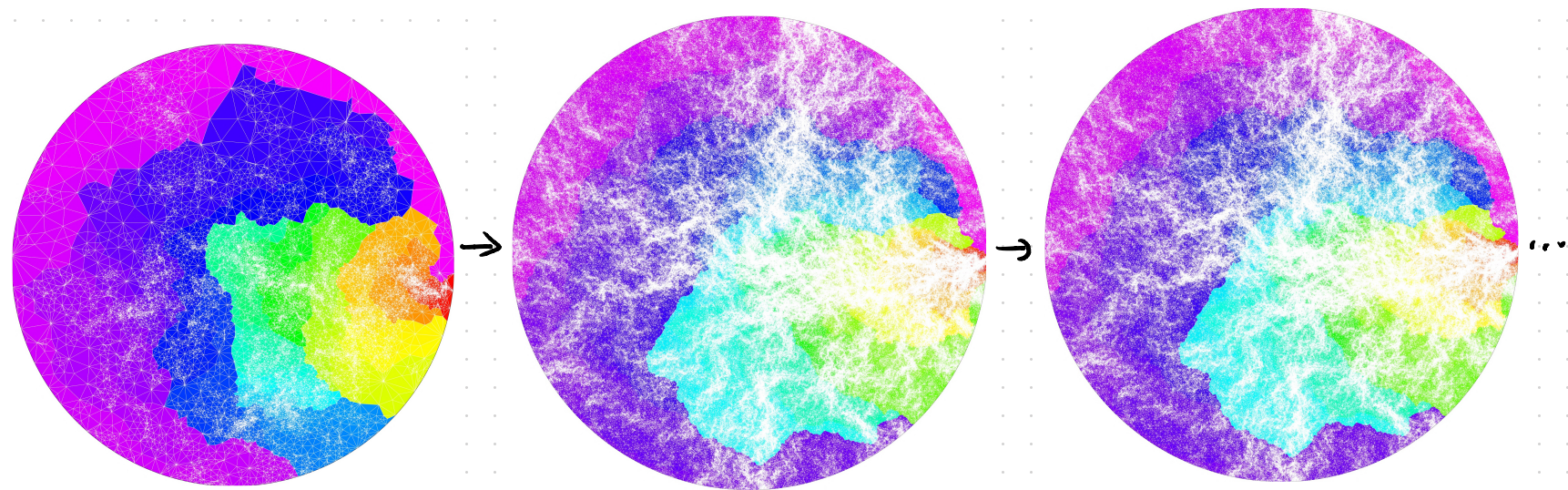
Counting meas. on  $V(M_n)$

(Gwynne-Miller 2019)

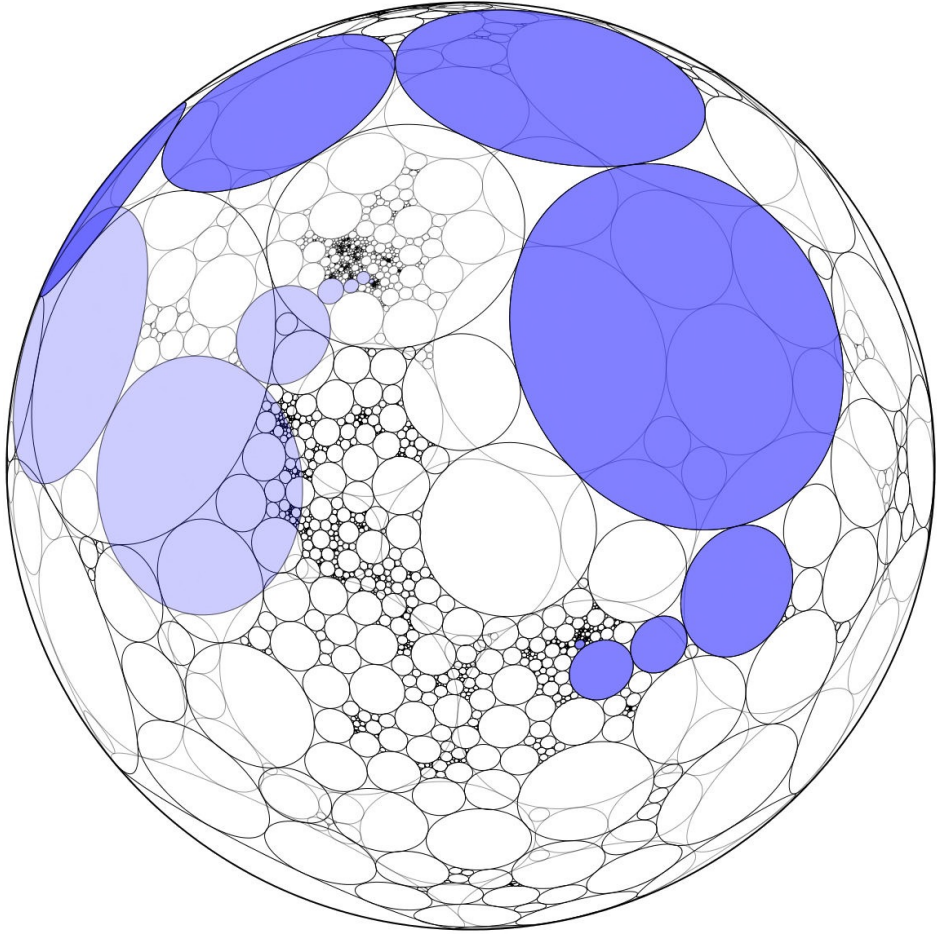
**Theorem**  $(V(M_n), n^{-1/4} d_n, n^{-1/2} \sum_n, n^{-1} \mu_n) \xrightarrow{\text{dist}} \text{BD}_1$

$\uparrow$  (NB  $\mu_n(V(M_n)) = |M_n|$  is random)

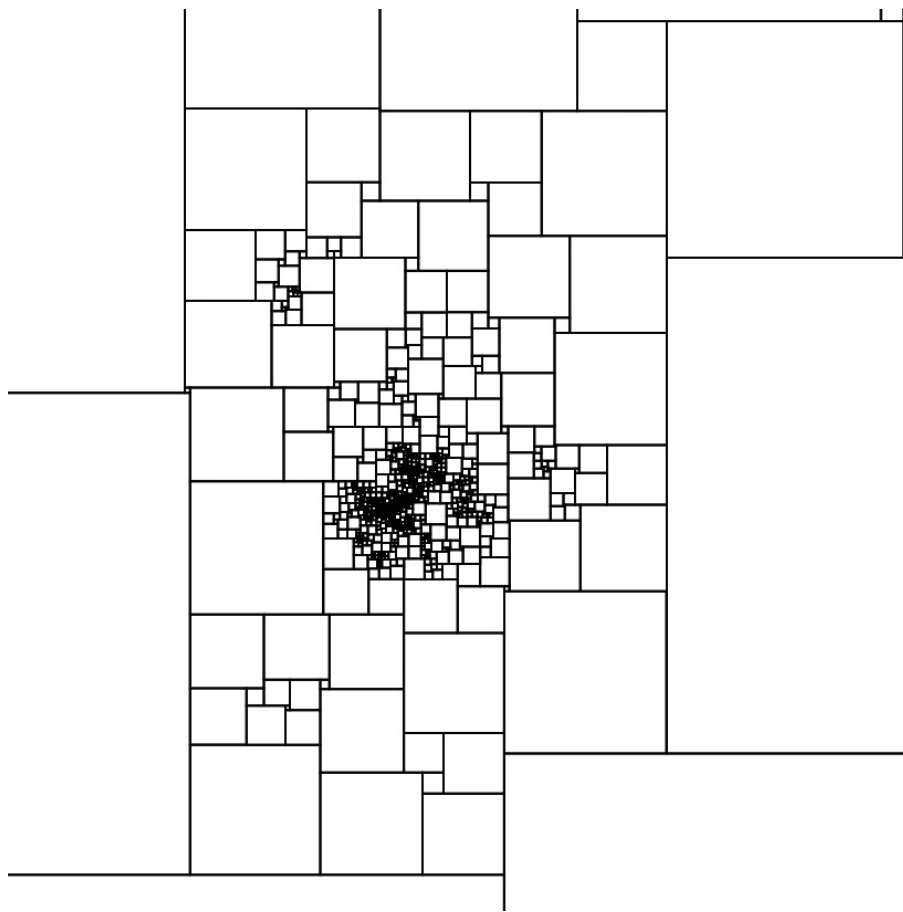
Missing: Robust convergence theory for random maps  
seen as embedded objects.



# Embedding rules: Circle Packing



# Embedding rules: Electrical network embedding



# Embedding rules: Spring embedding

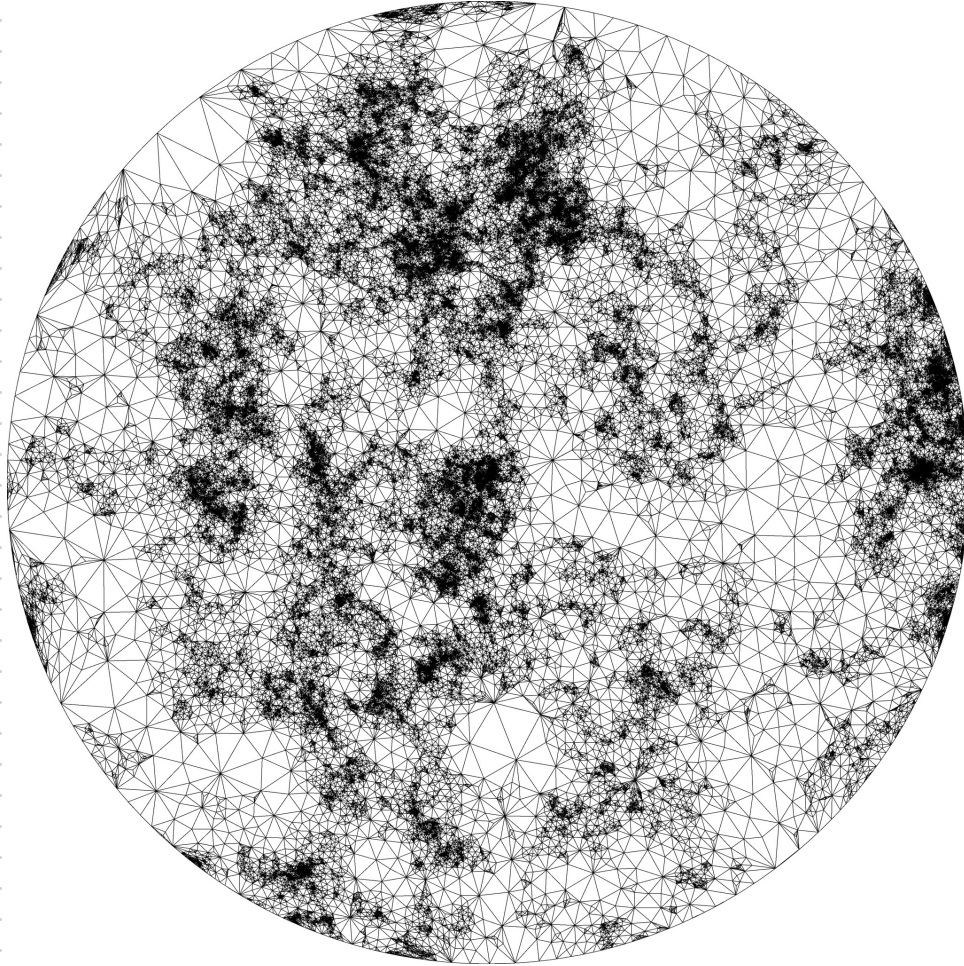


Image by  
Jason Miller

# Principal Challenge:

Prove Convergence  
for embedded maps

AREA MEASURE:

$n^{-1}$  · (Counting measure  
on  $\approx n$  inner vertices)

BOUNDARY MEASURE:

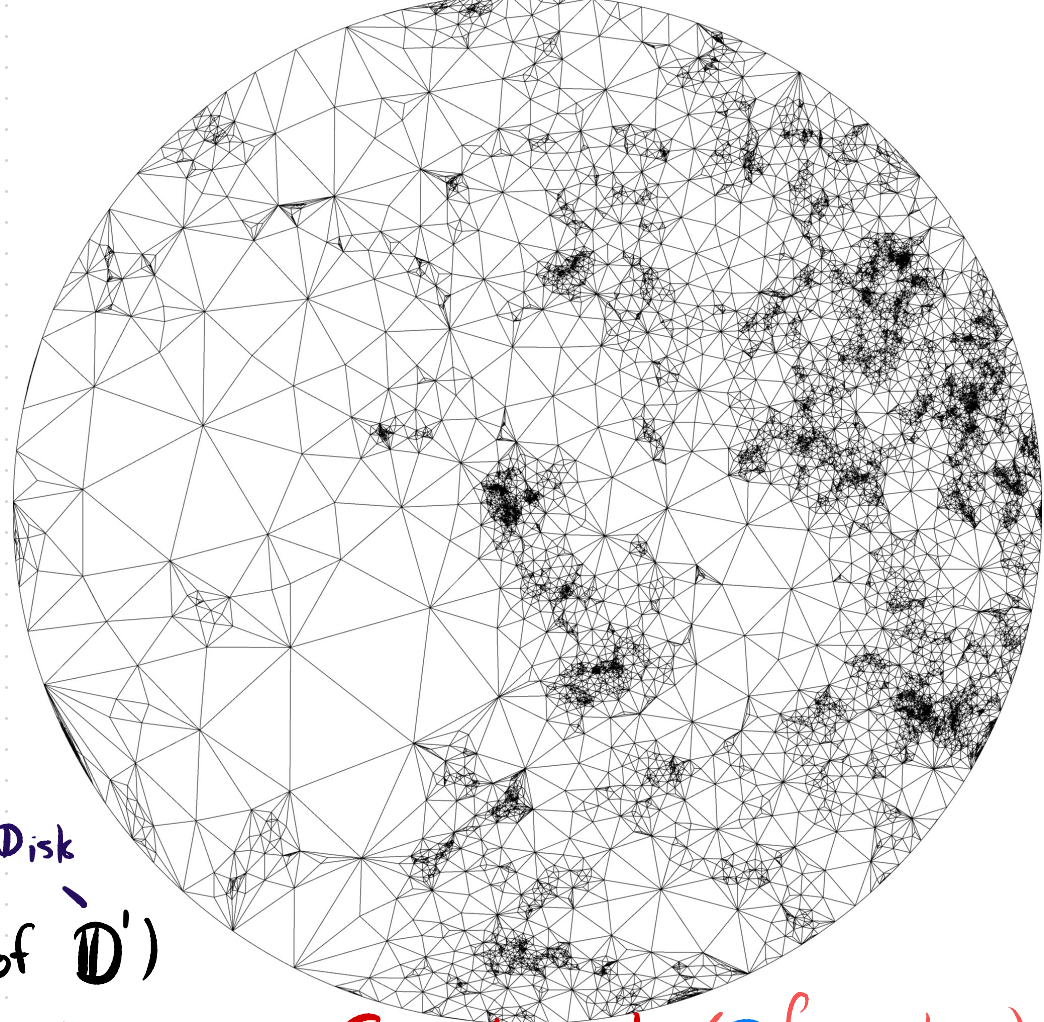
Uniform prob. measure on  
 $n^{1/2}$  boundary vertices.

DISTANCE:

$n^{-1/4}$  · Graph distance

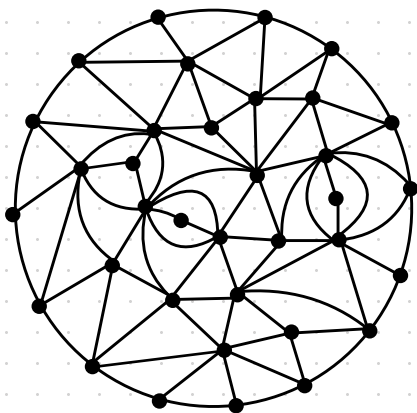
Unit Disk

(extend continuously to all of  $\mathbb{D}$ )



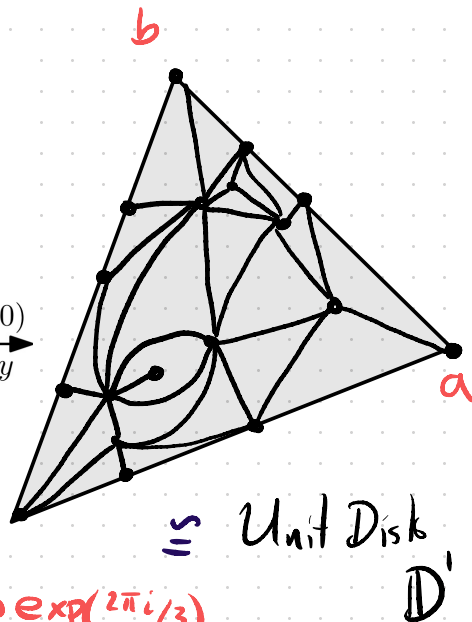
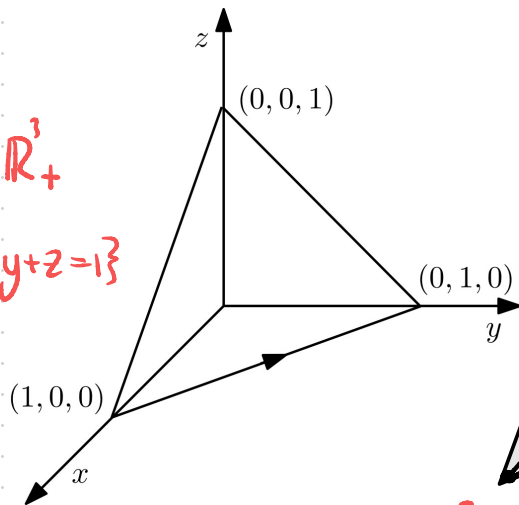
Candidate limit:  $\sqrt{8/3}$ -Liouville Quantum Gravity Disk ( $\mathbb{D}$  from above)

# The Cardy embedding of uniformly random triangulations



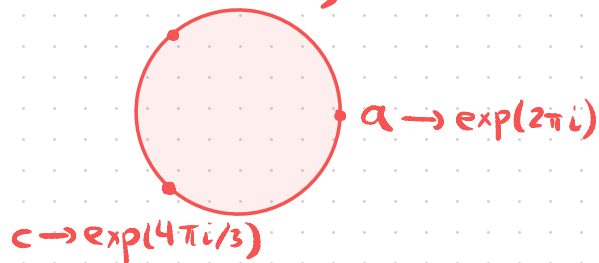
embed on  $\Delta \subset \mathbb{R}_+^3$

$$\Delta = \{(x, y, z) : x + y + z = 1\}$$



$\cong$  Unit Disk  $\mathbb{D}^1$

$$b \rightarrow \exp(2\pi i/3)$$



$$a \rightarrow \exp(2\pi i)$$

$$c \rightarrow \exp(4\pi i/3)$$

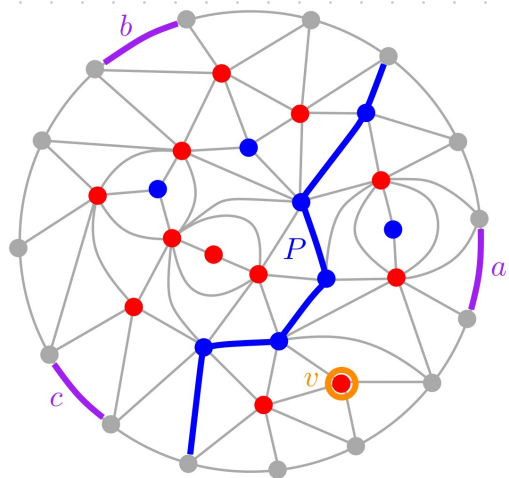
# Cardy Embedding

- $a, b, c$  chosen unif. at random from boundary edges (labeled in ccw order)
- Interior vertices coloured uniformly at random (red / blue)

$E_a(v) := \{ \exists \text{ blue path separating } a \text{ \& } v \text{ from } b \text{ \& } c \}$   $p_a(v) = \mathbb{P}(E_a(v))$

$E_b(v) := \{ \exists \dots b \text{ \& } v \text{ from } a \text{ \& } c \}$   $p_b(v) = \mathbb{P}(E_b(v))$

$E_c(v) := \{ \exists \dots c \text{ \& } v \text{ from } a \text{ \& } b \}$   $p_c(v) = \mathbb{P}(E_c(v))$



Embed  $v$  at position

$$C(v) := \frac{1}{p_a(v) + p_b(v) + p_c(v)} \cdot (p_a(v), p_b(v), p_c(v)) \in \Delta$$

(Holden, Sun 2023)

$\mu_n^*, \xi_n^*, d_n^* \rightarrow$  Pushforwards of  $\mu_n, \xi_n, d_n$  to  $\mathbb{D}^1$  by  $C$ .

Theorem  $(n^{-1/4} d_n^*, n^{-1/2} \xi_n^*, n^{-1} \mu_n^*) \xrightarrow{\text{dist}} \mathcal{D}$

First/only random map scaling limit for a natural/conformal embedding rule.

### 3. When latent geometry disappears

Tselil Schramm



# Erdős-Rényi: Random Graph

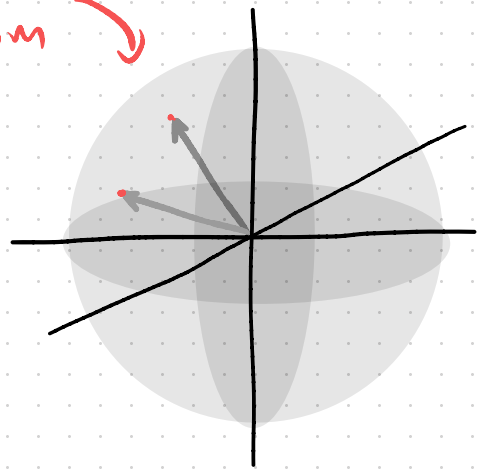
$G(n, p)$

- Vertex set  $[n]$
- Edges indep. present with prob.  $p$ .

$\{1, 2, \dots, n\}$

# $G_{\text{Geo}_d}(n, p)$

Sample  $v_1, \dots, v_n$   
 u.a.r. from  $S^{d-1}$



- Vertex set  $[n]$
  - Edge  $ij$  present if  $\langle v_i, v_j \rangle > \tau$
- $\tau = \tau(p)$  s.t.  $\mathbb{P}(\langle v_i, v_j \rangle > \tau) = p$ .

Edges correlated:  $\mathbb{P}(\langle v_i, v_k \rangle > \tau \mid \langle v_i, v_j \rangle > \tau, \langle v_j, v_k \rangle > \tau) > p$

## Question

For which values  $n, p, d$  are  $G(n, p)$  and  $\text{Geo}_d(n, p)$  statistically indistinguishable?

$$(d_{\text{TV}}(G(n, p), \text{Geo}_d(n, p)) \rightarrow 0)$$

"When does the latent geometry disappear?"

Devroye, Gyöngy, Lugosi, Udina (2011):

$d \gg \exp(n^2) \Rightarrow$  indistinguishable

# Bubeck-Ding-Eldan-Racz (2016)

$p = \Theta(1) \Rightarrow$  distinguishability threshold at  $d \asymp n^3$

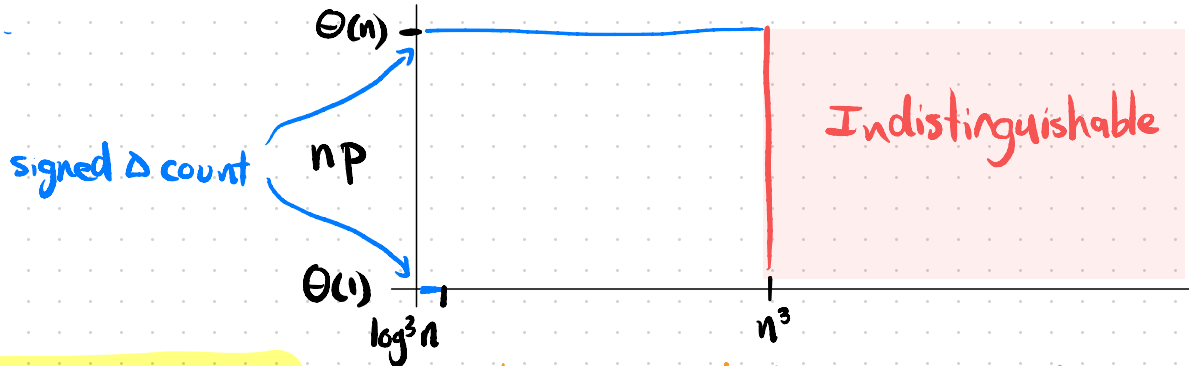
Key statistic: "signed triangle count"

$$T(G) = \sum_{u,v,w} (G_{uv} - p)(G_{uw} - p)(G_{vw} - p) \quad \sim \quad G_{xy} = \mathbb{1}_{xy \in E(G)}$$

$\frac{d}{n^3} \rightarrow \infty$   
then  $d_{TV} \rightarrow 0$

$\frac{d}{n^3} \rightarrow 0$  then  
 $d_{TV} \rightarrow 1$

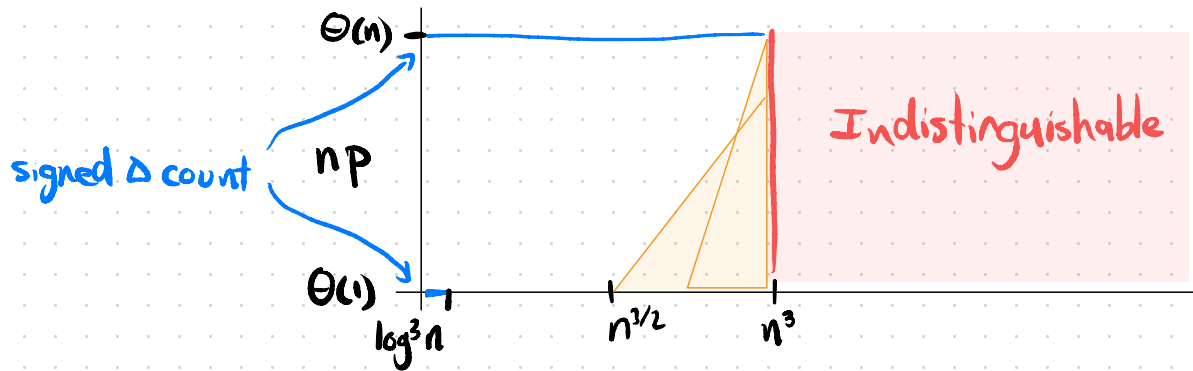
$p = \Theta(1/n)$ : if  $d/\log^3 n \rightarrow 0$  then  $d_{TV} \rightarrow 1$



**Conj (BDER)**:  $p = \Theta(1/n)$ : If  $d/\log^3 n \rightarrow \infty$  then  $d_{TV} \rightarrow 0$

Brennan-Bresler - Nagaraj (2020):

$\frac{\log n}{n^2} < p < \frac{1}{2}$  : If  $d \gg \min(pn^3, p^2 n^{7/2}) \log^5 n$  then  $d_{TV} \rightarrow 0$



**Conj (BDER)** :  $p = \Theta(1/n)$  : If  $d / \log^3 n \rightarrow \infty$  then  $d_{TV} \rightarrow 0$

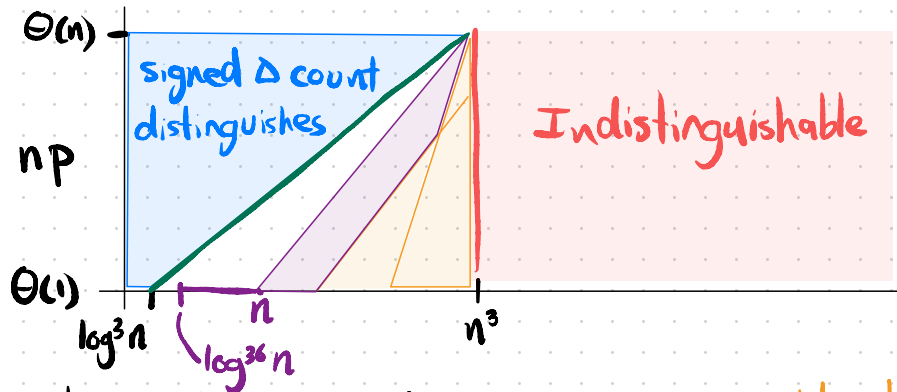
Theorem (Liu-Mohanty-Schramm-Yang):

①  $p = \Theta(1/n)$ : If  $d / \log^{36} n \rightarrow \infty$  then  $d_{TV} \rightarrow 0$

②  $\forall \alpha > 0 \exists c > 0$ : if  $\frac{\alpha}{n} < p < \frac{1}{2}$ ,  $d / (p^2 n^3) \gg \log^7 n$  then  $d_{TV} \rightarrow 0$

③ If  $n^2 \ll p \ll c < 1$  then

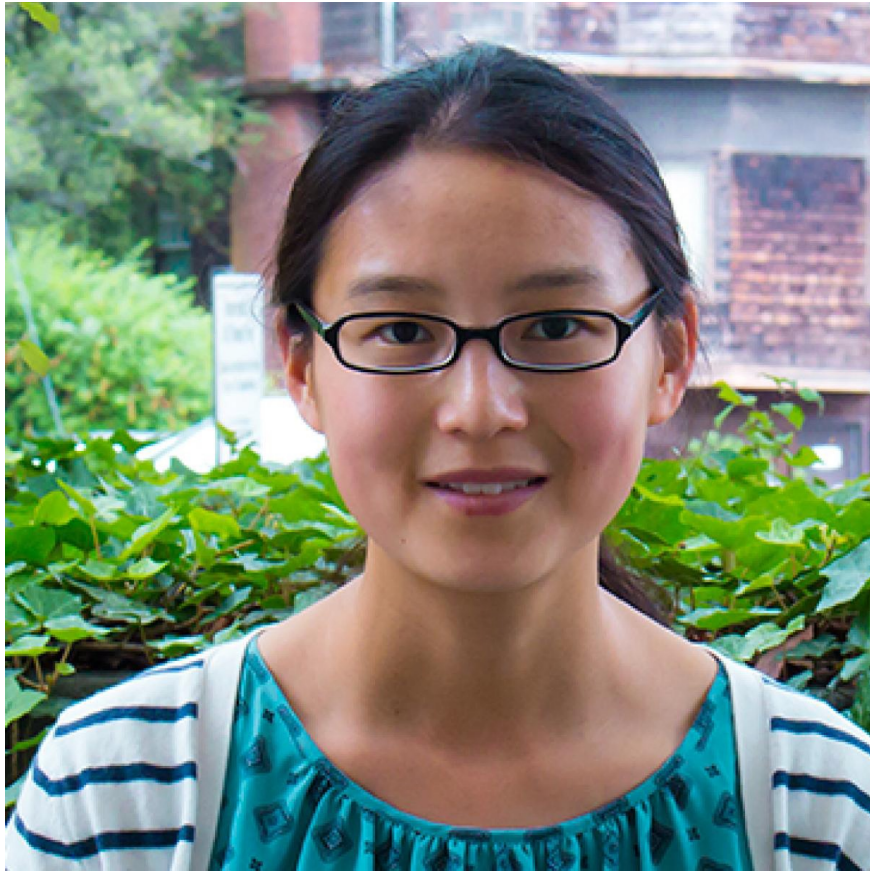
$d \ll (nH(p))^3 \Rightarrow d_{TV} \rightarrow 1$ . (Statistic: signed triangle count)



Conj (Liu-Mohanty-Schramm-Yang): True threshold at  $d \asymp (nH(p))^3$   
 $H(x) = (-x \log x - (1-x) \log(1-x))$

# 4. Locality for critical percolation on expanders.

Nike Sun



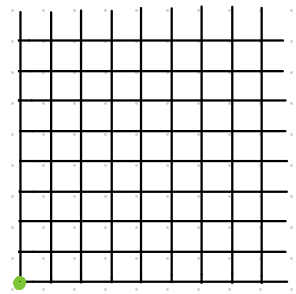
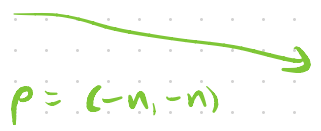
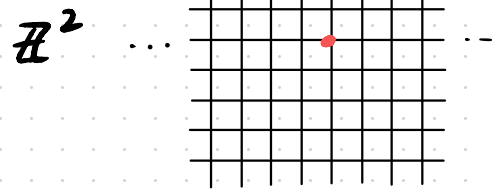
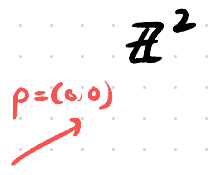
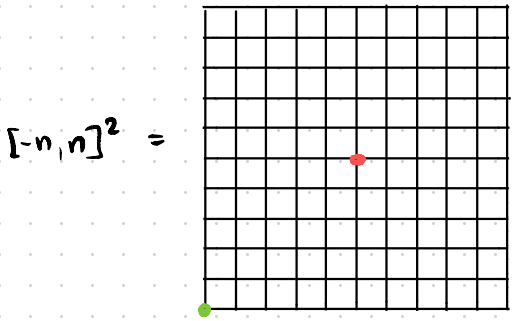
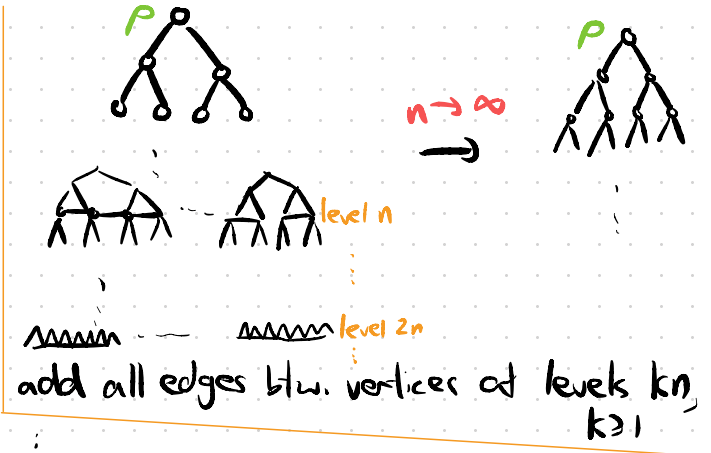
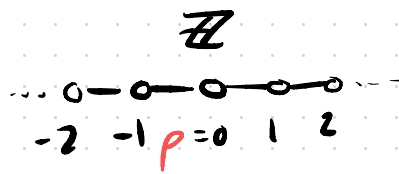
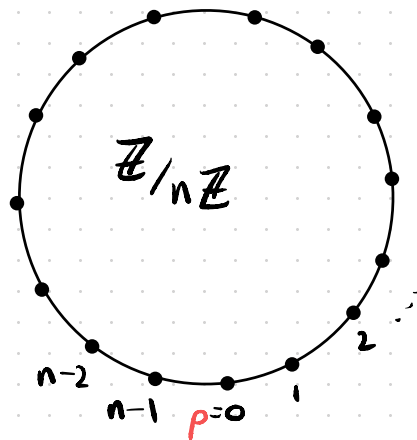
# Local Convergence

**Rooted graph**: Triple  $(V, E, \rho)$  s.t.  $(V, E)$  is a locally finite graph.  
vertices / edges / root  $\in V$

Say a sequence  $(G_n, 1 \leq n \leq \infty)$  of rooted graphs  
**converges**  $(G_n \rightarrow G_\infty)$  if  $(V_n, E_n, \rho_n)$

$\forall r \in \mathbb{N} \exists n_0$  s.t.  $B_{G_n}(\rho_n, r) \cong B_{G_\infty}(\rho_\infty, r)$  for  $n \geq n_0$ .  
root-preserving isom.

# Examples



"The success of a limit theory depends on how many interesting parameters are continuous with respect to the convergence notion."

-Balazs Szegedy  
arXiv 1502.07861

Def:  $G = (V, E)$  is **transitive** if  $\forall p, p' \in V, (V, E, p) \cong (V, E, p')$

For transitive graph sequences, the limit (if it exists) does not depend on the choice of root.

For a graph  $G$ , let  $G_p$  be **Bernoulli( $p$ )** bond percolation on  $G$ :

Keep each edge independently with probability  $p$ .

Define  $p_c(G) = \inf\{p \in [0, 1] : \mathbb{P}(G_p \text{ contains an } \infty \text{ comp}) > 0\}$

**Conjecture (Schramm)**:

If  $(G_n, 1 \leq n < \infty)$  are infinite transitive graphs s.t.

a)  $G_n \rightarrow G_\infty$  and b)  $\sup_n p_c(G_n) < 1$ ,

then  $p_c(G_n) \rightarrow p_c(G_\infty)$

Q: What is an analogue of this conjecture for finite graphs?

Transitive  $\rightsquigarrow$  Uniformly random root.

(limit still infinite)

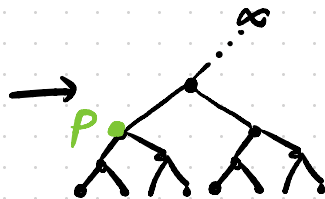
(The symmetry of a "uniform random choice" can sometimes be adequate to replace the symmetry of transitivity)

Say a sequence  $(G_n, 1 \leq n \leq \infty)$  of random rooted graphs converges in distribution if  $\exists$  a coupling  $(G'_n, 1 \leq n \leq \infty)$  of  $(G_n, 1 \leq n \leq \infty)$

$$\text{s.t. } P(G'_n \rightarrow G_\infty) = 1$$

$$\equiv \forall r > 0, \forall \text{ fixed rooted graphs } G, P(B_{G_n}(p_n, r) = G) \rightarrow P(B_{G_\infty}(p_\infty, r) = G)$$

Example  $T_n =$   randomly rooted



"Canopy tree"  
 $p$  at  $\text{Geom}(1/2)$   
distance along spine

## Expansion

For  $G=(V,E,p)$  finite, let  $h(G) = \min \left\{ \frac{|E_G(A,A^c)|}{|A|} : A \subseteq V, |A| \leq \frac{|V|}{2} \right\}$   
↳ finite graphs

Say  $(G_n, n \geq 1)$  is an expander sequence if  $\inf (h(G_n), n \geq 1) > 0$

**Theorem (Ren + Sun, 2022)** Perc. threshold is cts. for expanders w/ bounded average degree: if  $(G_n, n \geq 1)$  is a unif. rooted expander seq. such that  $G_n \xrightarrow{\text{dist}} G_\infty = (V, E, p)$  with  $\mathbb{E}[\deg(p)] < \infty$ ,

then  $p_c(G_n, n \geq 1) \rightarrow p_c(G_\infty)$ , in that:

a)  $p_c(G_\infty)$  a.s. constant

b) If  $p < p_c$  then  $\mathbb{P}(|C_{\max}(G_{n,p})|/n > p) \rightarrow 0$

c)  $\forall p > p_c \exists \alpha > 0$  s.t.  $\mathbb{P}(|C_{\max}(G_{n,p})| > \alpha n) \rightarrow 1$ .

• Improves past results of Benjamini, Nachmias, peres, Sankar

• Result can fail w/o expansion assumption