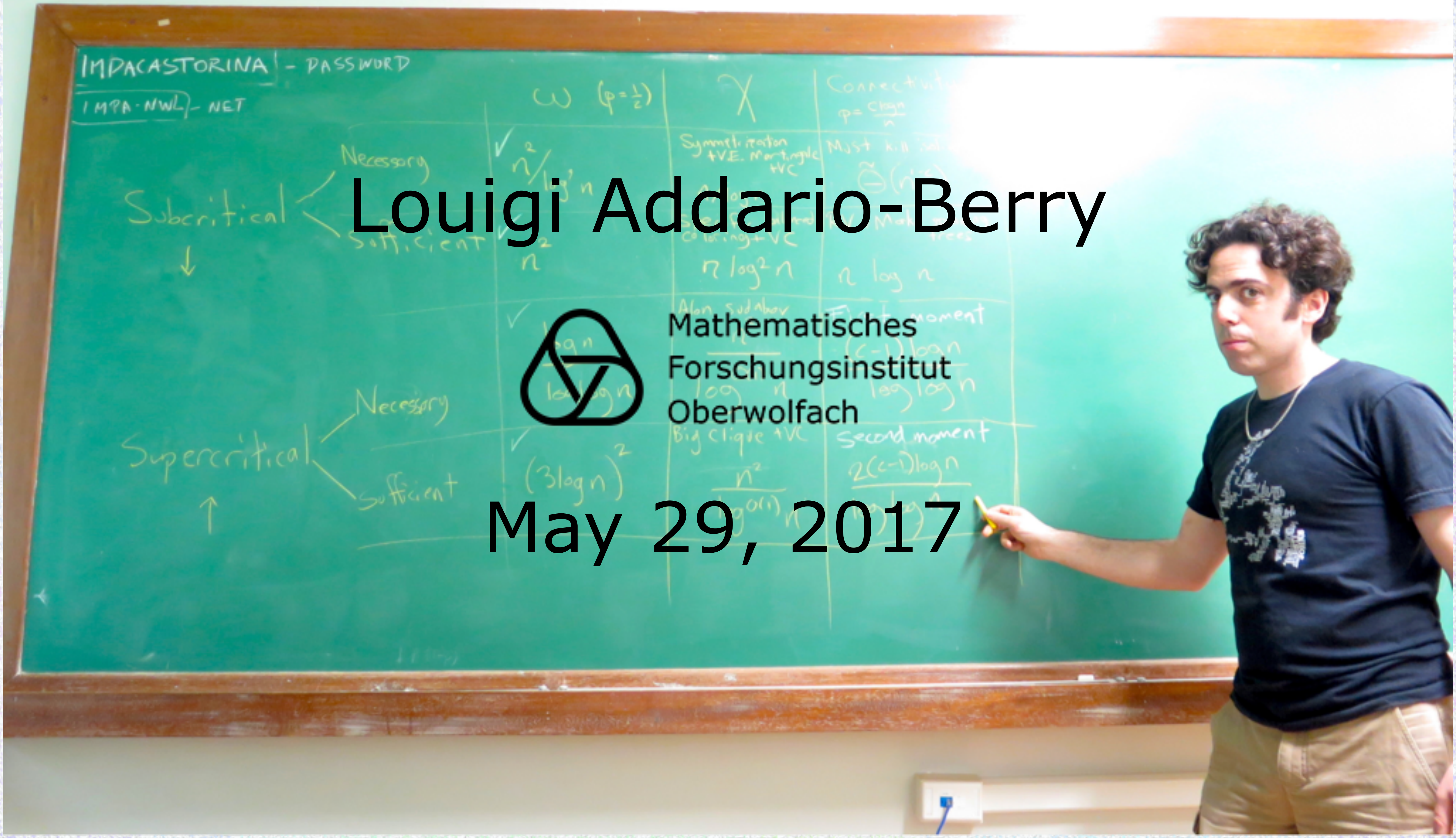
# Voronoi cells in random maps

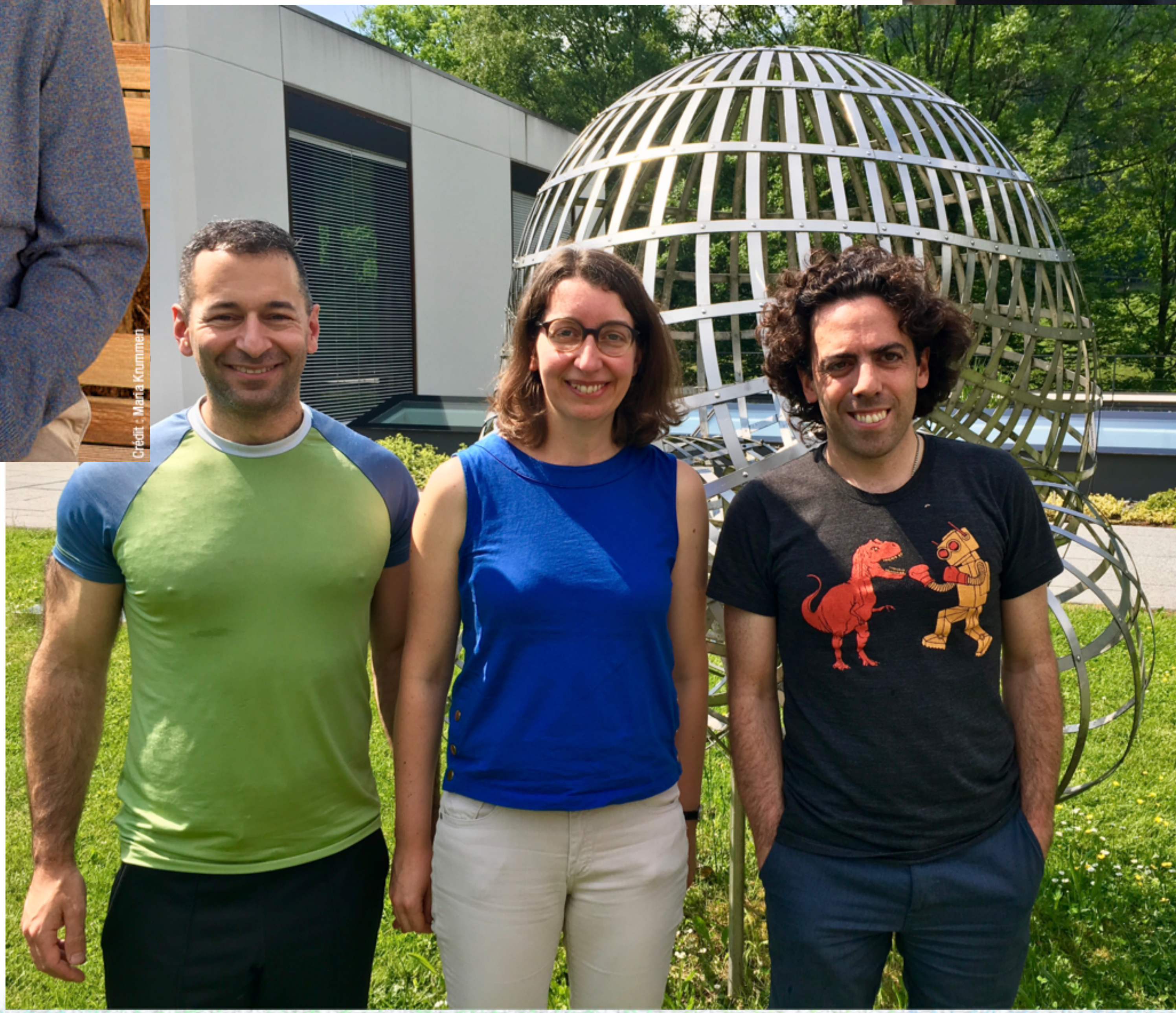


Louigi Addario-Berry

Mathematisches
Forschungsinstitut
Oberwolfach

May 29, 2017


# Joint work

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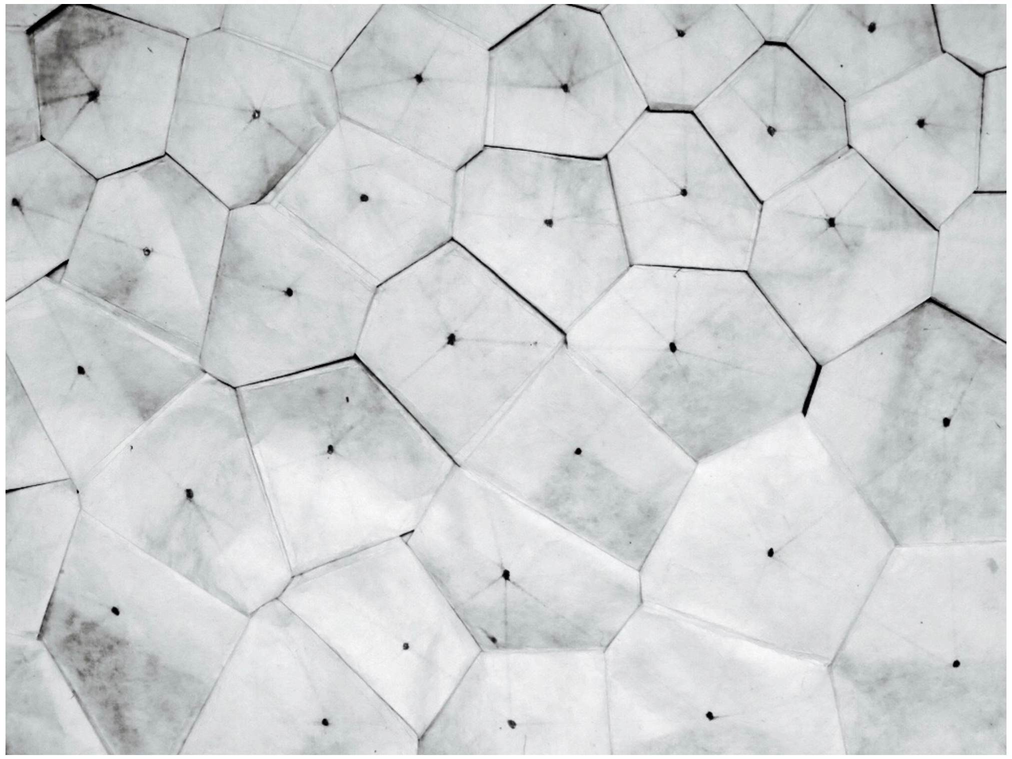

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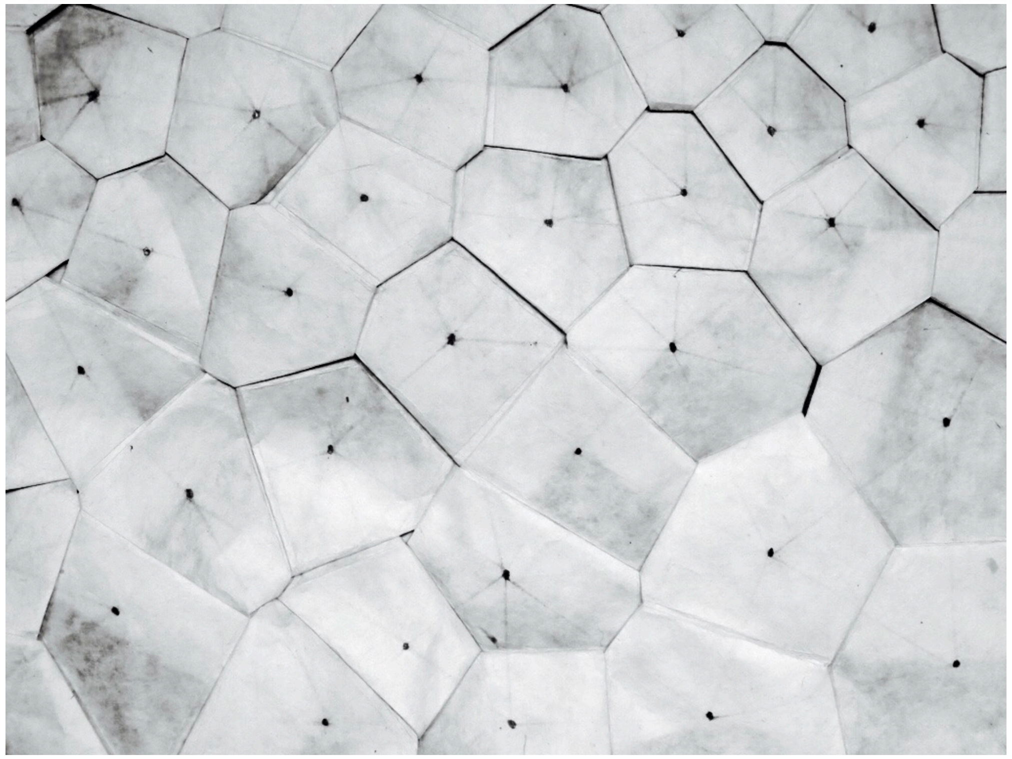

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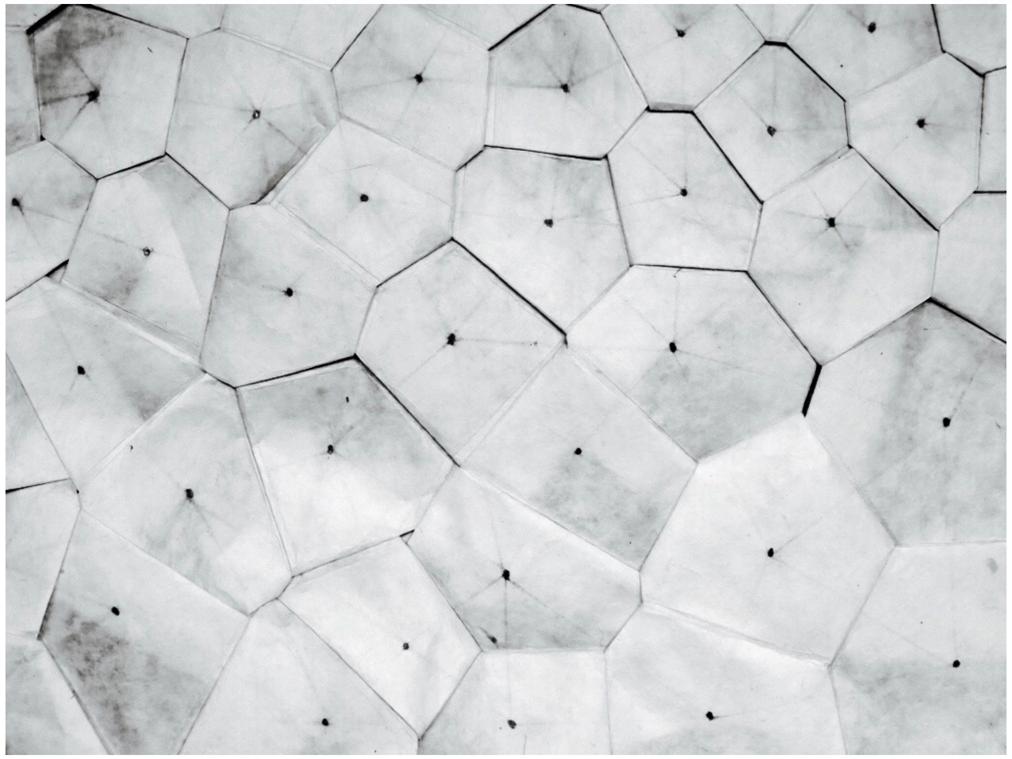

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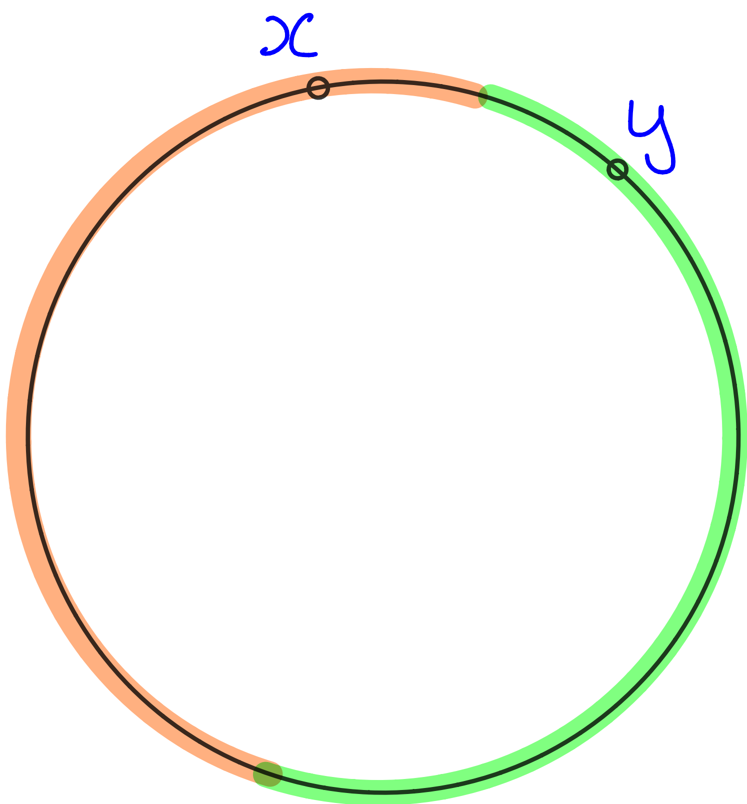

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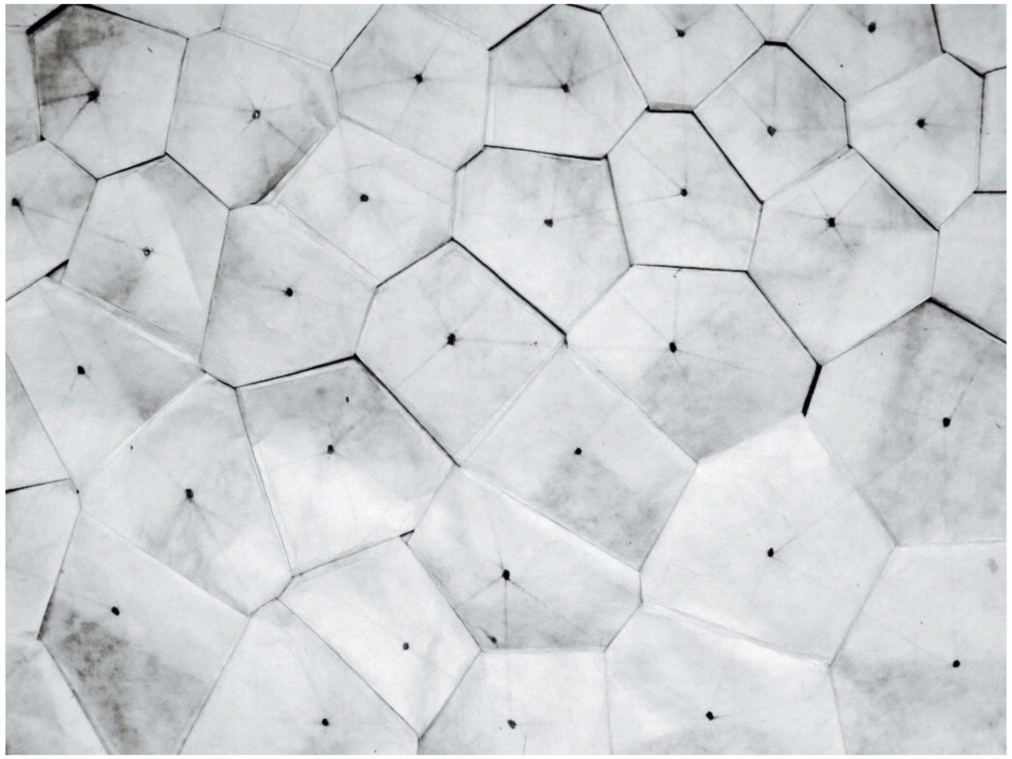

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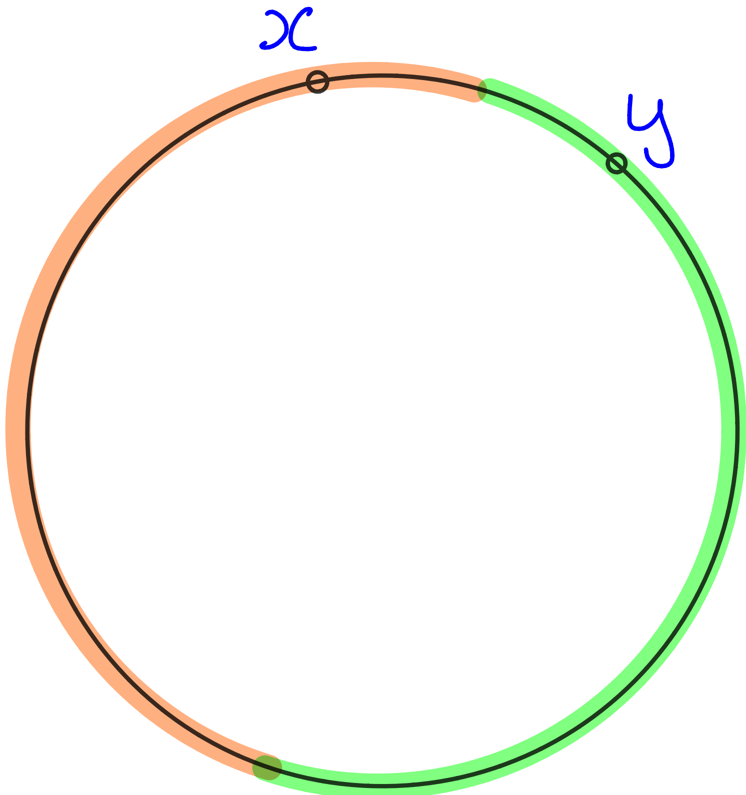


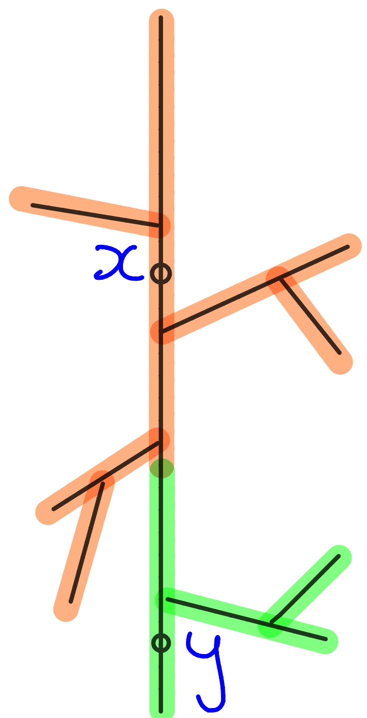

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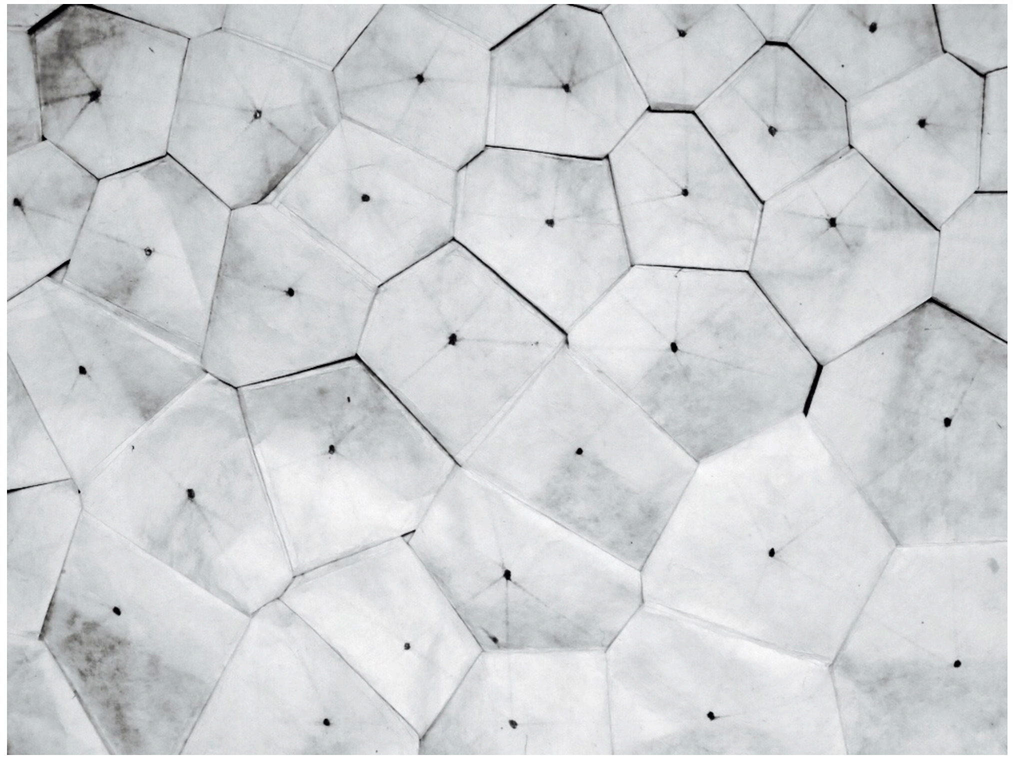

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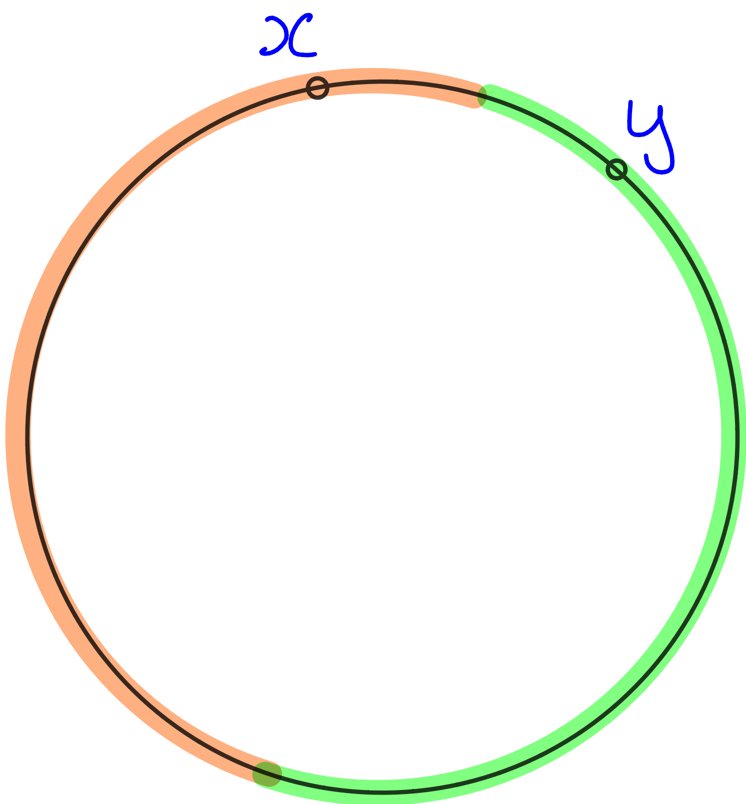


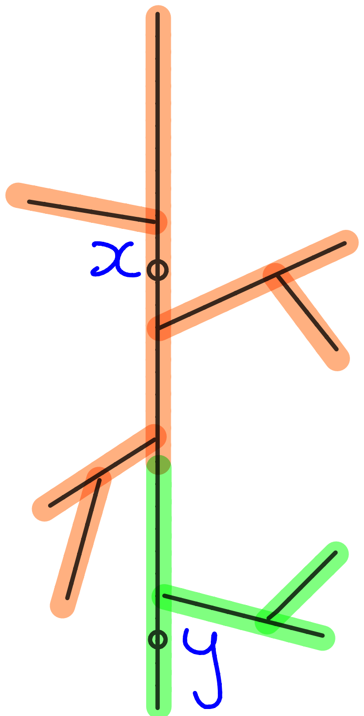


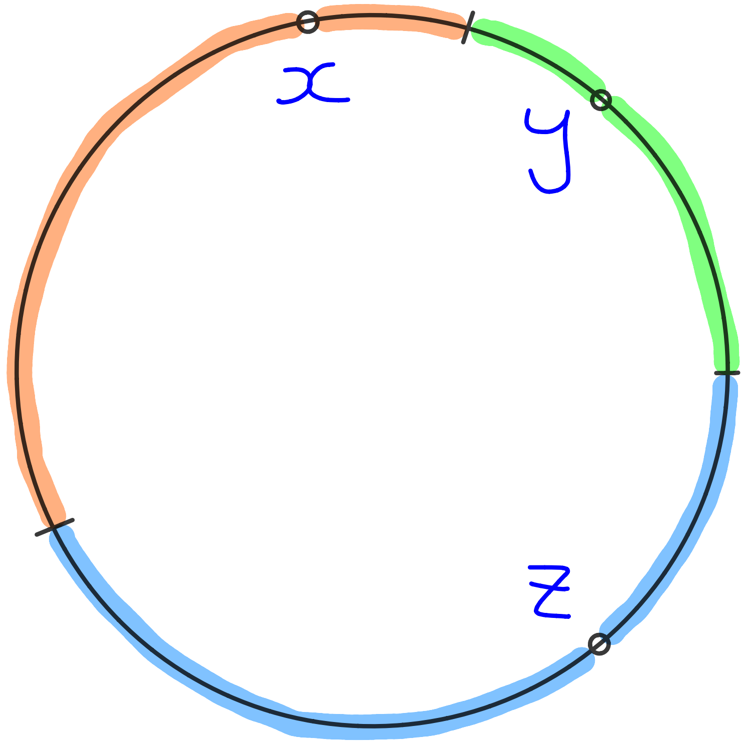

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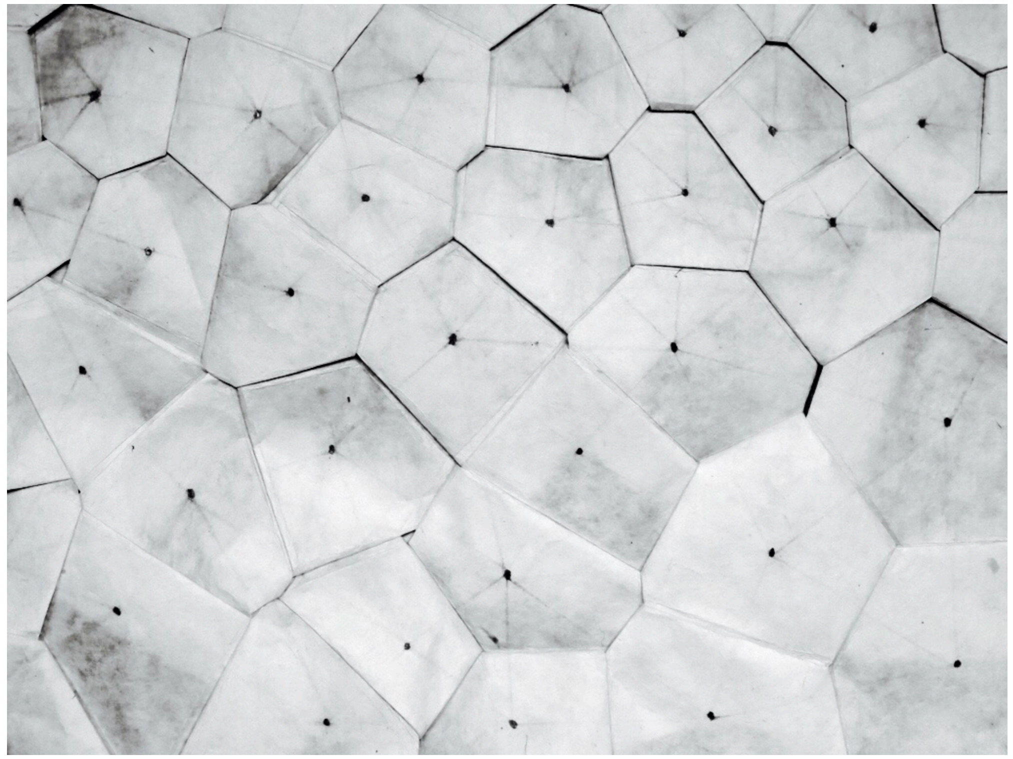

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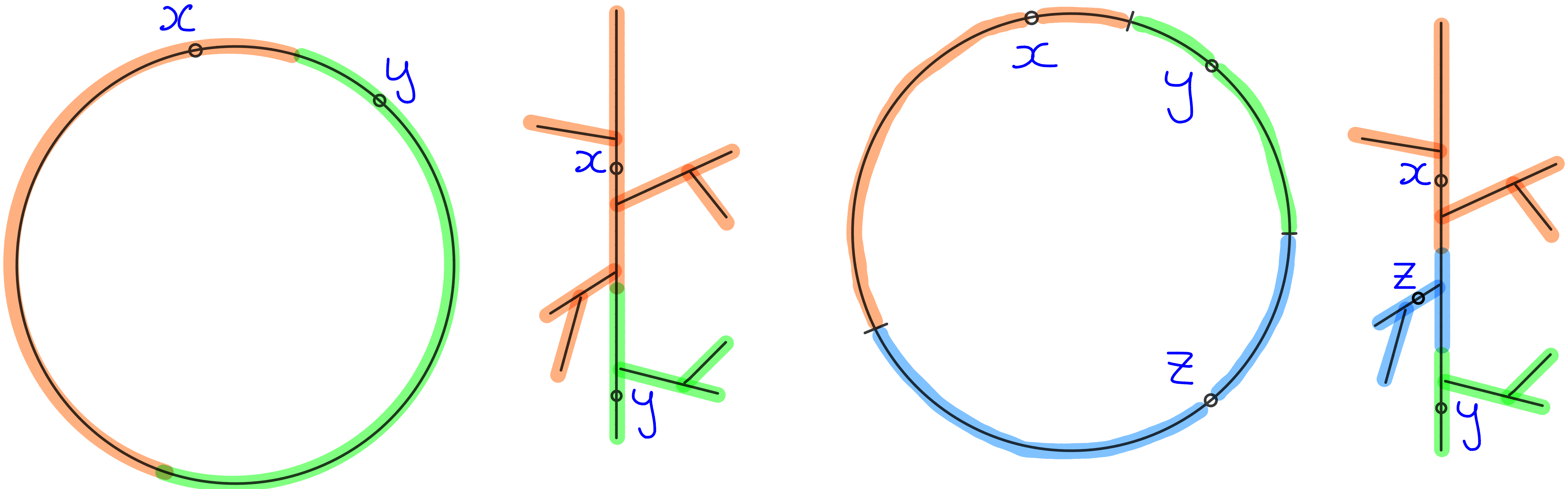

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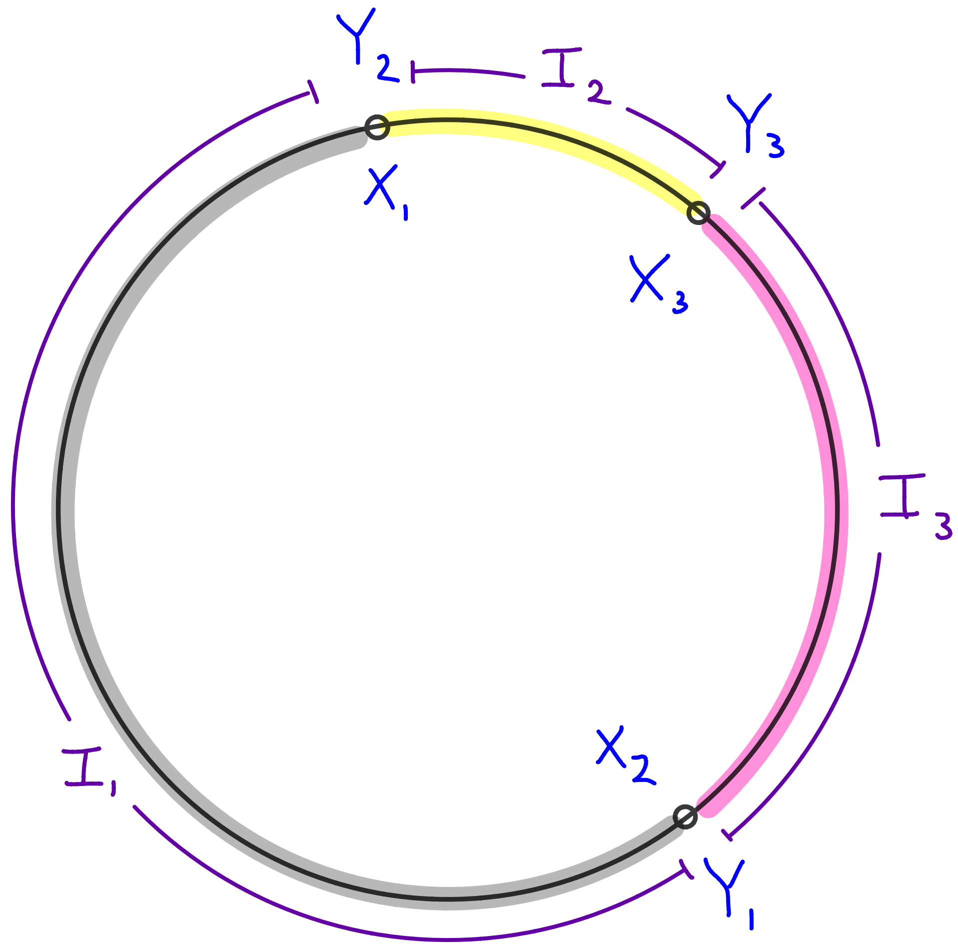
List in cyclic order as $Y_1, Y_2, \ldots, Y_n$.



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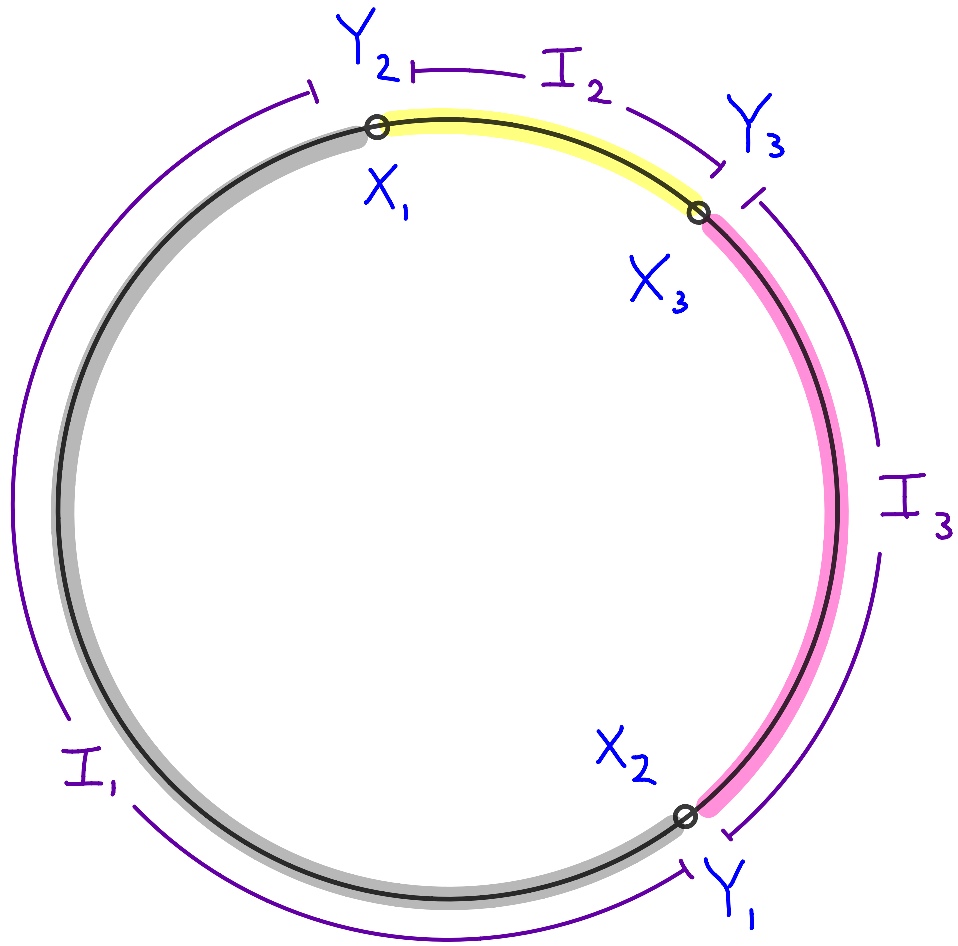

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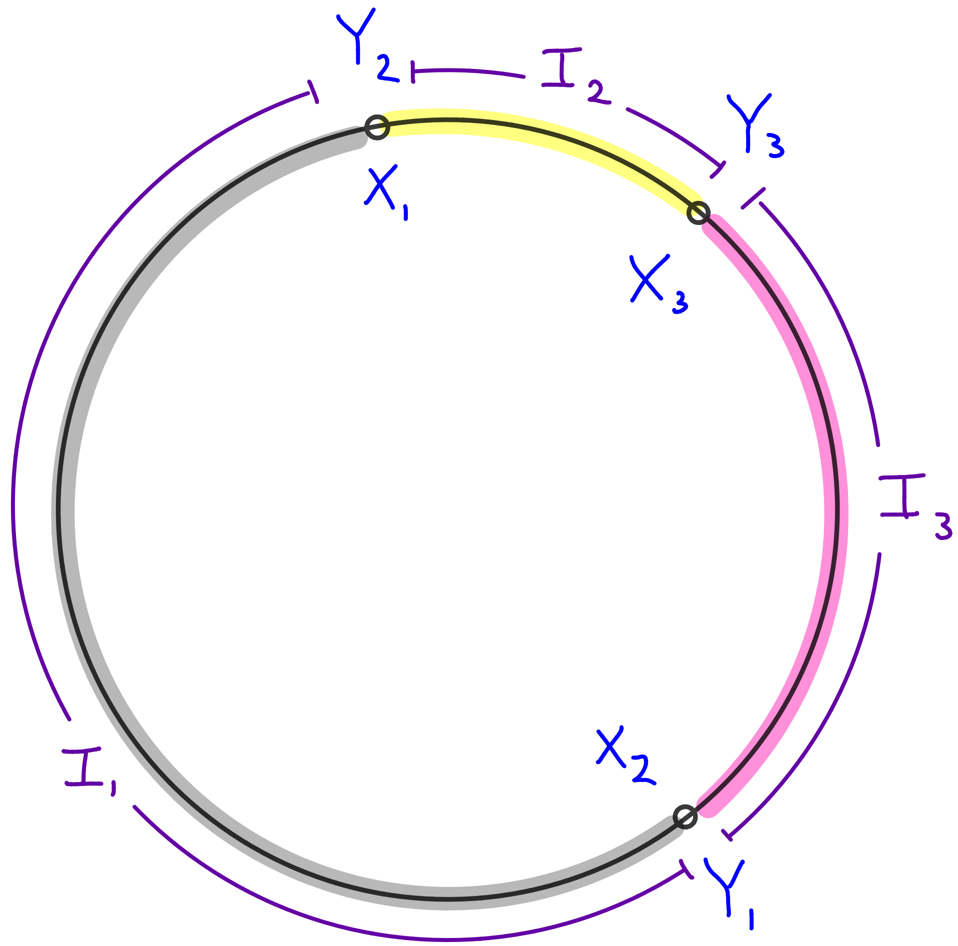

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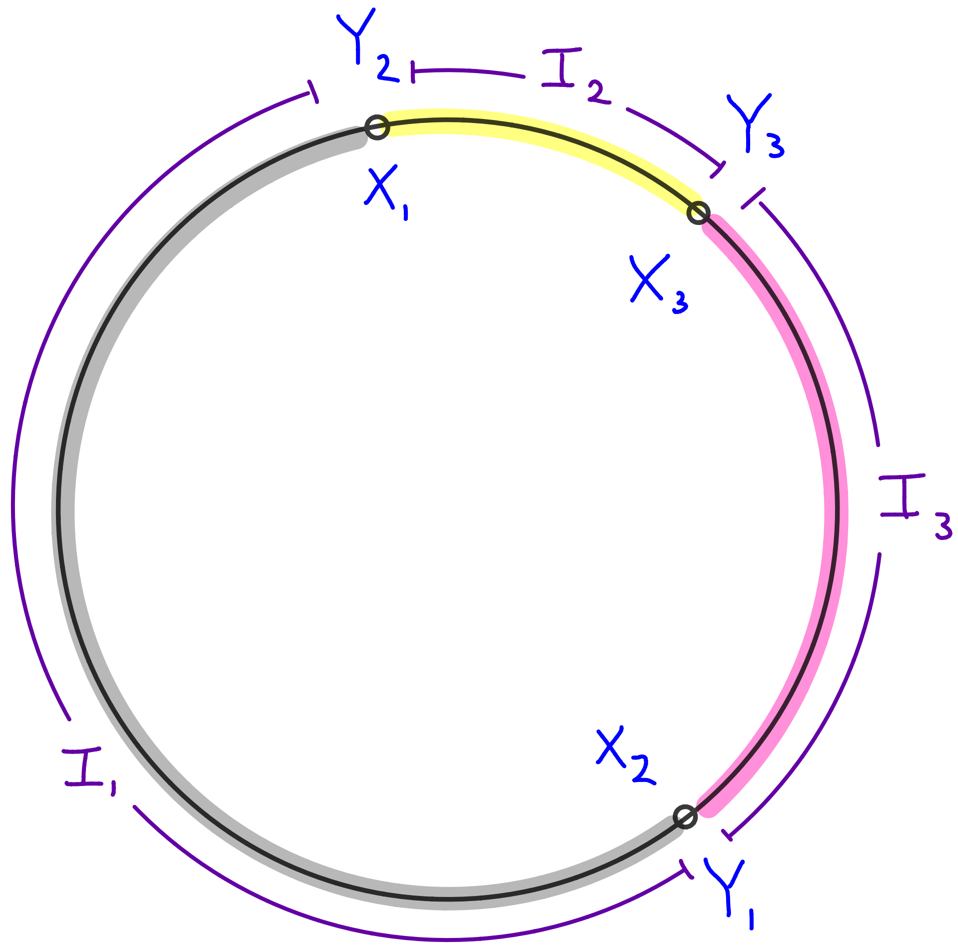


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$\underline{\underline{n=3}}$: for $\varepsilon < \frac{1}{2}$, $\mathbb{P}(V(Y_i) < \varepsilon) = (2\varepsilon)^2$, i.e., $V(Y_i) \sim \frac{1}{2} \cdot \mathrm{Beta}(2,1) \overset{d}{=} \frac{1}{2} \cdot \max(U_1, U_2)$.

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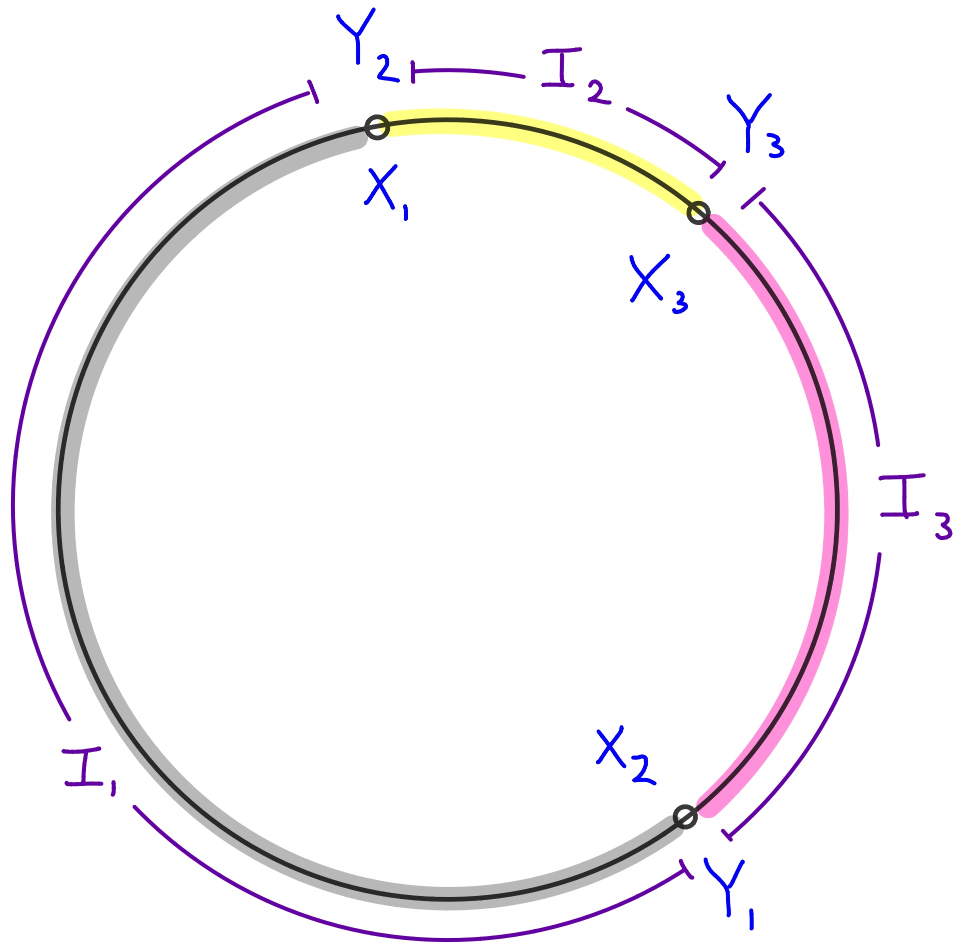


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Exchangeability gives $(V(X_i), 1 \leq i \leq n) \stackrel{d}{=} \left(\frac{1}{2}(I_i + I_{i+1}), 1 \leq i \leq n\right)$

**Theorem:** Let $S$ be any compact surface without boundary.

Let $M$ be a unicellular continuum random map on $S$.

Let $\{1, 2, \ldots, k\}$ be iid samples from the mass measure of $M$.

Then $(V(1), \ldots, V(k)) \sim \mathrm{Dir}(1, 1, \ldots, 1)$, i.e., this random vector

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⚠ Not the same thing as the Brownian map ⚠

$CRM(S,f)$ is the scaling limit of large $f$-faced random maps on $S$.

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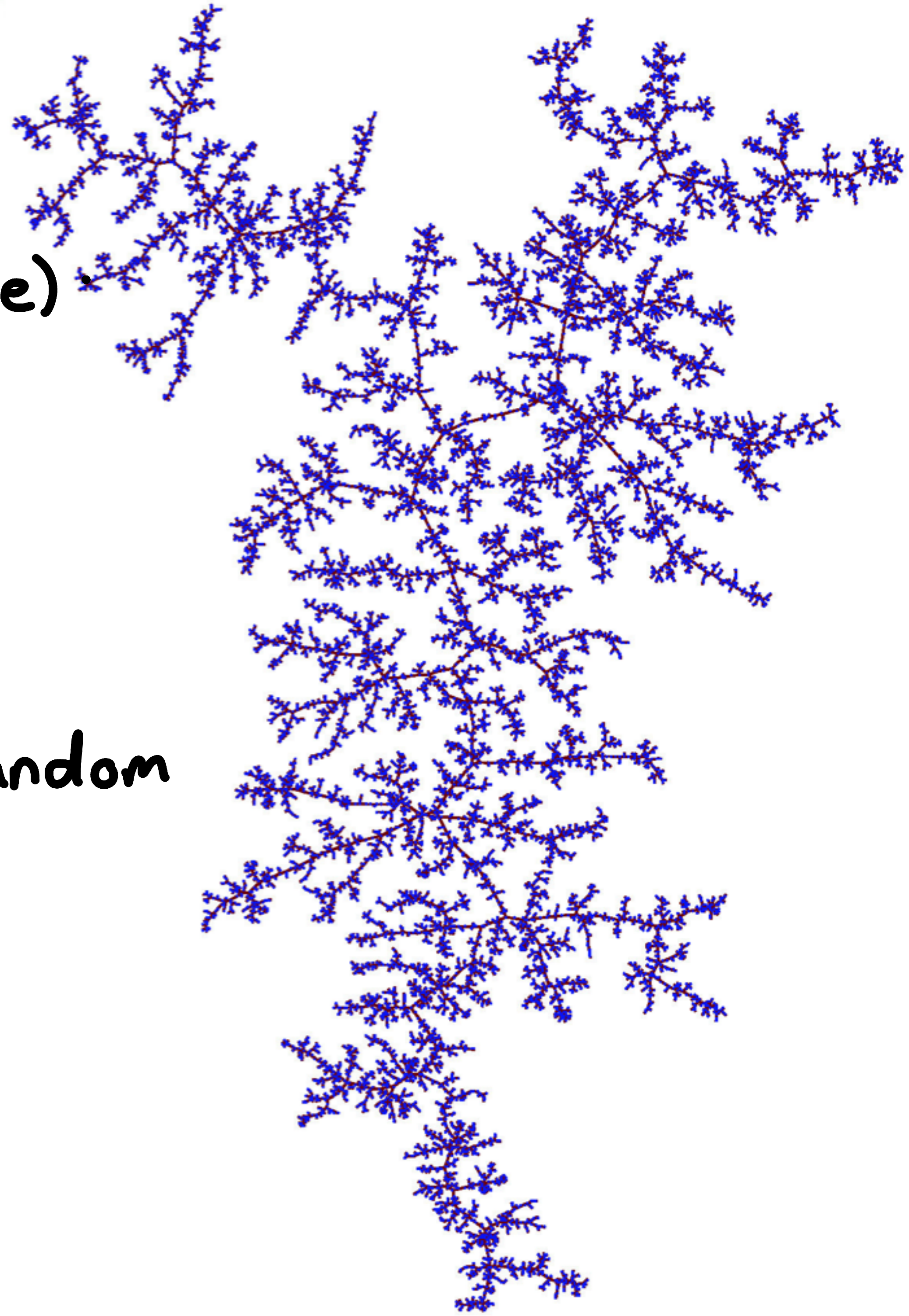

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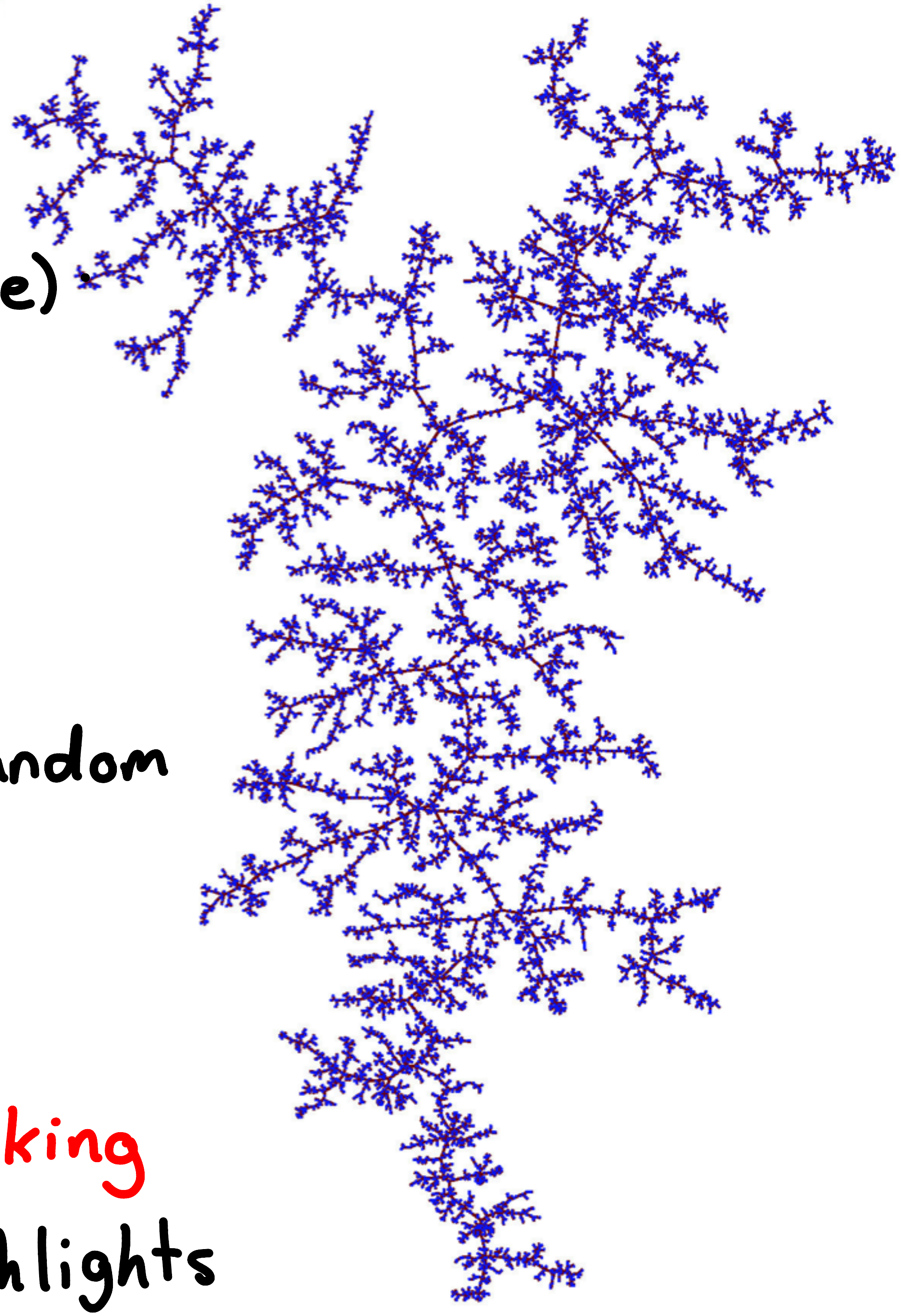

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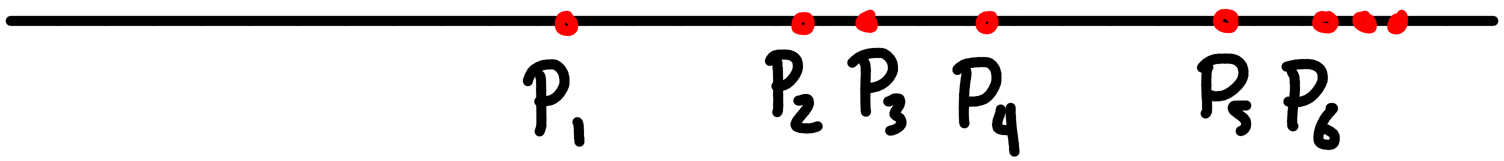
We next give the so-called stick-breaking construction of this object, as it highlights many useful properties.

# Stick-breaking constructions

Take an inhomogeneous Poisson process on $[0,\infty)$ with rate $\lambda(t)=t$.

Atoms $P_1, P_2, \ldots$



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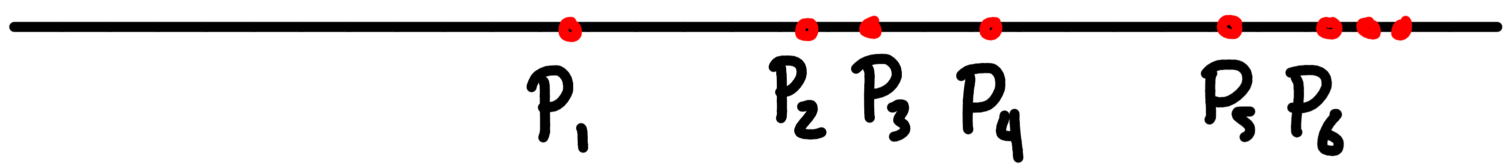


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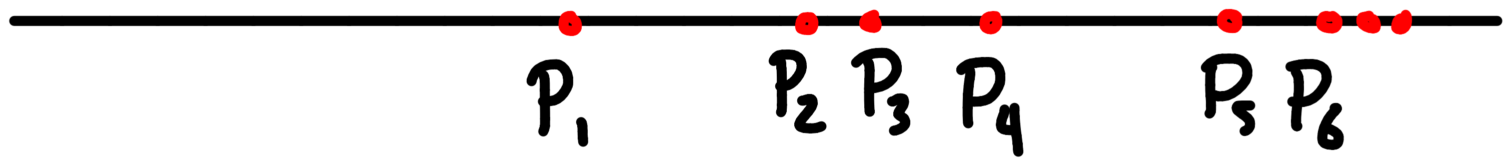


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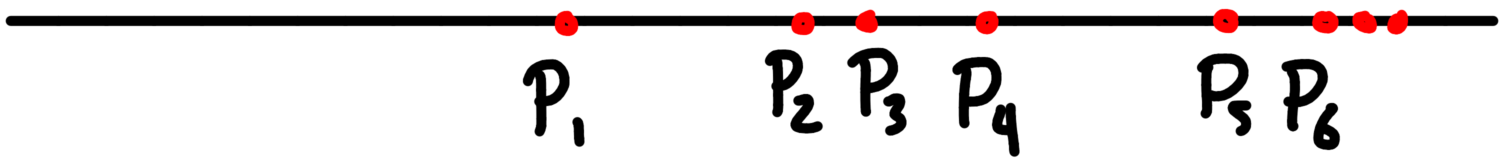


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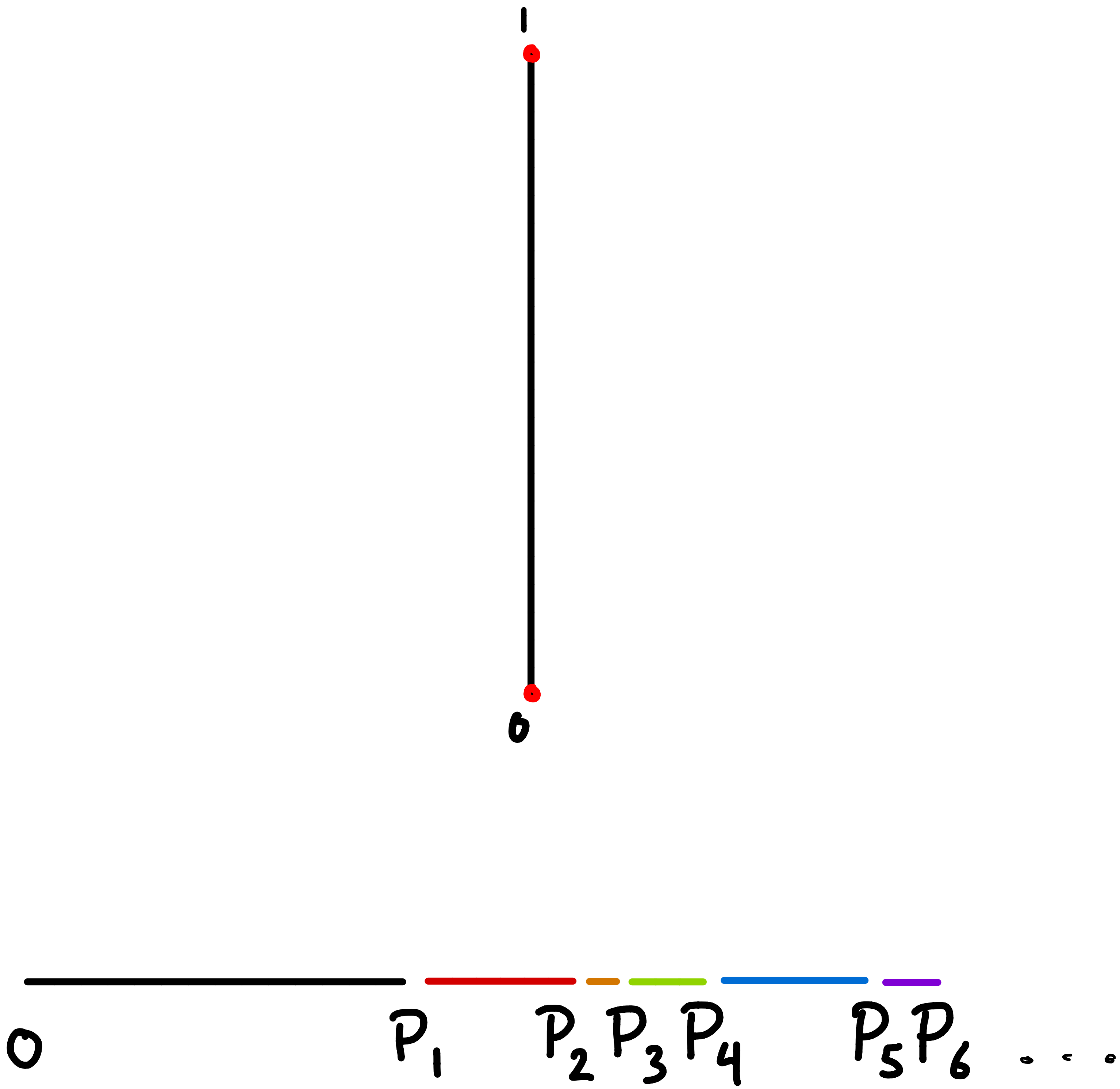

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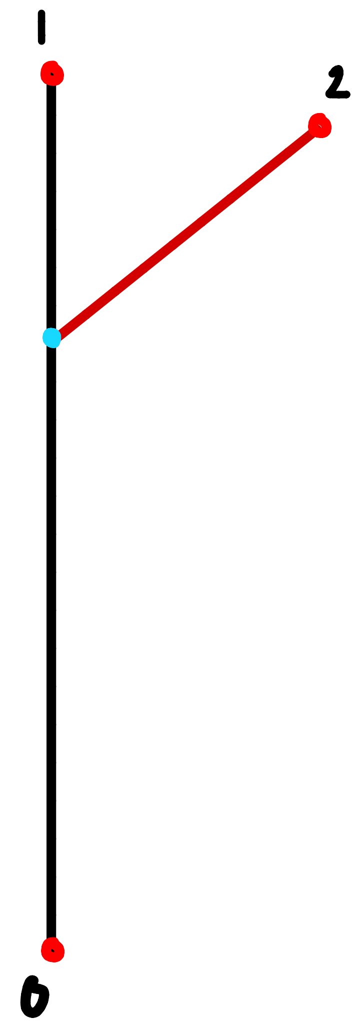


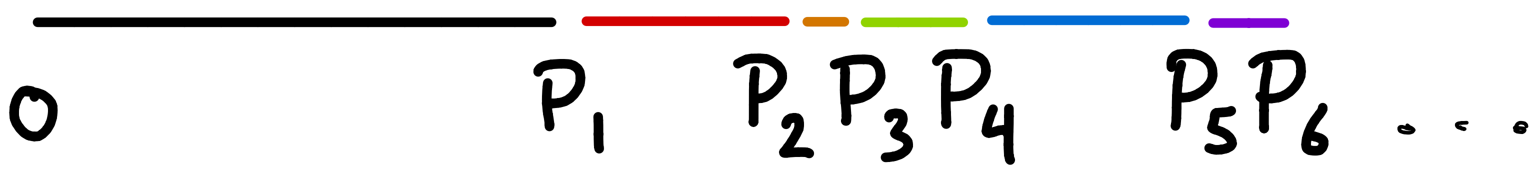

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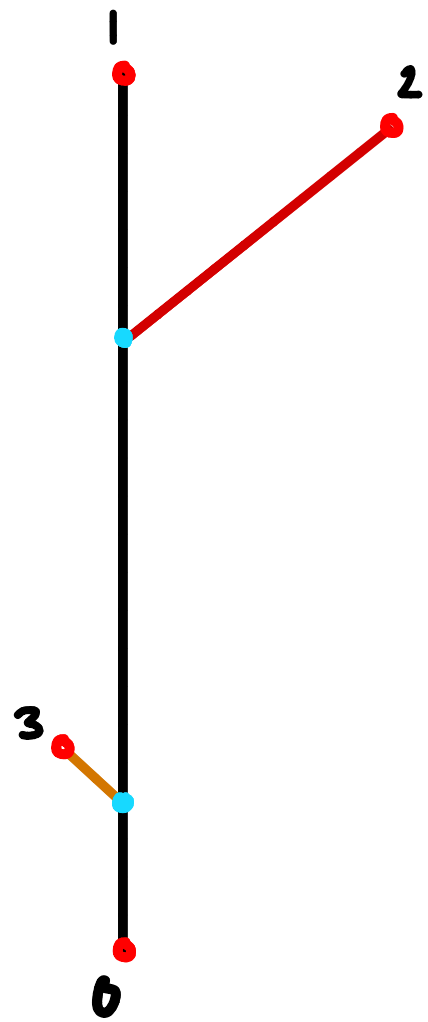


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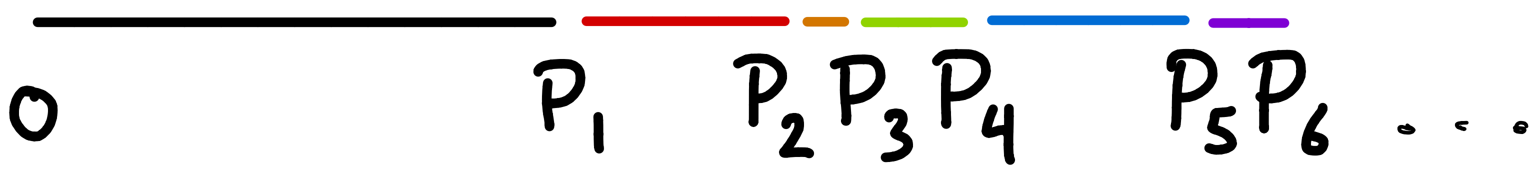

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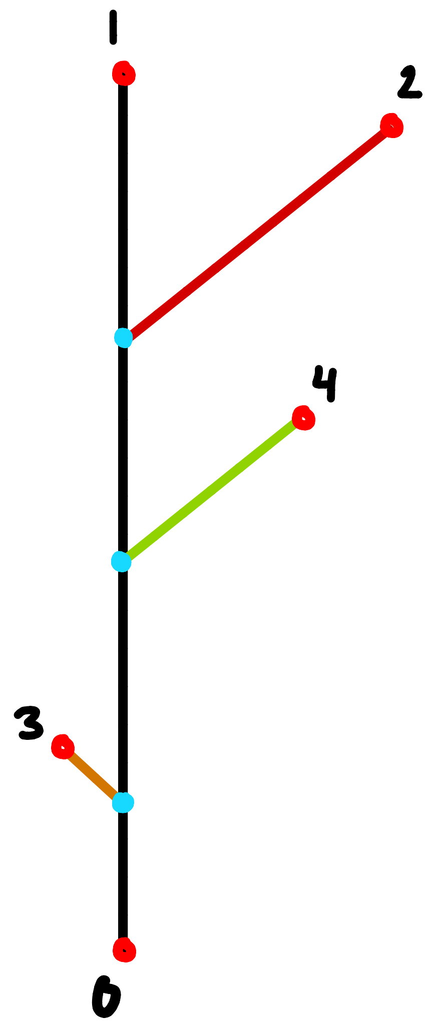


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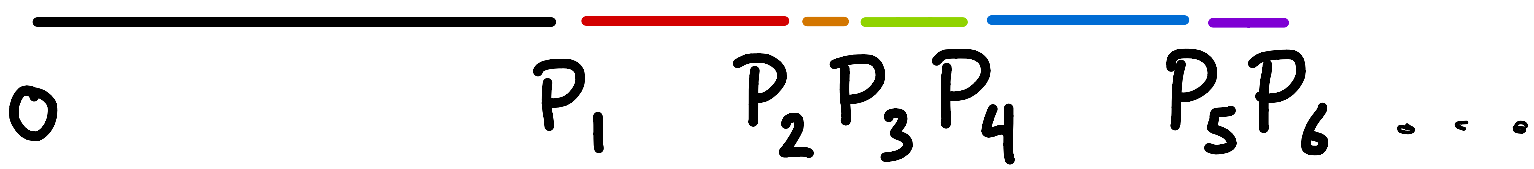

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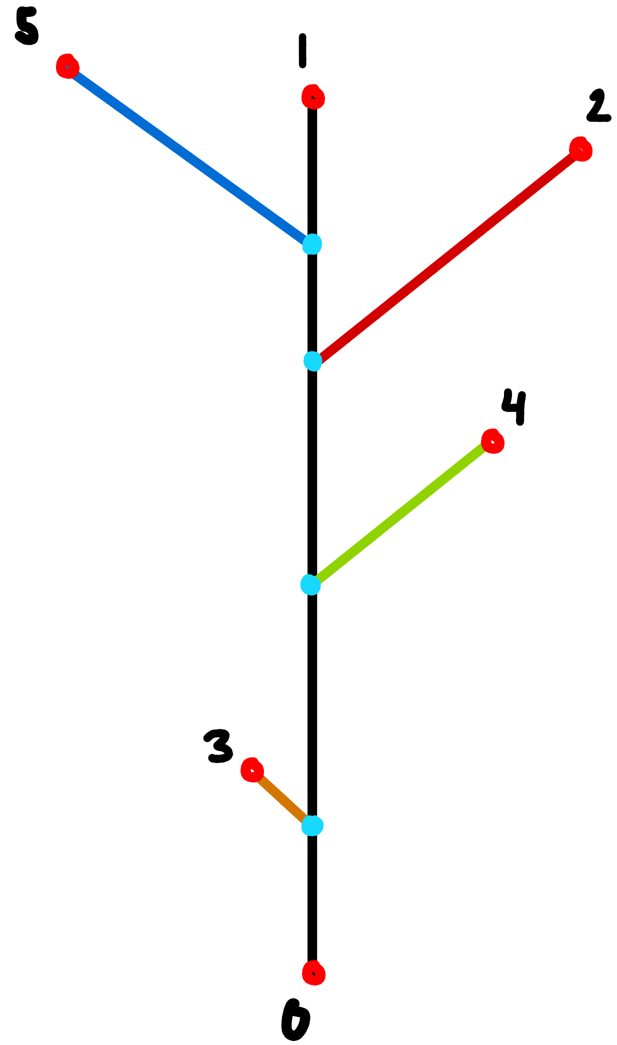


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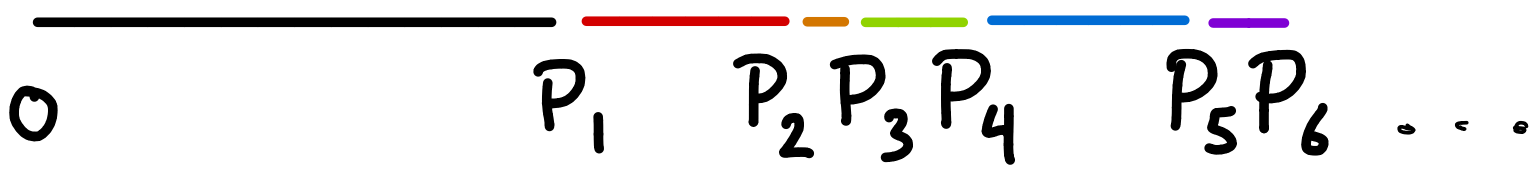

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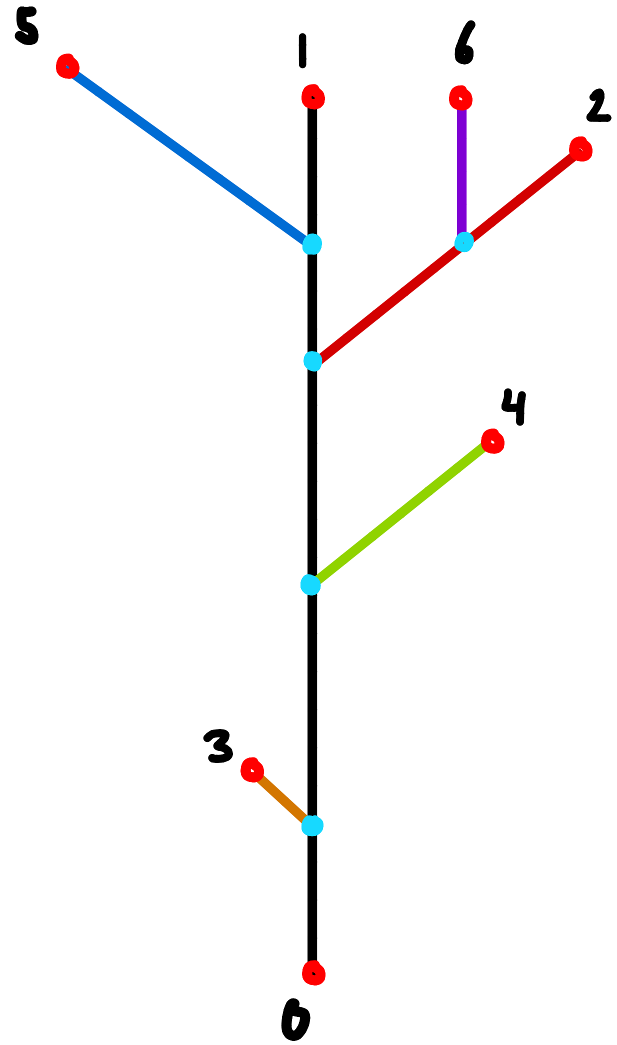


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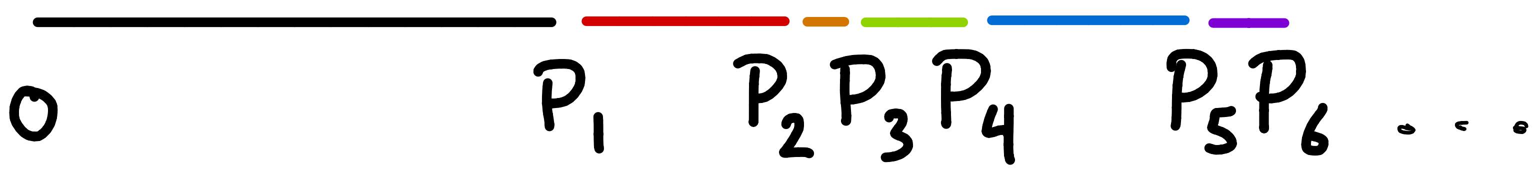

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- Start from $[0, P_1)$ and proceed as follows.
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- Start from $[0, P_1)$ and proceed as follows.
- For $i \geq 2$, attach $[P_{i-1}, P_i)$ at a uniformly random location on the existing tree $T_{i-1}$ to form $T_i$
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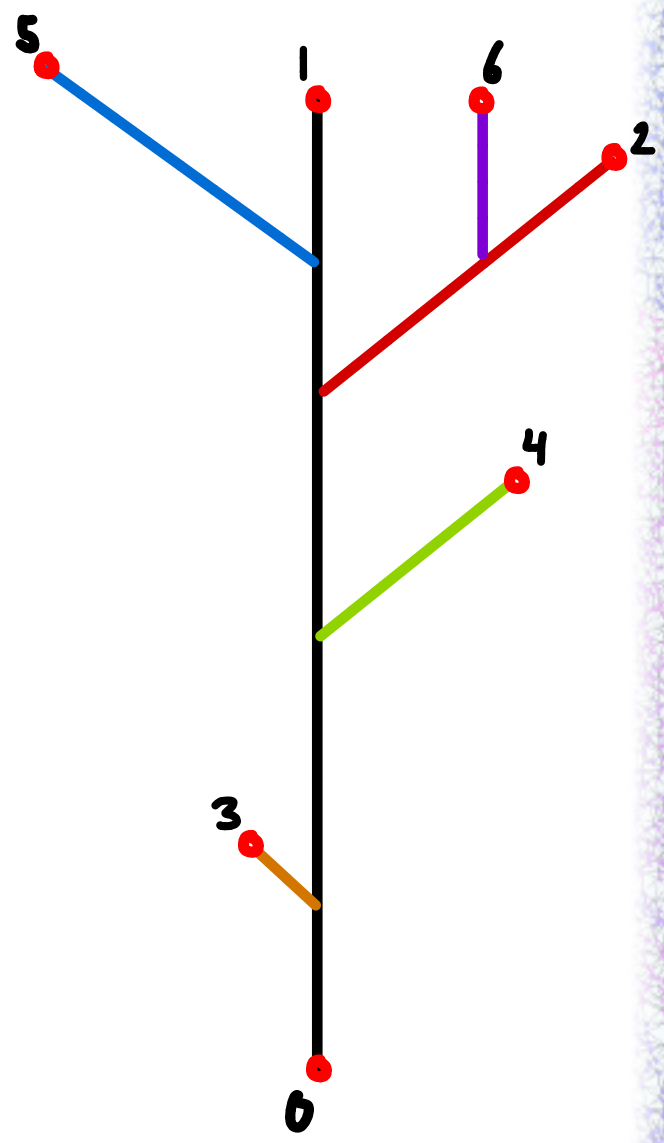
- $\mathcal{T}$ is naturally endowed with a probability measure $\mu$ which is the limit of the empirical leaf measures.

  Empirical leaf measure on $T_n$ is $\mu_n = \frac{1}{n+1} \sum_{i=0}^{n} \delta_i$.

# Properties of the construction



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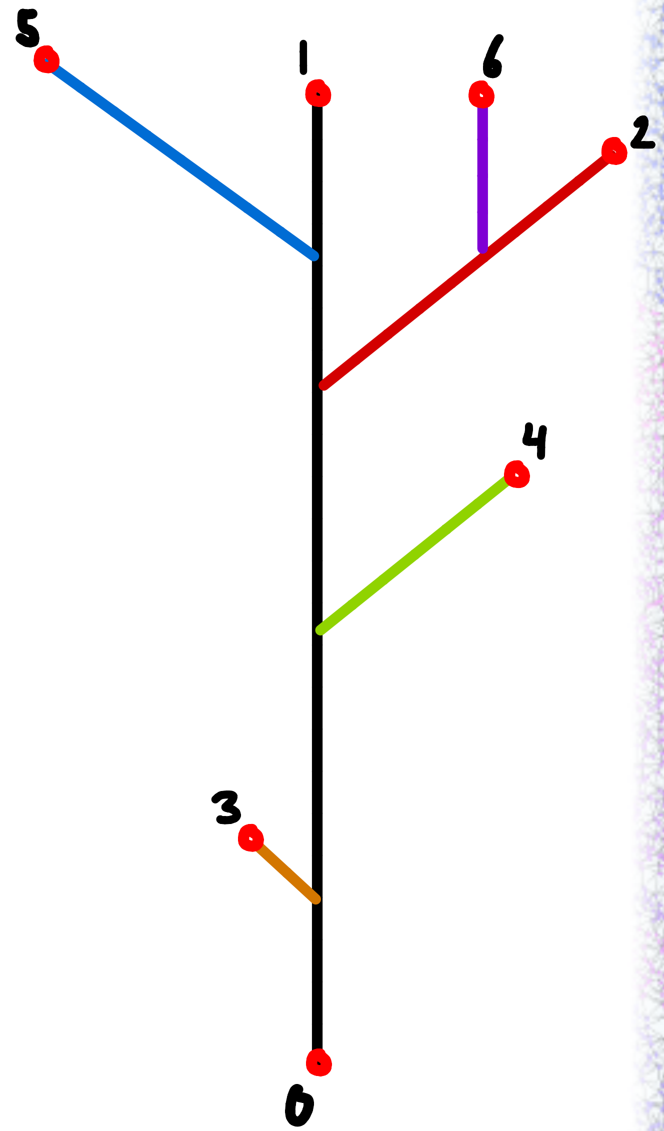

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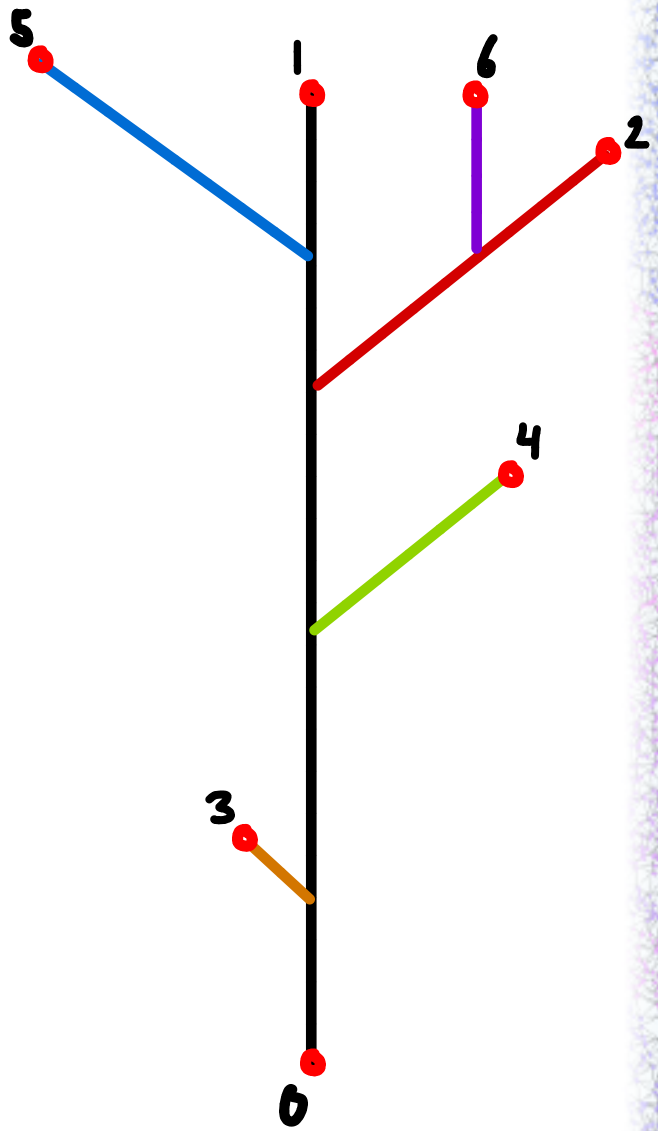
The branch  $(n+1)$ attaches to a uniform side of a uniform edge in $T_n$, to form $T_{n+1}$.



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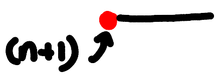
The branch 

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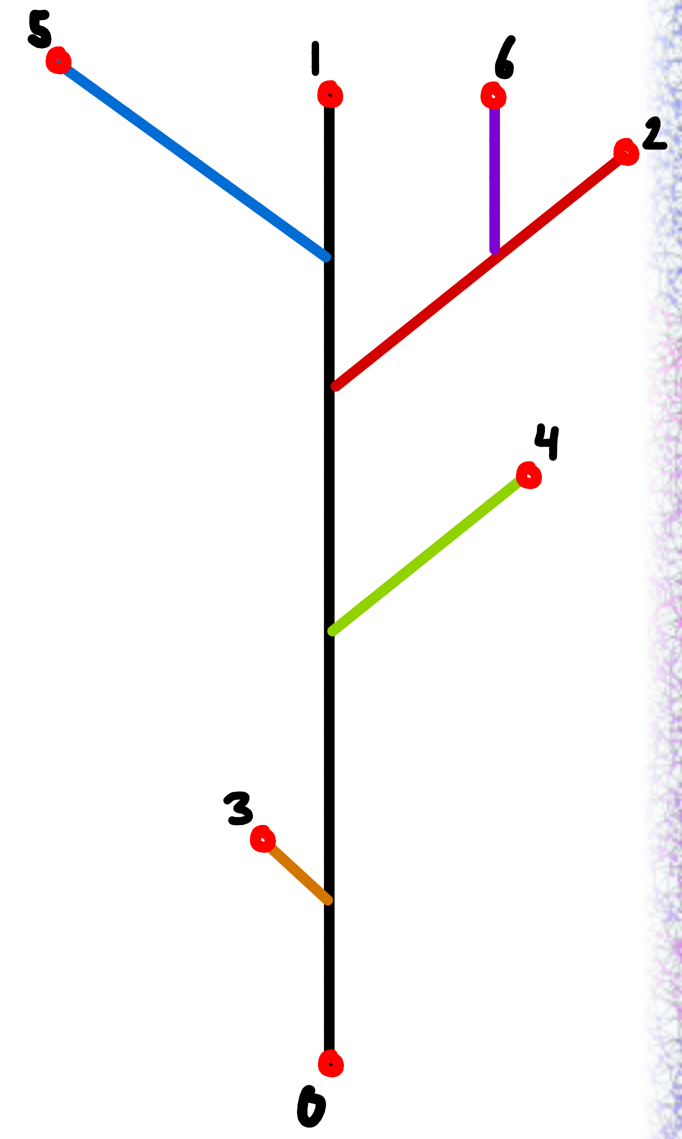
The left-right choices we make during the stick-breaking construction endow the leaves of $T_n$ with a cyclic (clockwise) order

$0<5<1;\ 1<6<4;\ 4<3<6$



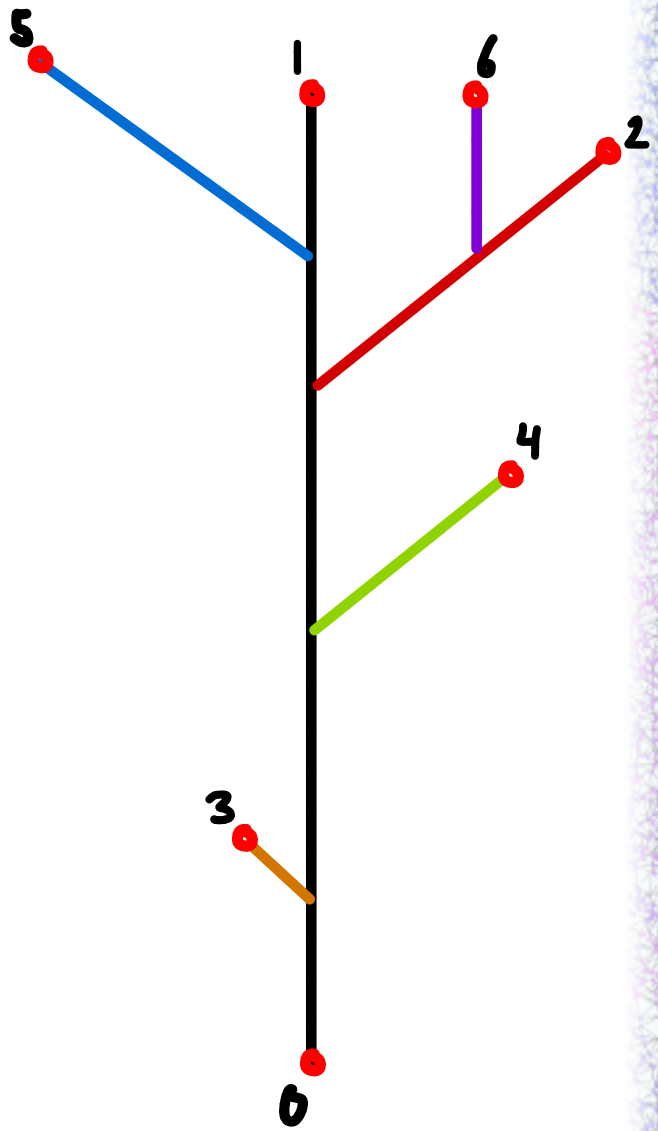
# Properties of the construction

Let $T_n$ be the combinatorial tree obtained by ignoring edge lengths in $\mathcal{T}_n$.

Then $(T_n, n \geq 0)$ is a Markov process called Rémy's algorithm.

The branch  $(n+1)$ attaches to a uniform side of a uniform edge in $T_n$, to form $T_{n+1}$.

The left-right choices we make during the stick-breaking construction endow the leaves of $\mathcal{T}_n$ with a cyclic (clockwise) order




**With respect to this order, the leaf labels in $\mathcal{T}_n$ are a uniformly random cyclic permutation of $\{0, 1, \ldots, n\}$, independent of the shape of $T_n$.**

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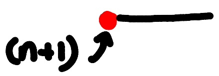
The branch 

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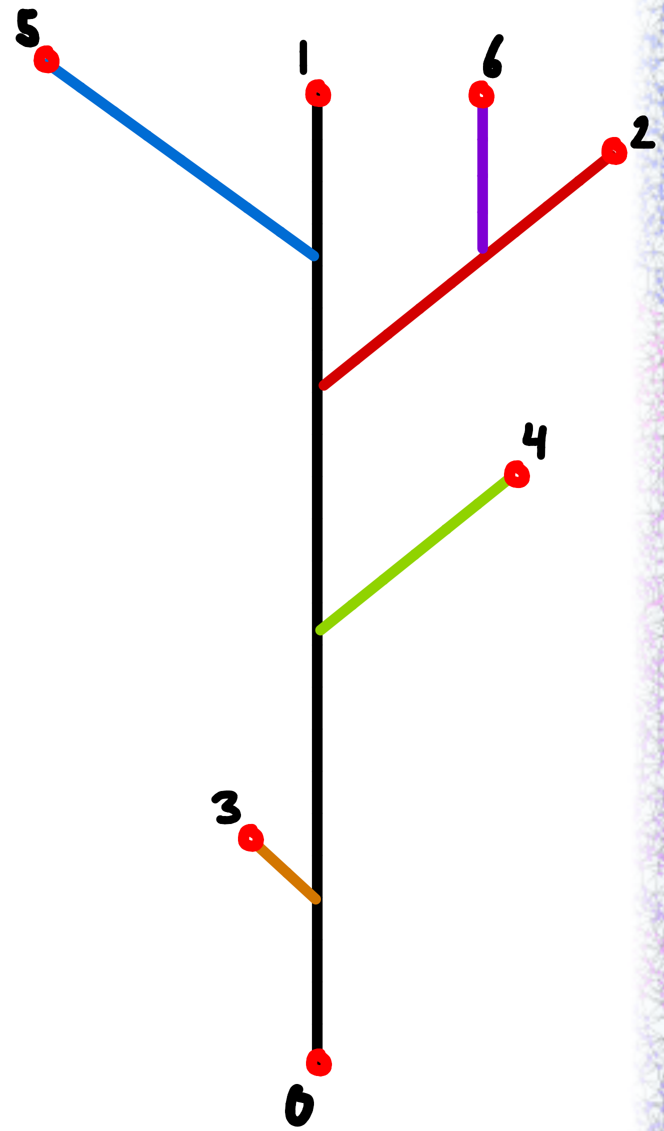


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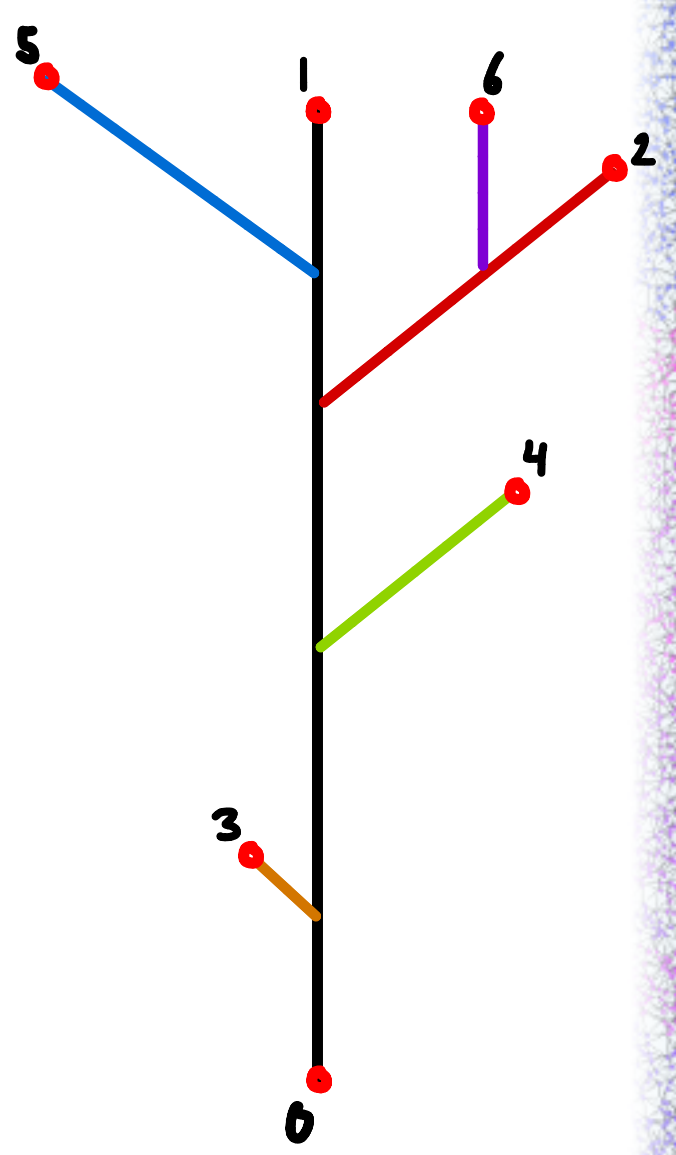


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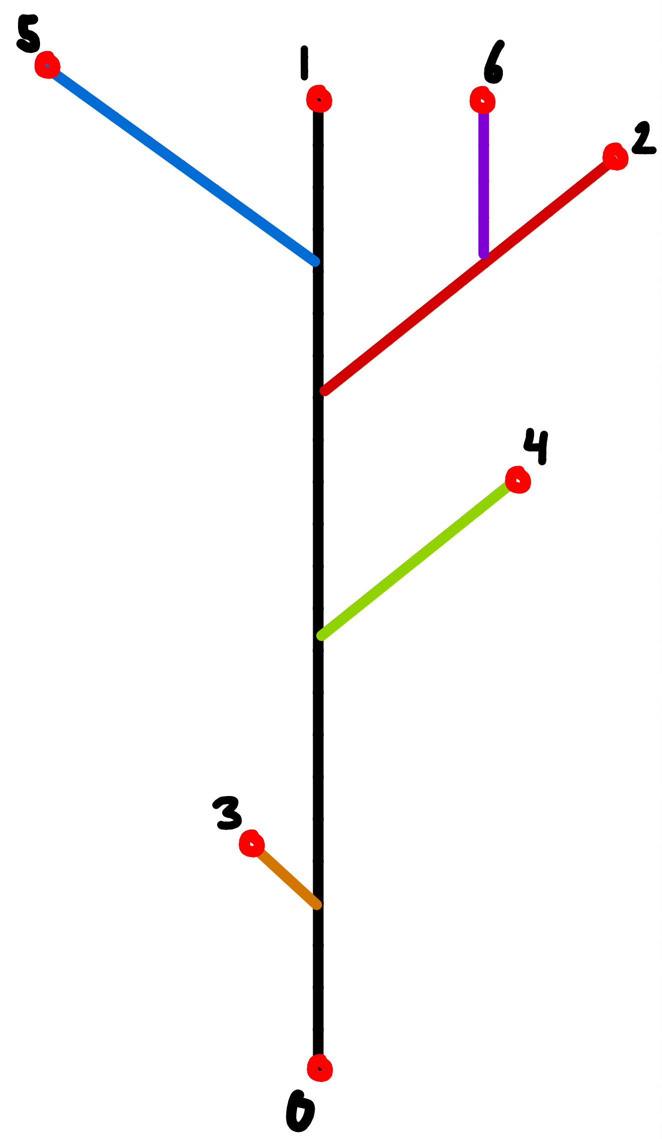
(Aside: The cyclic order extends naturally to the leaves of the CRT $\mathcal{T}$.)

# Properties of the construction

- Fix $k \in \mathbb{N}$, let $n$ be large.



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- Fix $k \in \mathbb{N}$, let $n$ be large.
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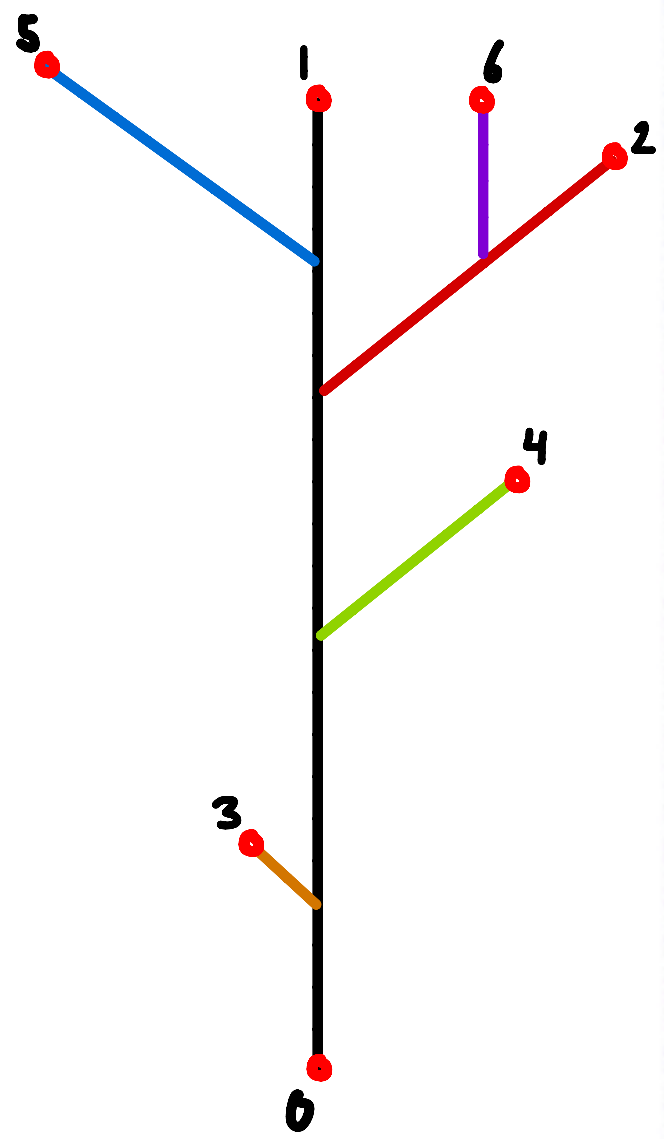

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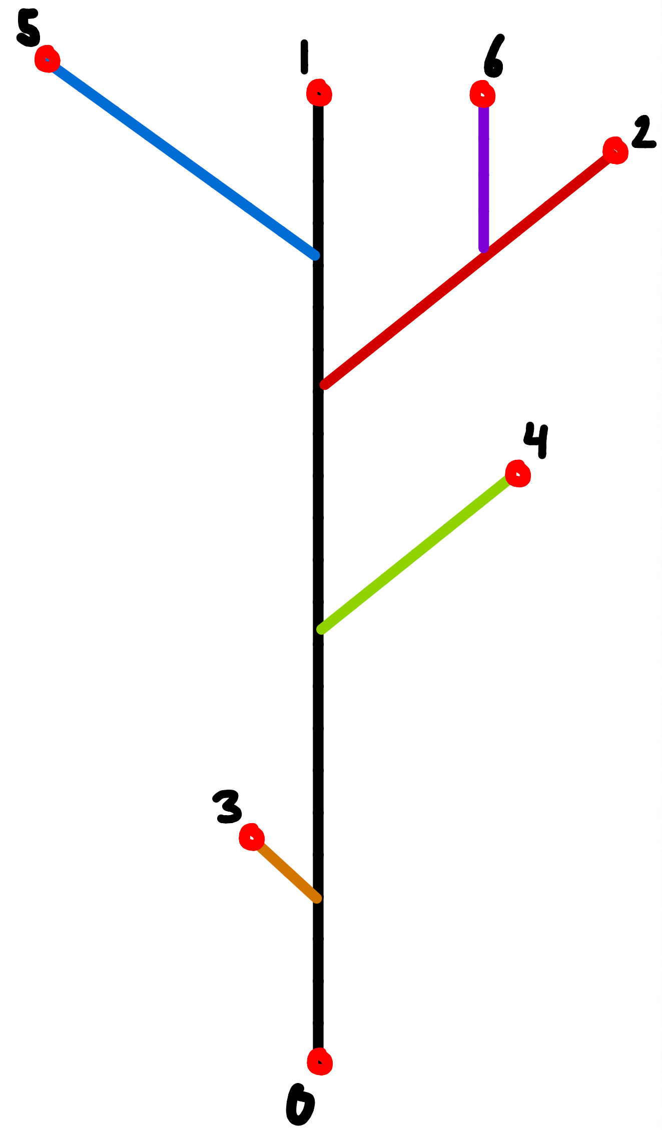

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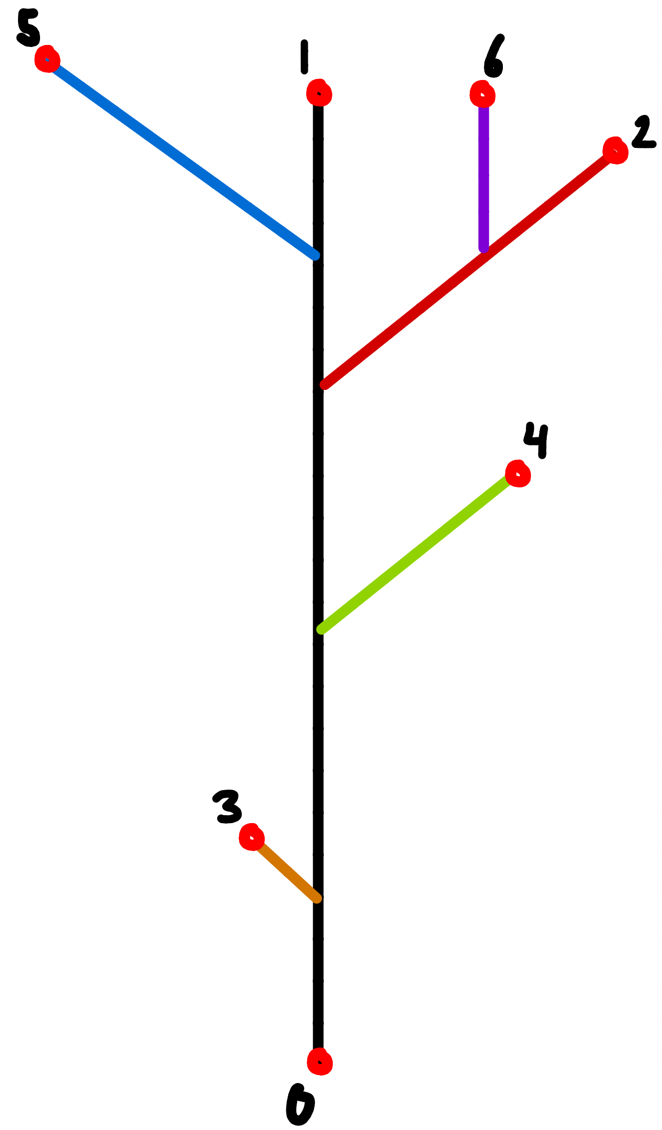
- Leaf labels a u. permutation $\Rightarrow$

$$\left( \frac{i(0)}{n}, \frac{i(1)}{n}, \dots, \frac{i(k)}{n} \right) \xrightarrow[n\to\infty]{d}$$ $k$ independent Uniform $[0,1]$ random variables



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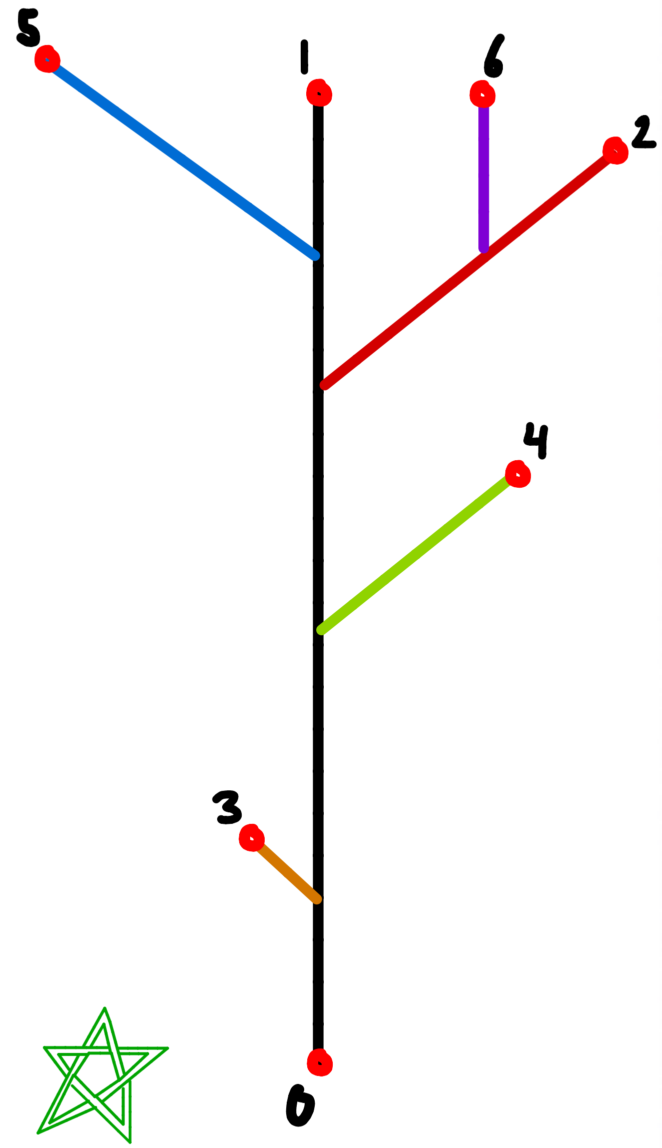
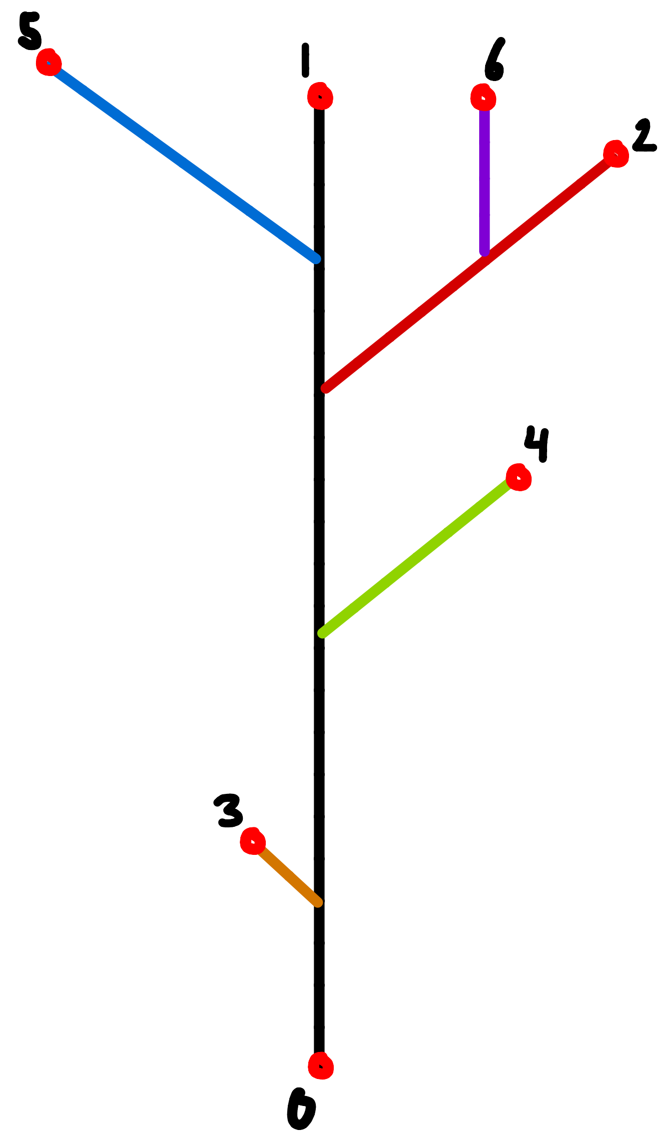



So for any $k$, the sequence $(0, 1, \dots, k)$ of elements of $\mathcal{T}$ are IID samples from $\mu$

**Theorem:** (2 Voronoi cells in the CRT)

$(V(0), V(1)) \overset{d}{=} (U, 1-U)$ where $U \sim \text{Uniform}[0,1]$



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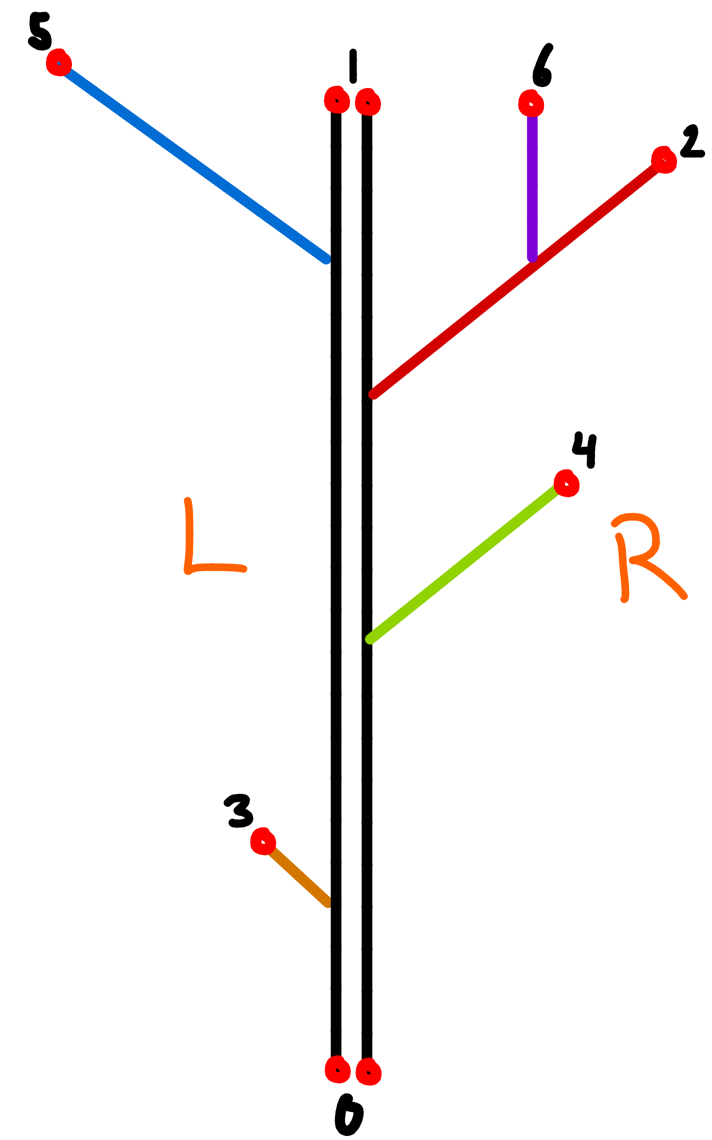

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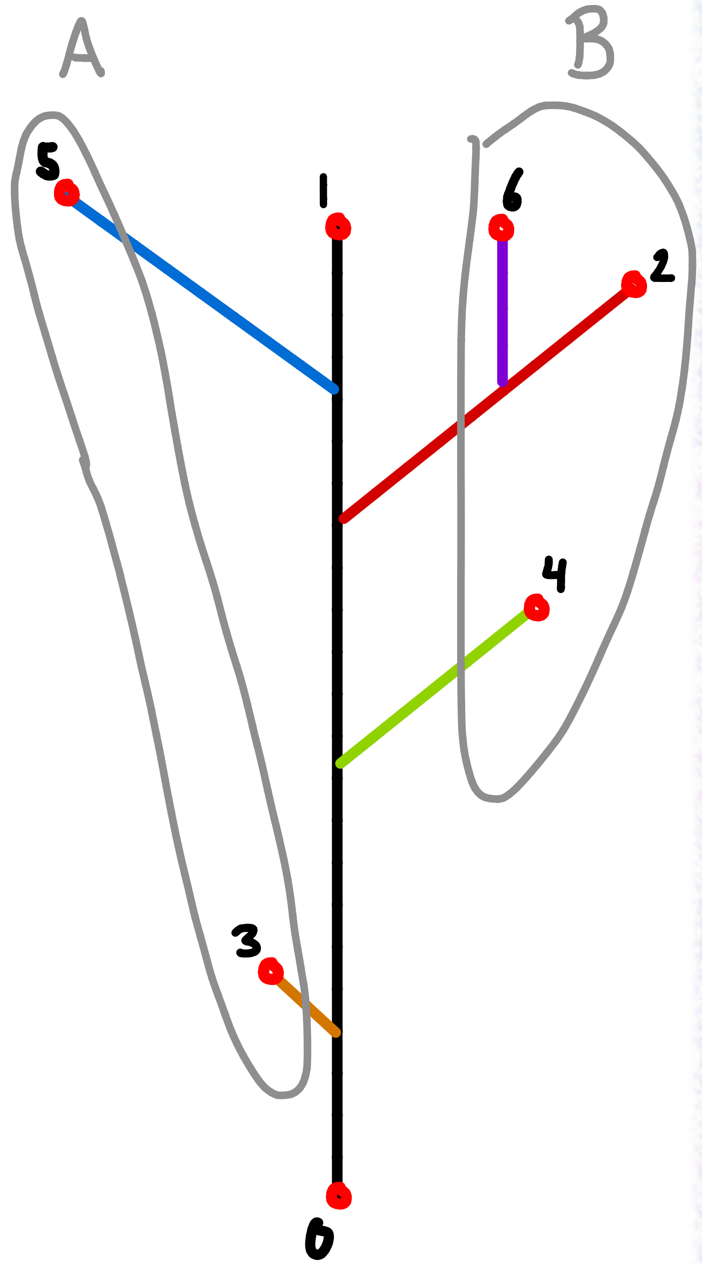

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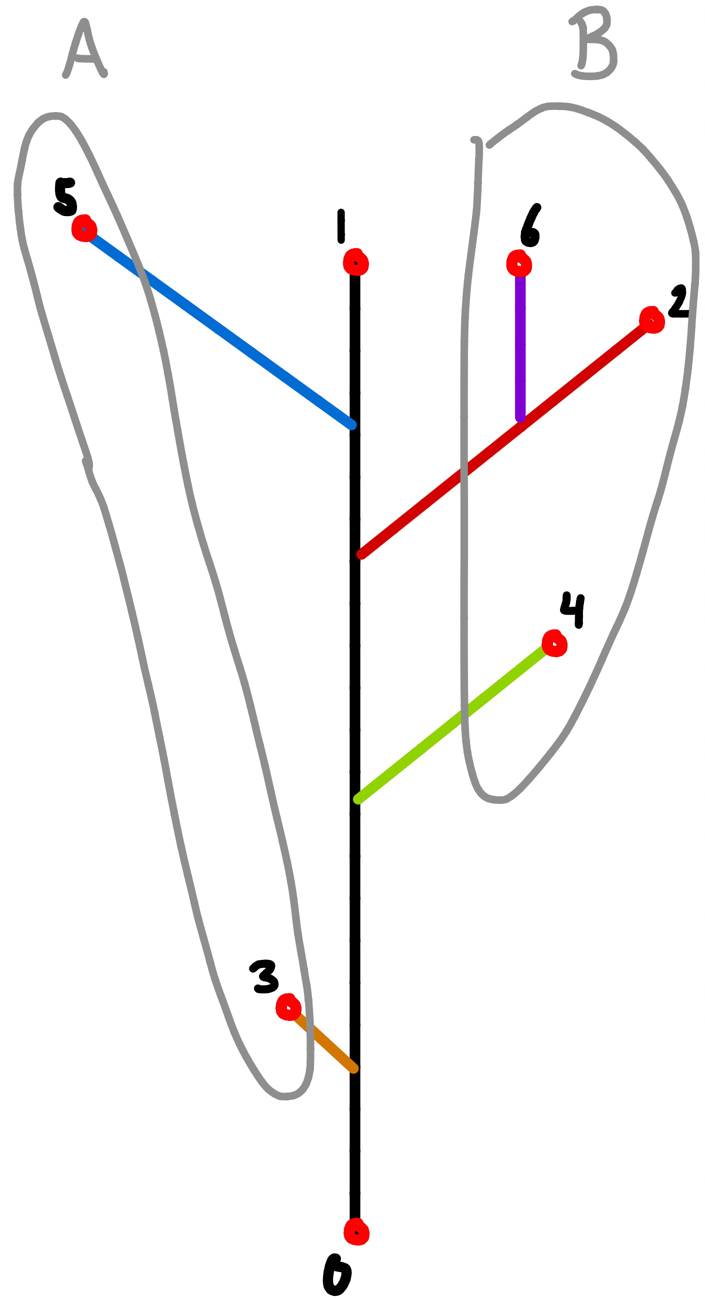
Call these "cyclic intervals" around $\mathcal{T}$.



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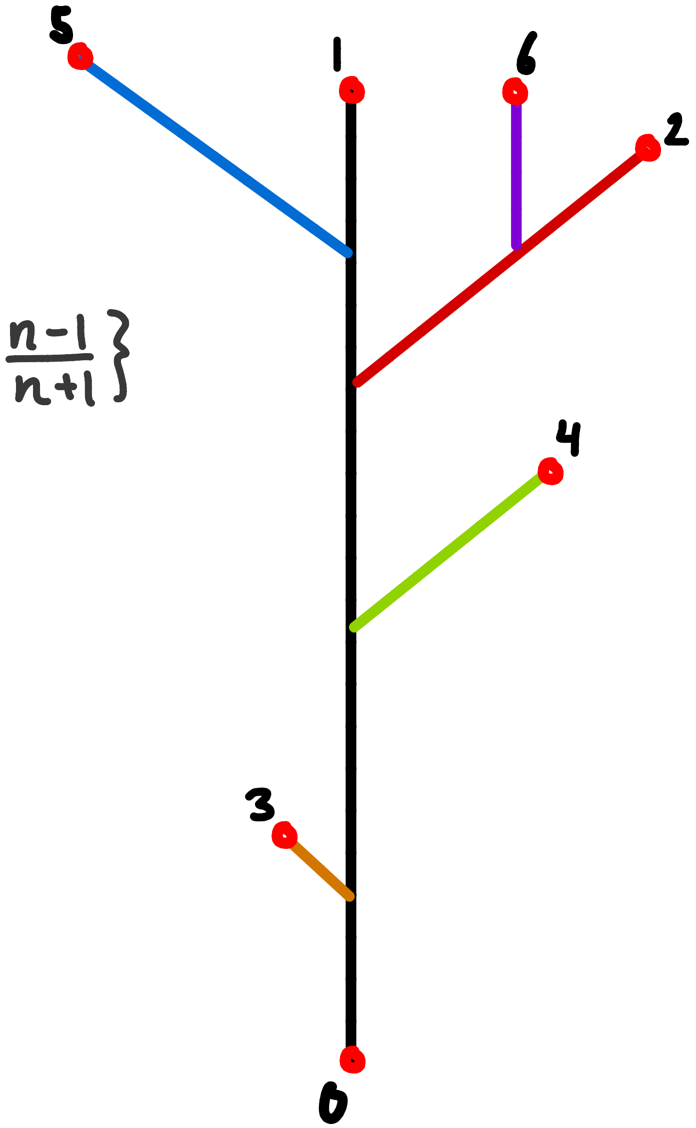


Then

$$\mu_n(A) = \frac{1}{n+1} \cdot \#\{i : 0 \preccurlyeq i \preccurlyeq 1 \text{ in } T_n\} \sim \text{Uniform}\left\{\frac{0}{n+1}, \frac{1}{n+1}, \dots, \frac{n-1}{n+1}\right\}$$

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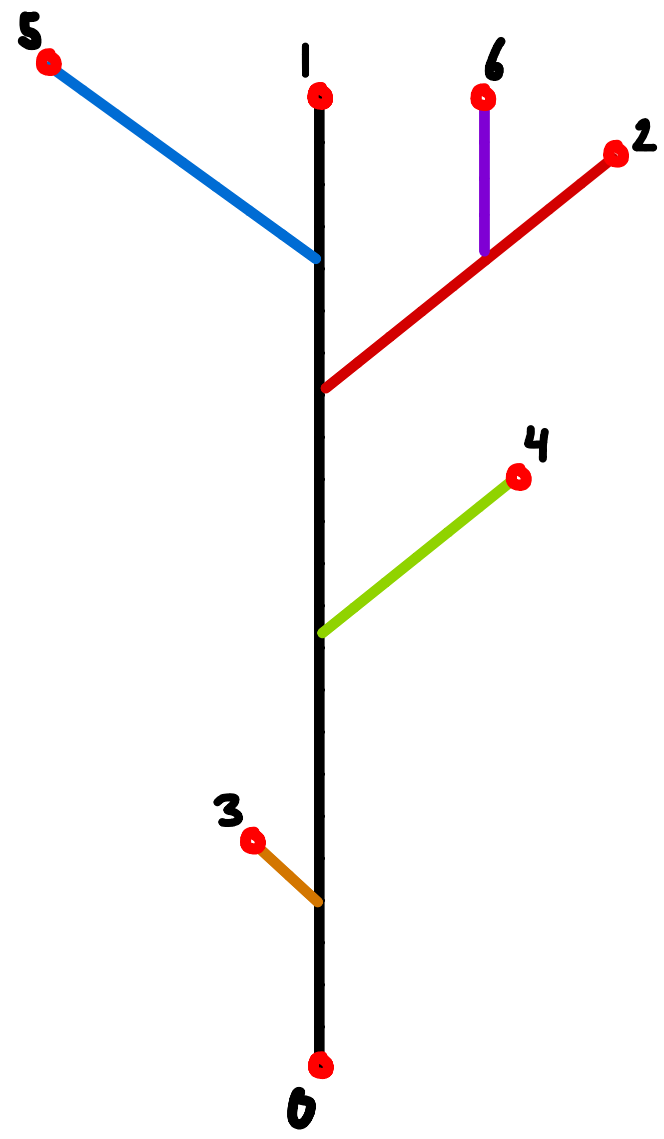
Since $\mu_n \to \mu$, get $(\mu(A), \mu(B)) \overset{d}{=} (U, 1-U)$



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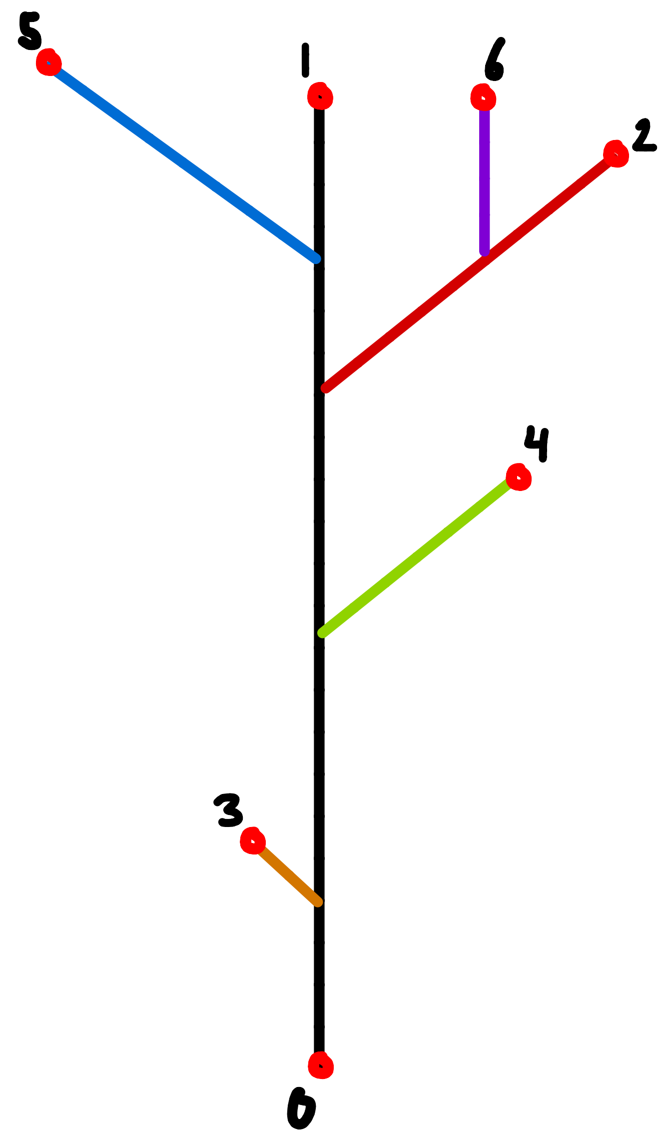


So can prove the theorem by showing Voronoi cells & cyclic intervals have the same distribution.

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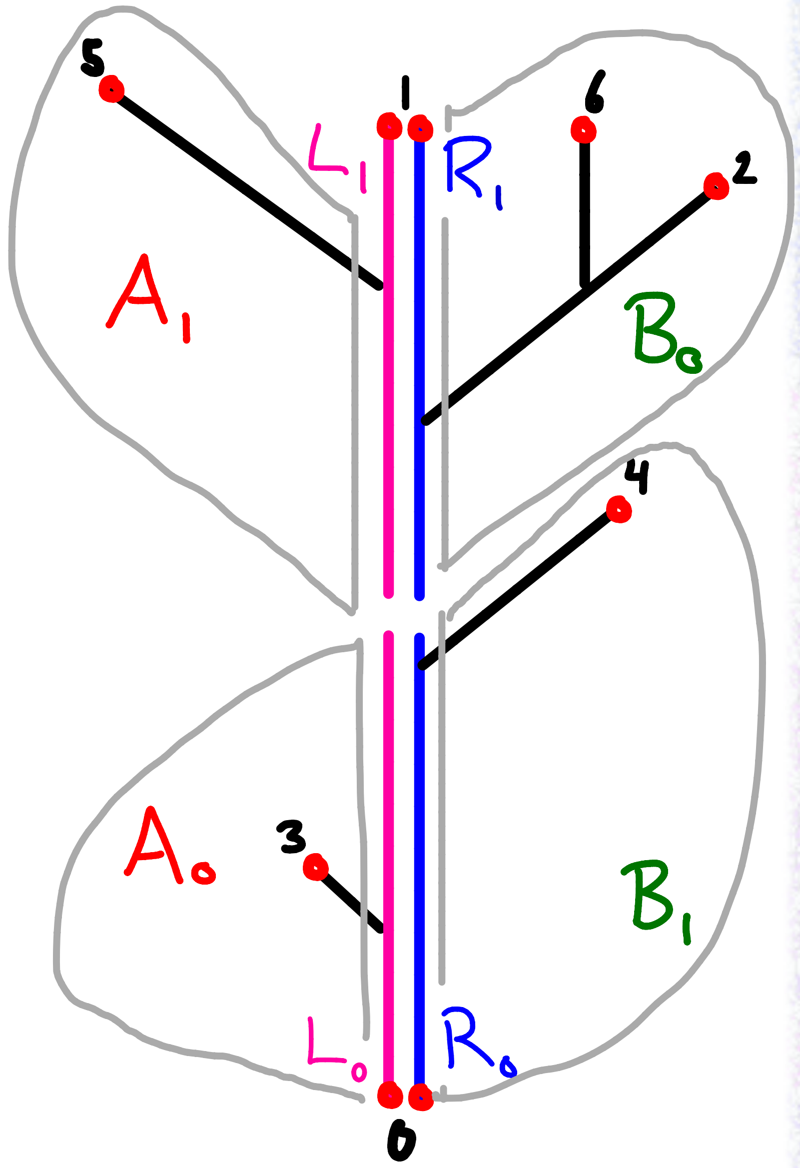


Divide the two sides of $[[0,1]]$ at the midpoint to obtain $L_0, L_1, R_0, R_1$

Define $A_0, A_1, B_0, B_1$ accordingly.

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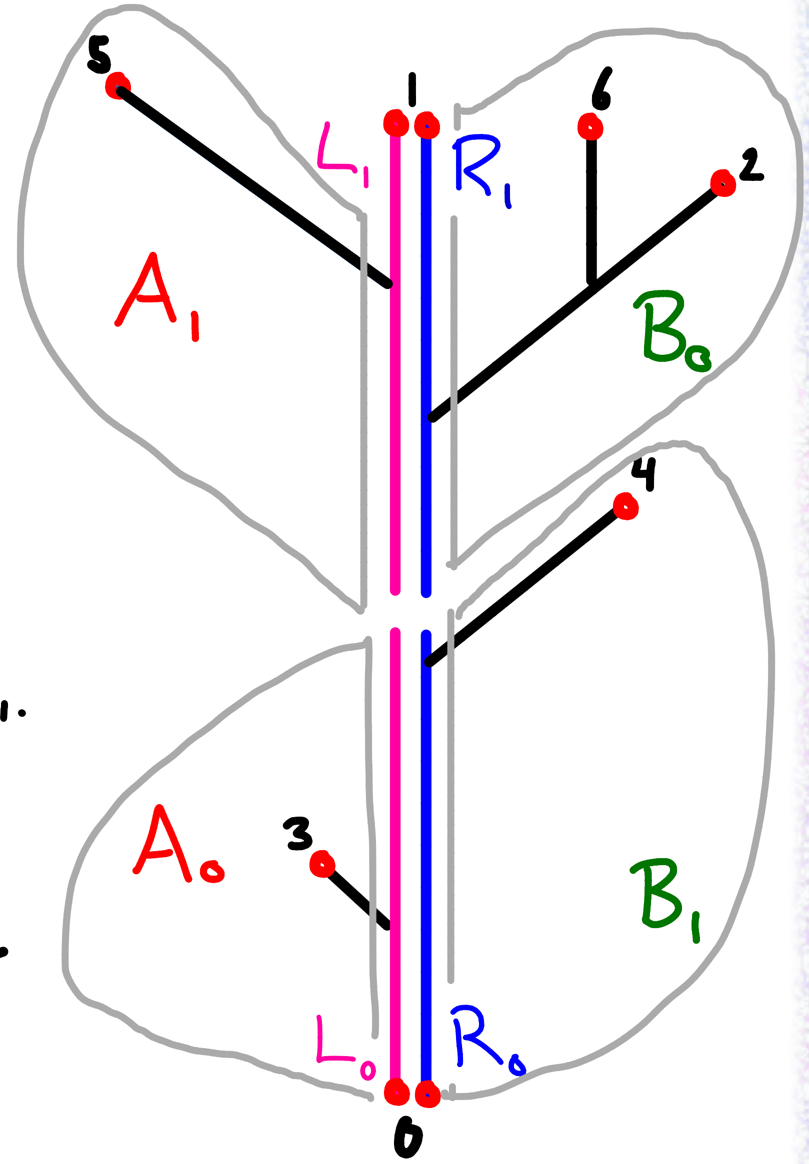


**By stick-breaking construction, immediate that $(\mu(A_0), \mu(A_1), \mu(B_0), \mu(B_1))$ exchangeable.**

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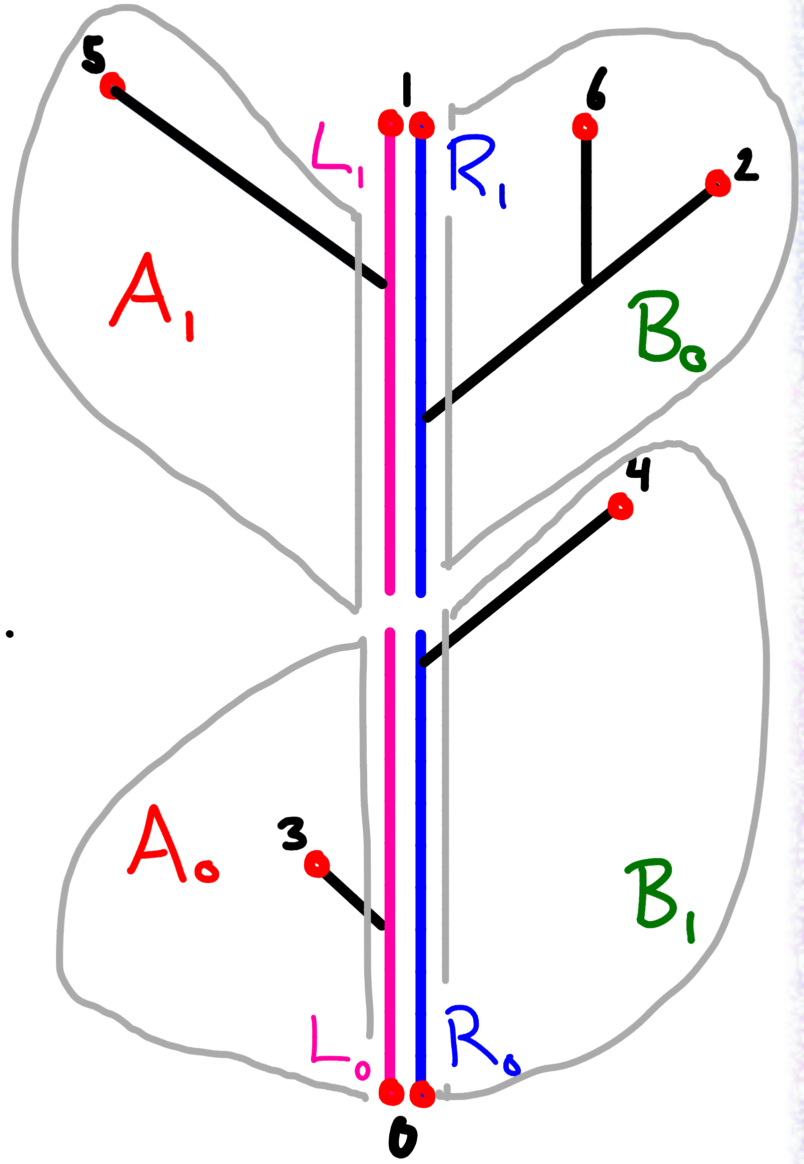


<u>Thus</u> $(\mu(A), \mu(B)) = (\mu(A_0) + \mu(A_1), \mu(B_0) + \mu(B_1))$

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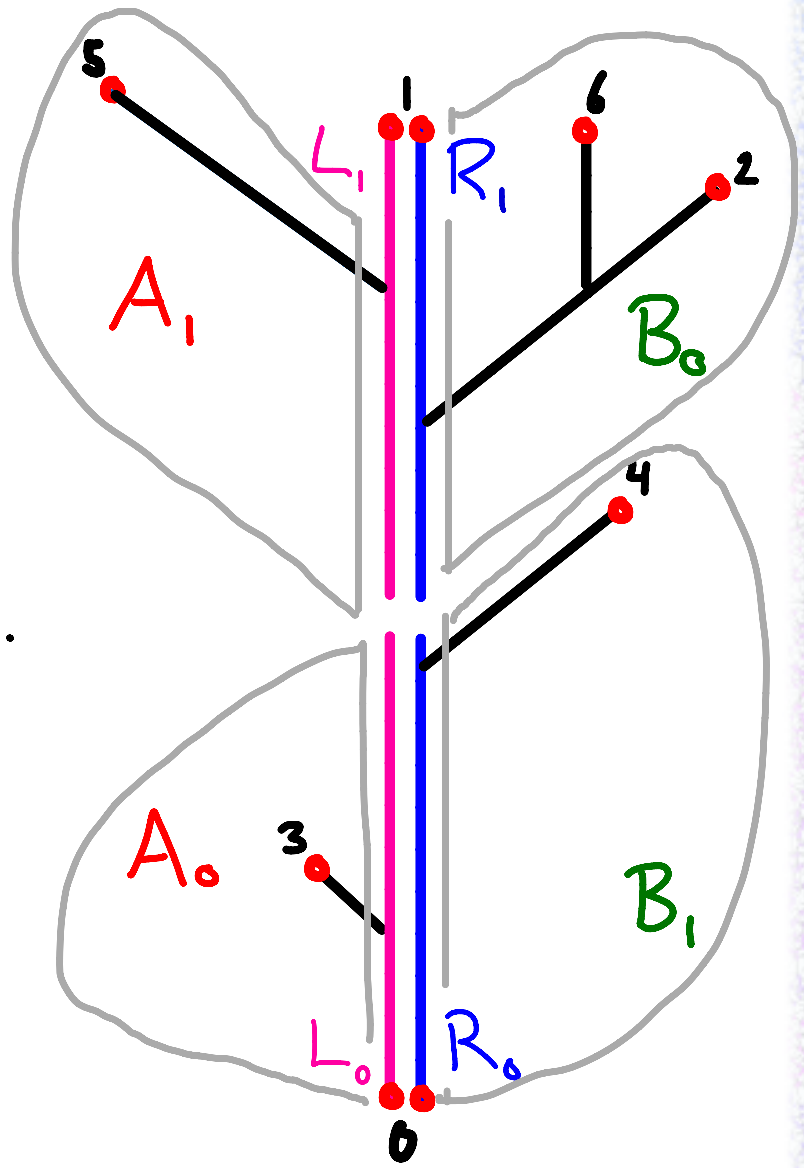


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$$\overset{d}{=} (\mu(A_0) + \mu(B_0), \mu(A_1) + \mu(B_1)) = (V(0), V(1)) \quad \blacksquare$$

# Branch lengths in $\mathcal{T}$

To make something similar work for $k > 2$ points, need a little more information.

Recall: $P_2$ = second point of P.P.P. with rate $\lambda(t) = t$.

# Branch lengths in $T$

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For $s \in [0, t)$, $\mathbb{P}(P_1 \in ds \mid P_2 = t) = \frac{s}{t^2/2}$ i.e. $P_1 \overset{d}{=} P_2 \cdot \max(U_1, U_2)$

where $U_1, U_2$ are Unif$[0,1]$ independent of each other and of $P_2$

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The branch —• 2 attaches to a uniform location on —• 1

$$P_1\cdot\mathcal{U} \overset{d}{=} P_2\cdot\max(\mathcal{U}_1,\mathcal{U}_2)\cdot\mathcal{U} \overset{d}{=} P_2\cdot\min(\mathcal{U}_1,\mathcal{U}_2)$$

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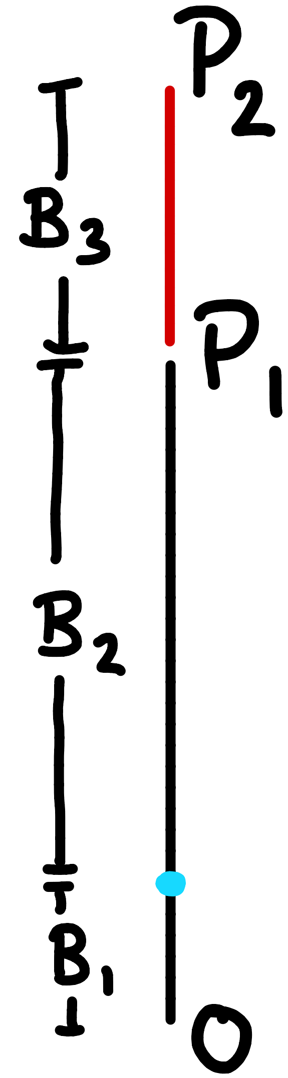

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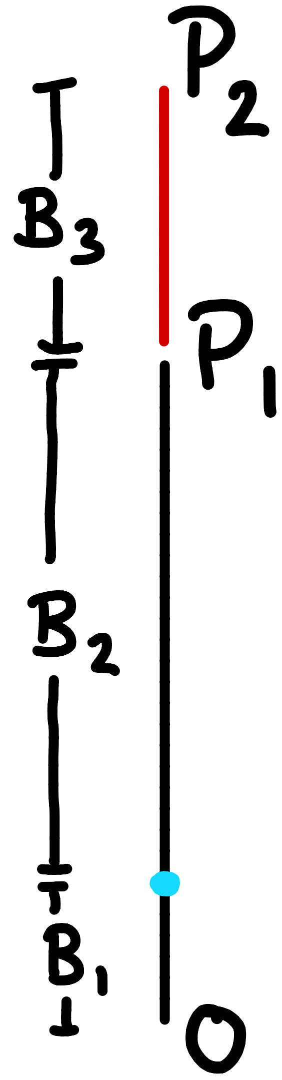
The branch $\text{—}\bullet\, 2$ attaches to a uniform location on $\text{—}\bullet\, 1$

$P_1 \cdot U \overset{d}{=} P_2 \cdot \max(U_1, U_2) \cdot U \overset{d}{=} P_2 \cdot \min(U_1, U_2)$

We have derived the following identity in law for the branch lengths in $T_2$:

$$(P_1 \cdot U, P_1, P_2) \overset{d}{=} P_2 \cdot (\min(U_1, U_2), \max(U_1, U_2), 1).$$



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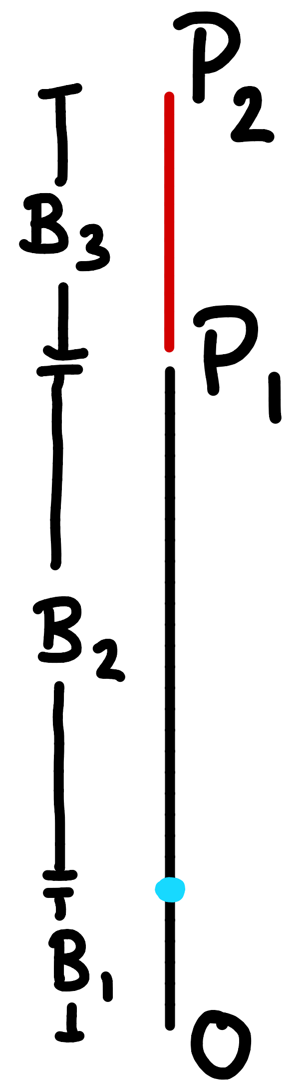

# Branch lengths in $\mathcal{T}$

To make something similar work for $k>2$ points, need a little more information.
Recall: $P_2$ = second point of P.P.P. with rate $\lambda(t)=t$.

For $s\in[0,t]$, $\mathbb{P}(P_1\in ds \mid P_2=t)=\frac{s}{t^2/2}$ i.e. $P_1 \overset{d}{=} P_2\cdot \max(\mathcal{U}_1,\mathcal{U}_2)$
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The branch $—\!\bullet 2$ attaches to a uniform location on $—\!\bullet 1$

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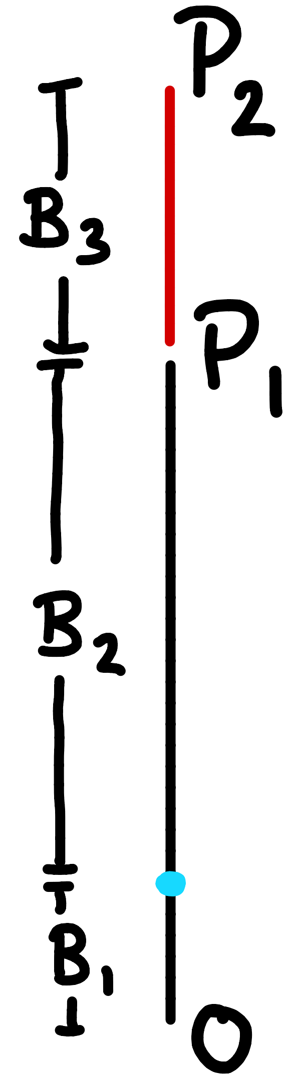


A more useful way to write this: Let $E_1,E_2,E_3$ be Exp$(1)$, with sum $G_3$
Then $\frac{1}{P_2}\cdot(B_1,B_2,B_3) \overset{d}{=} \frac{1}{G_3}\cdot(E_1,E_2,E_3)$.

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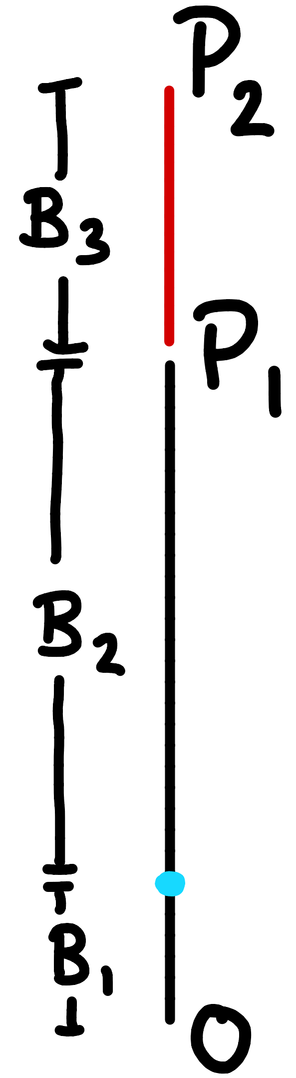


More generally: **Prop**: The branch lengths $(B_1,\dots,B_{2k-1})$ in $\mathcal{T}_k$

satisfy $\frac{1}{P_k}(B_1,\dots,B_{2k-1}) \stackrel{d}{=} \frac{1}{G_{2k-1}}\cdot(E_1,\dots,E_{2k-1})$

where $(E_i,\ i\geq 1)$ are IID Exp(1), $G_m=\sum_{i\leq m} E_i$.

# Branch lengths in $\mathcal{T}$

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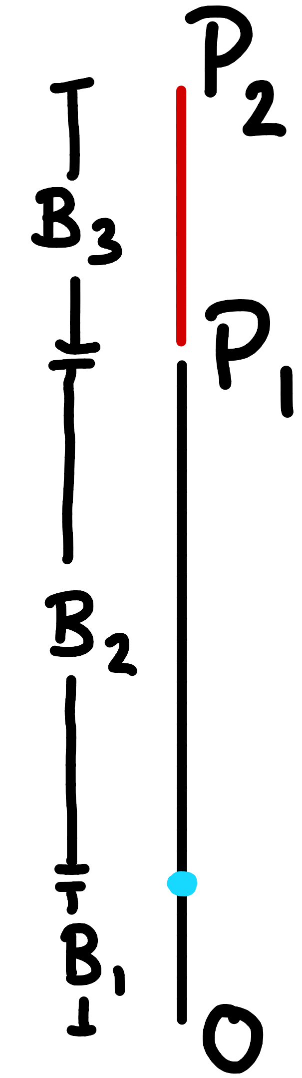


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More generally:

**Prop**: The branch lengths $(B_1, \dots, B_{2k-1})$ in $\mathcal{T}_k$ satisfy $\frac{1}{P_k}(B_1, \dots, B_{2k-1}) \overset{d}{=} \frac{1}{G_{2k-1}} \cdot (E_1, \dots, E_{2k-1})$

where $(E_i, i \geq 1)$ are IID Exp(1), $G_m = \sum_{i \leq m} E_i$.

**Proof**: Induction. ∎

Thm: ($k$ Voronoi cells in the CRT)

List the points $\{0,1,\dots,k-1\}$ according to the cyclic order in $T_k$ starting from $0$, as $(\ell_0, \ell_1, \dots, \ell_{k-1}, \ell_k = \ell_0)$.

Thm: (K Voronoi cells in the CRT)

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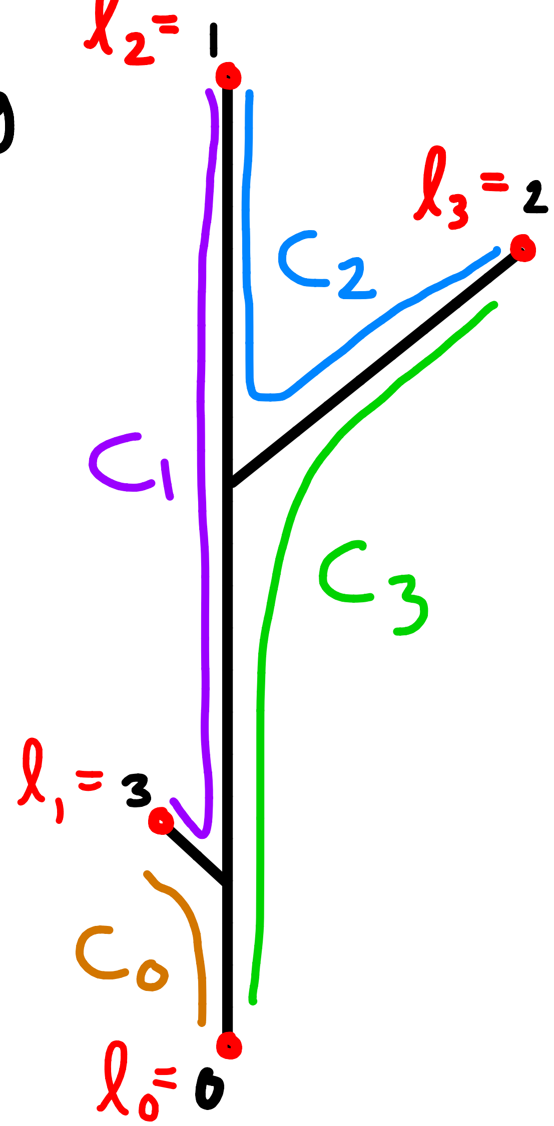

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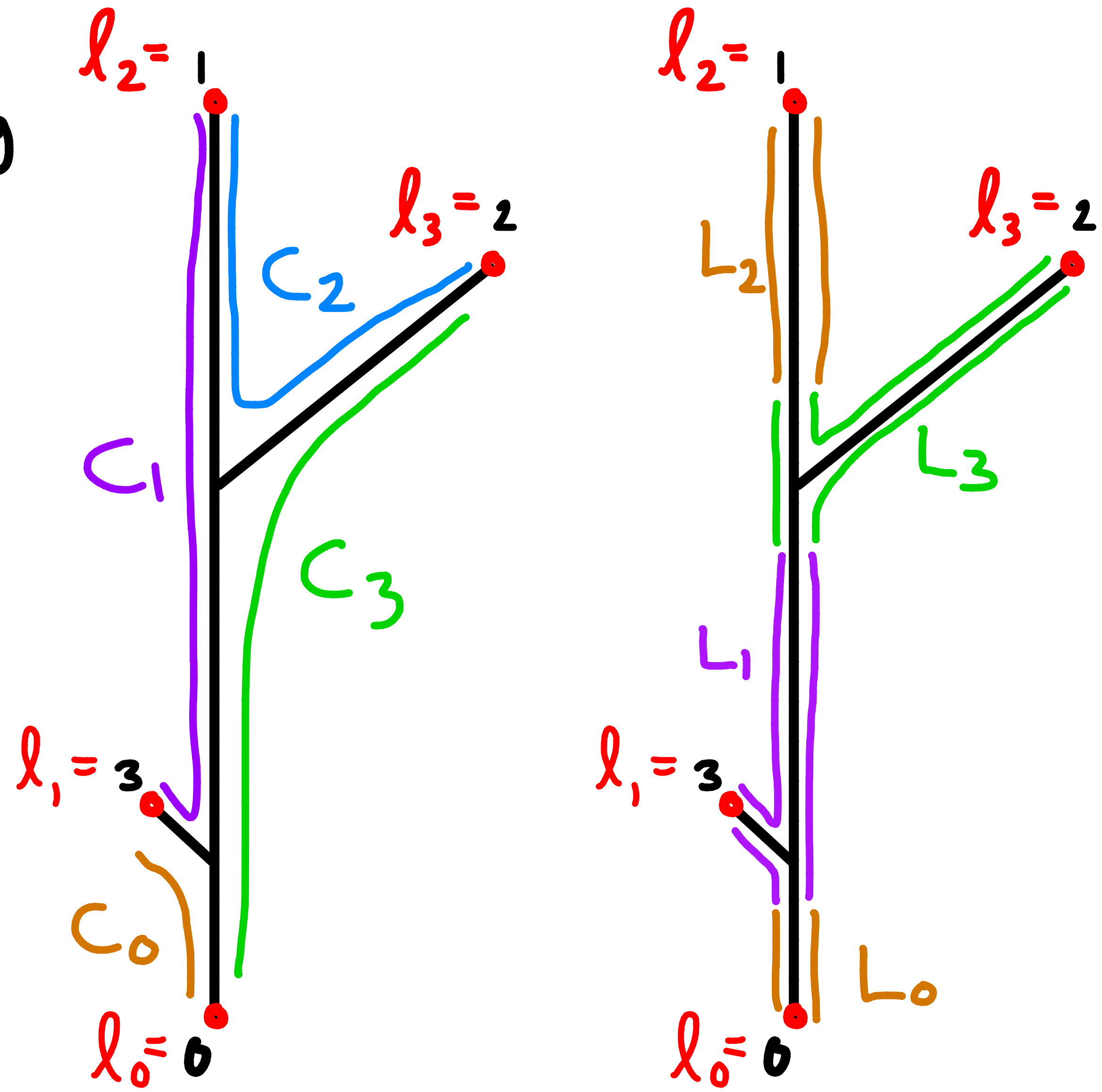

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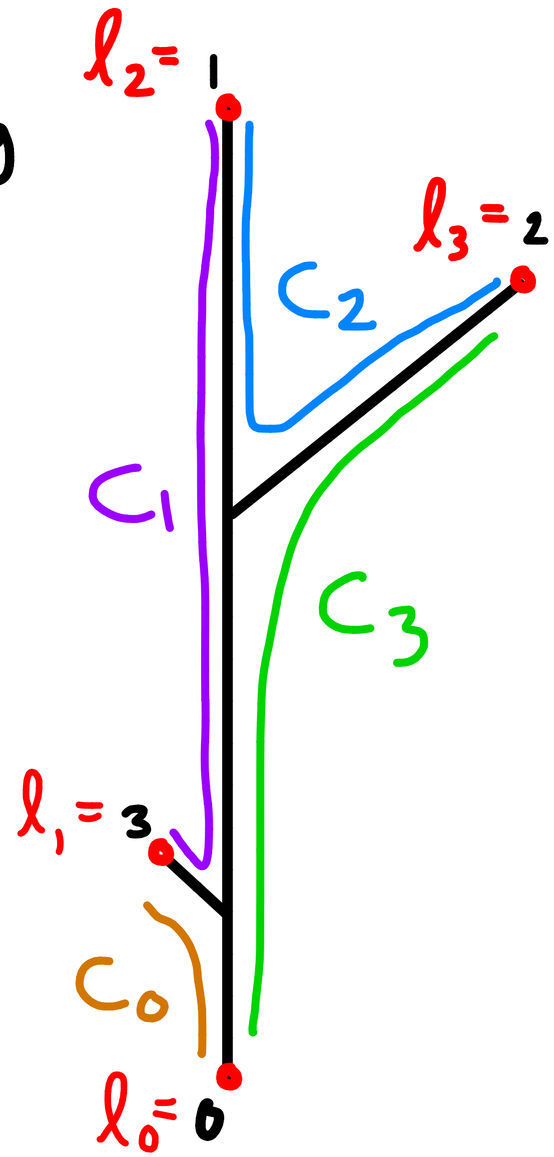


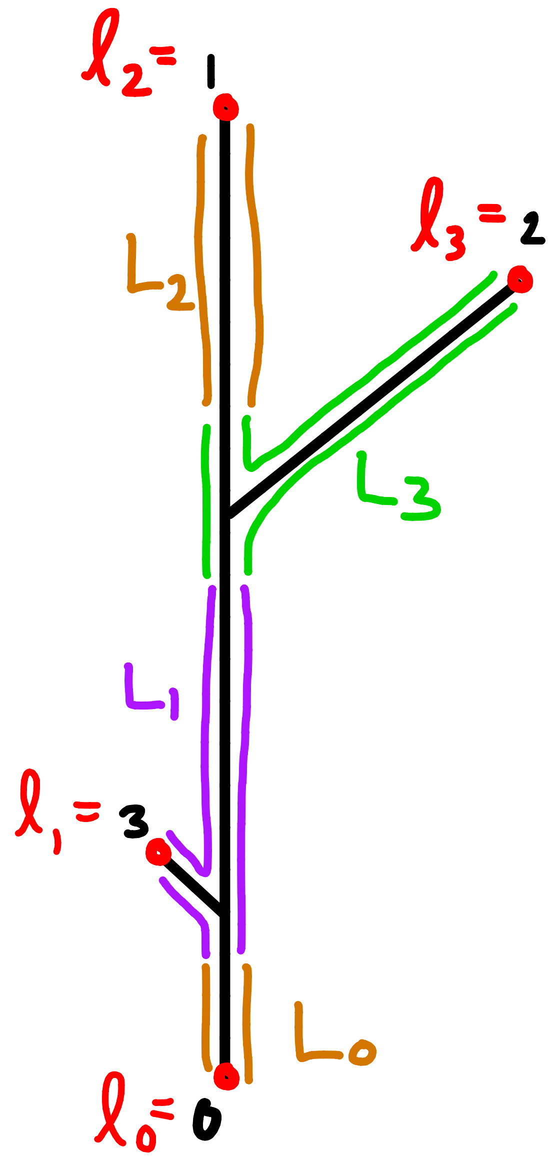

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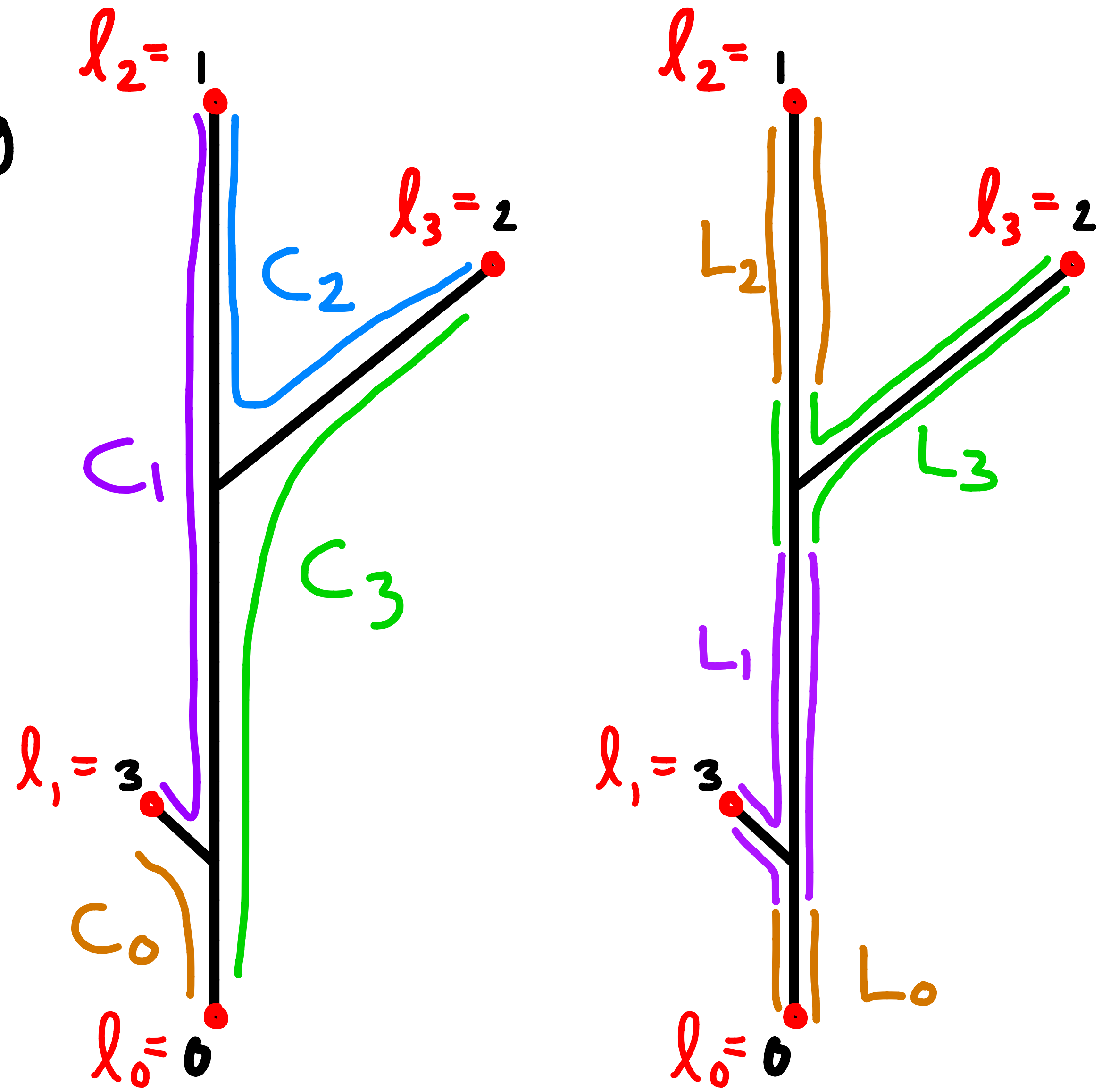
Then $(C_0, \dots, C_{k-1}) \overset{d}{=} (L_0, \dots, L_{k-1})$.

**Corollary:** $(V_0, V_1, \dots, V_k) \sim \mathrm{Dir}(1,1,\dots,1)$.



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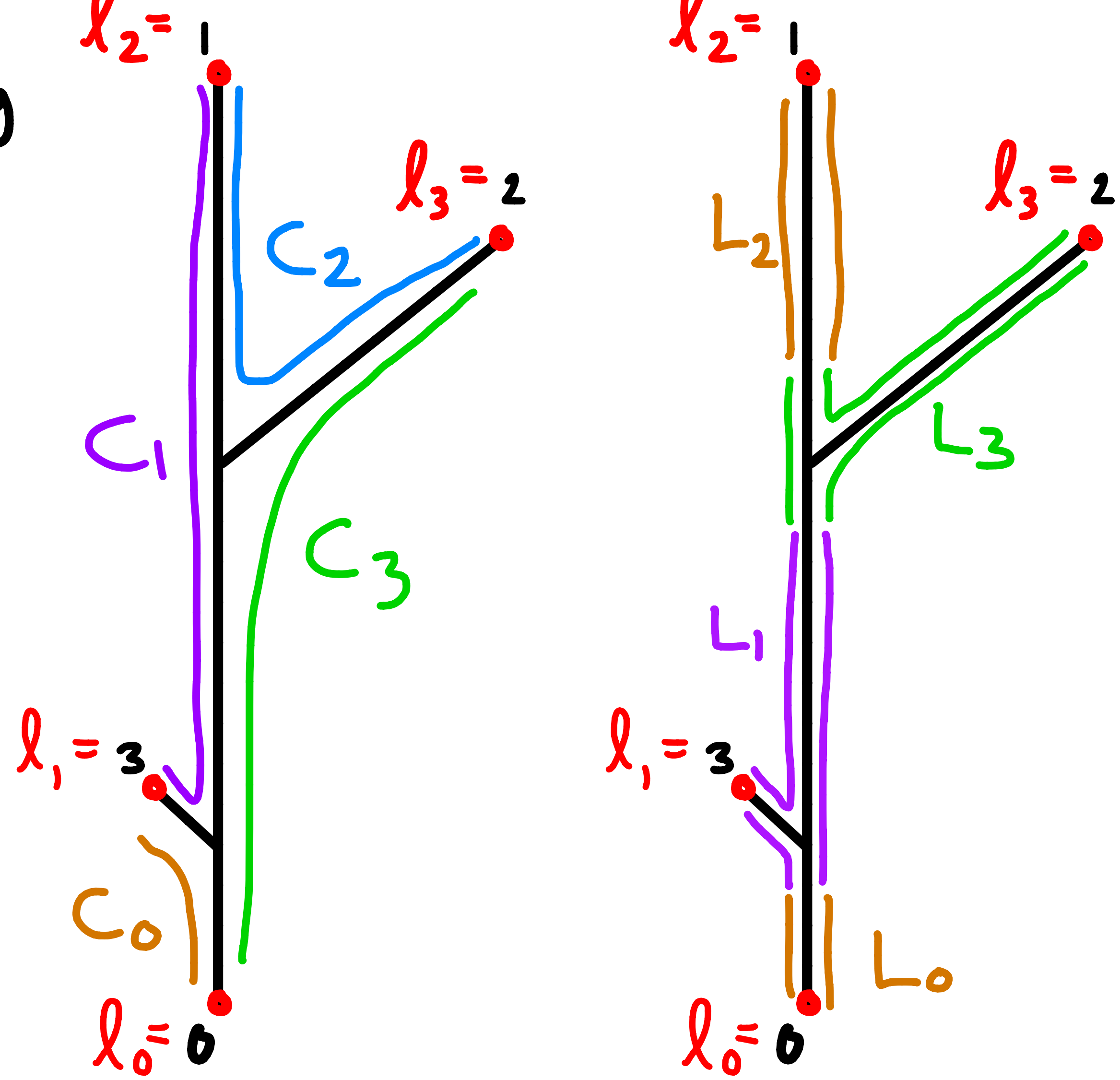


Then $(C_0, \dots, C_{k-1}) \overset{d}{=} (L_0, \dots, L_{k-1})$.

**Corollary:** $(V_0, V_1, \dots, V_k) \sim \mathrm{Dir}(1,1,\dots,1)$.

**Proof:** Let $I_j$ = total mass attaching to $C_j$. Since $\{0,1,\dots,k-1\}$ are uniform samples from $\mu$, we have $(C_0, C_1, \dots, C_{k-1}) \sim \mathrm{Dir}(1,1,\dots,1)$.

**Thm:** ( $k$ Voronoi cells in the CRT)

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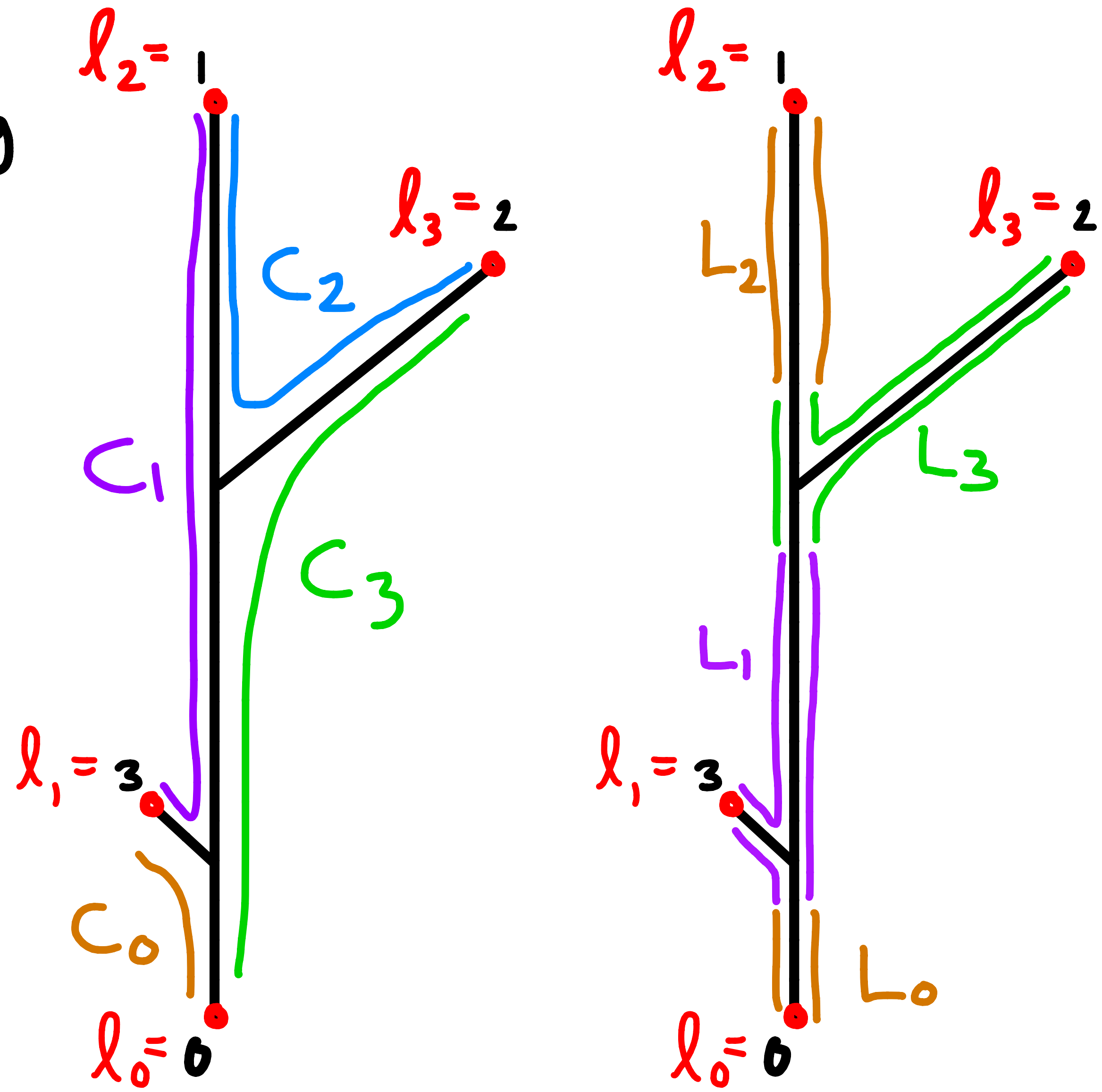


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Thm: (K Voronoi cells in the CRT)

List the points $\{0,1,\ldots,k-1\}$ according to the cyclic order in $T_k$ starting from $0$, as $(l_0, l_1, \ldots, l_{k-1}, l_k = l_0)$.

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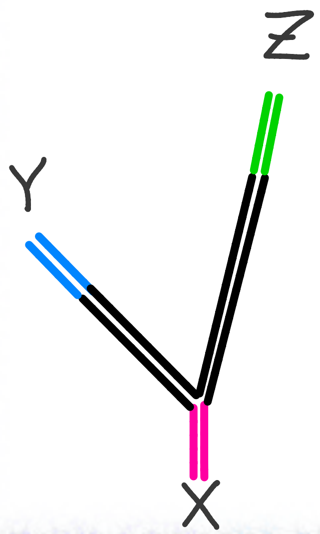

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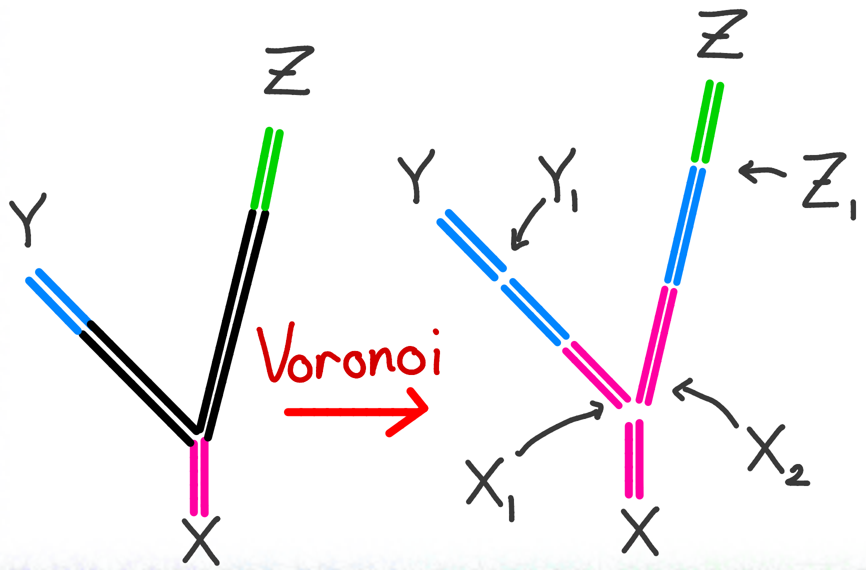

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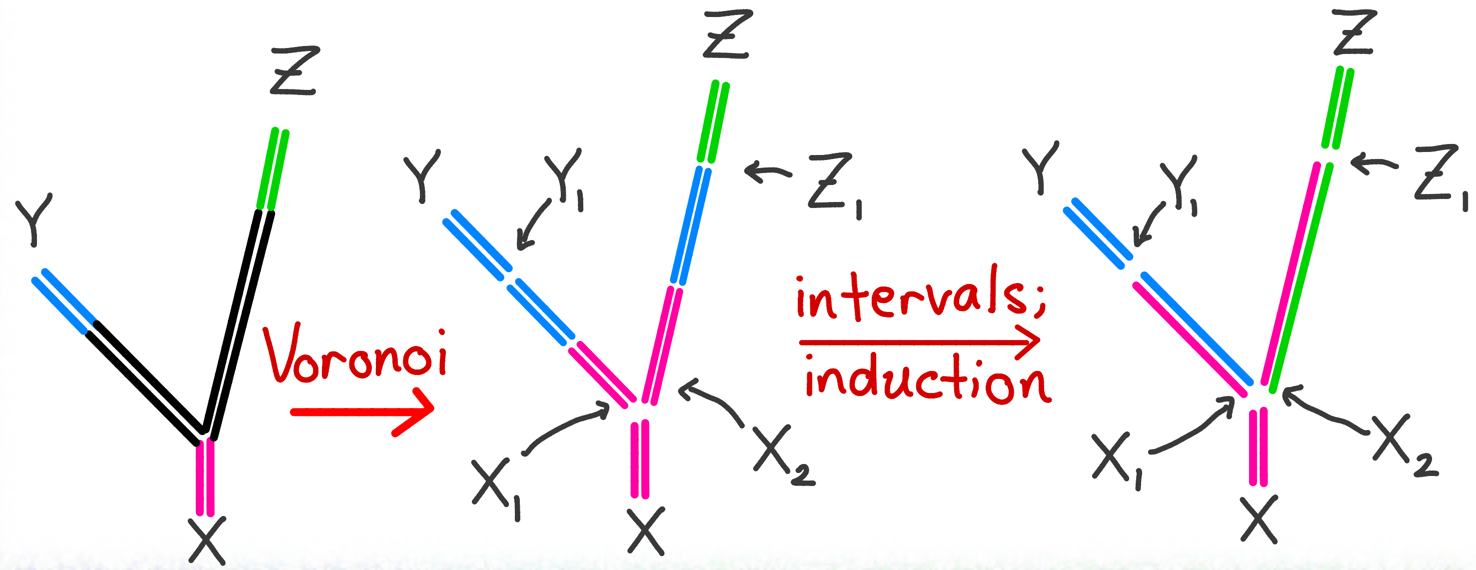

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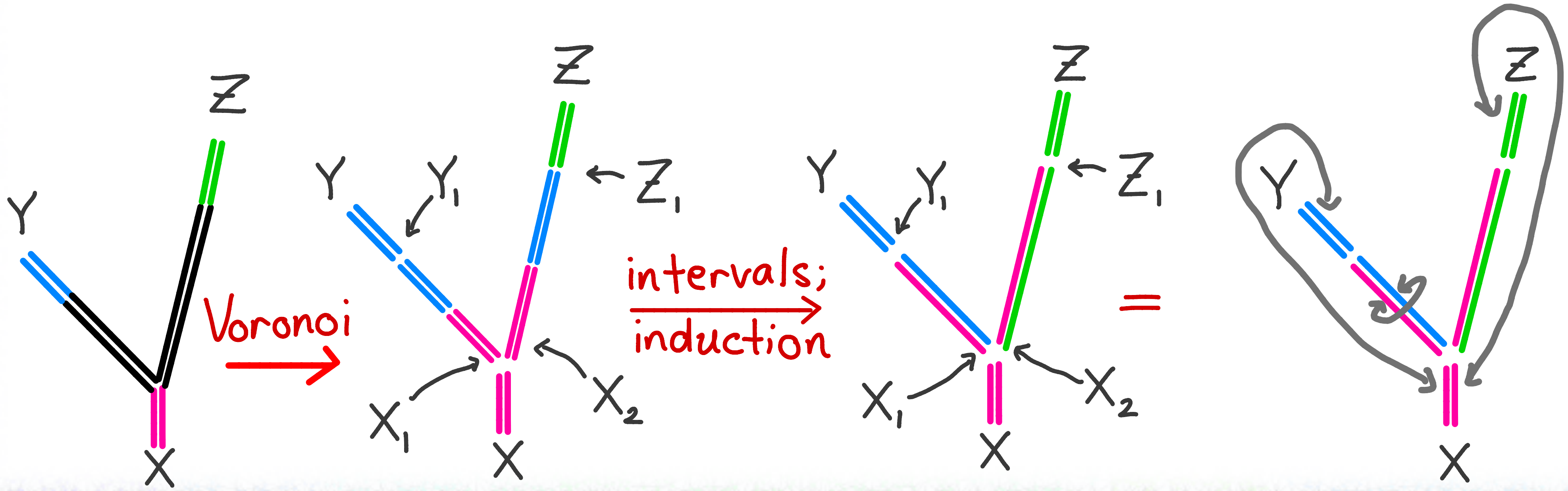

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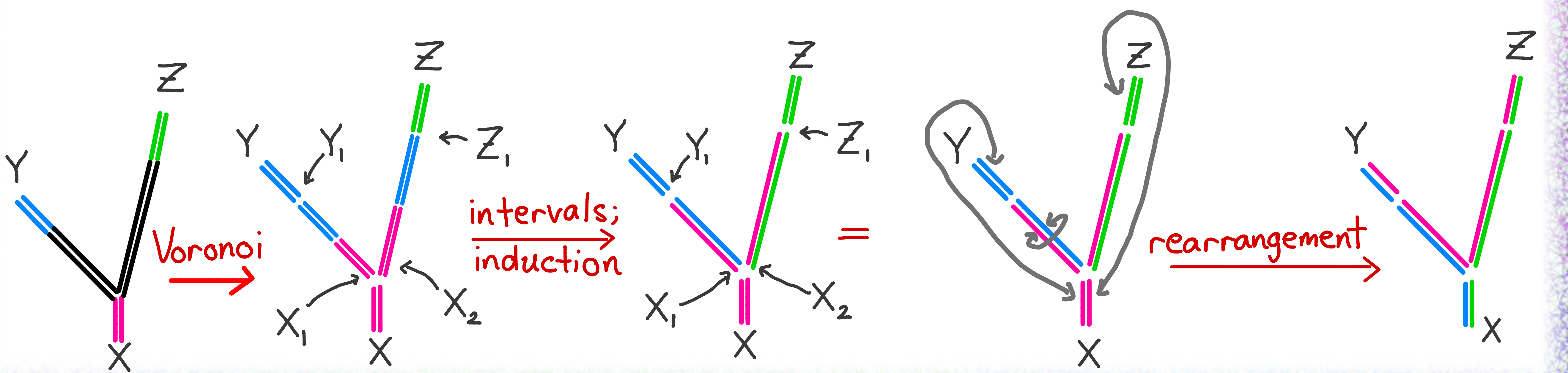

Proof contd. (general $k$): Always apply induction by removing shortest branch incident to a leaf.

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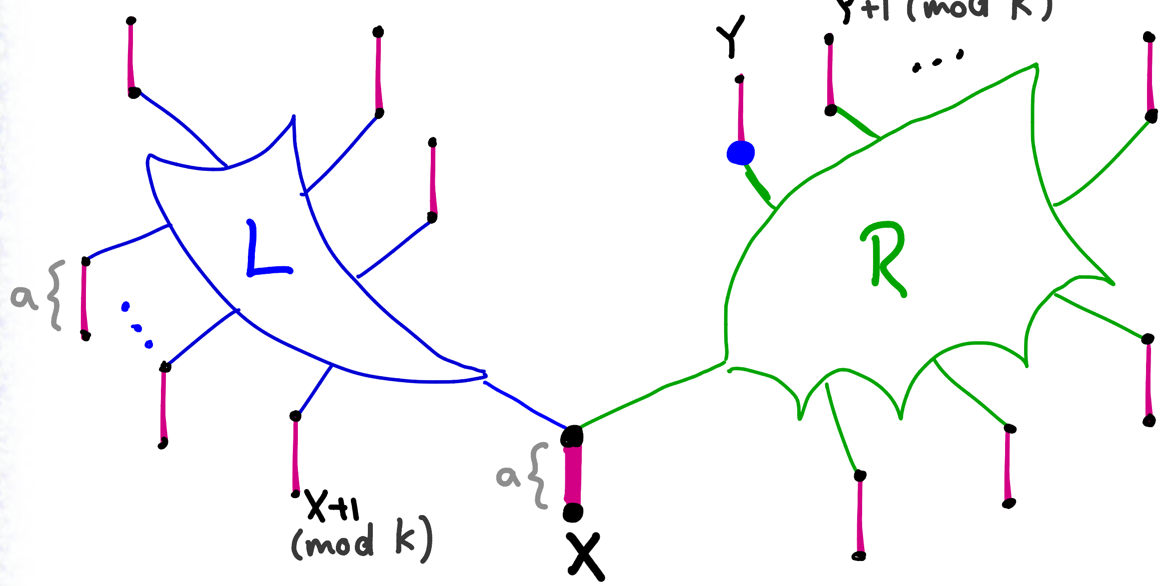

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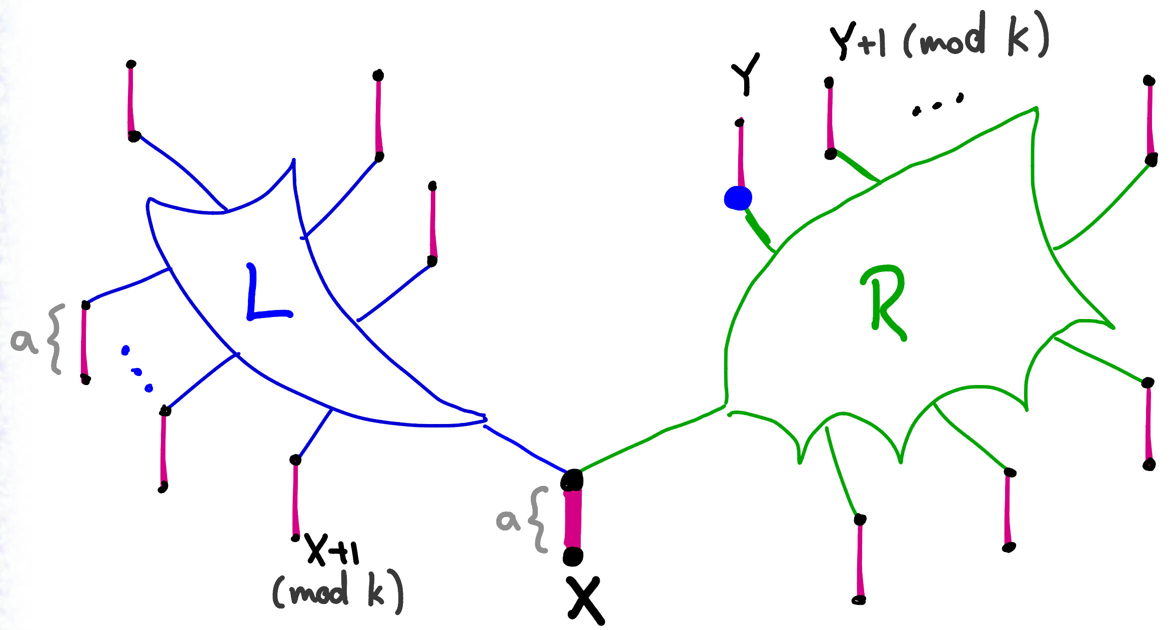


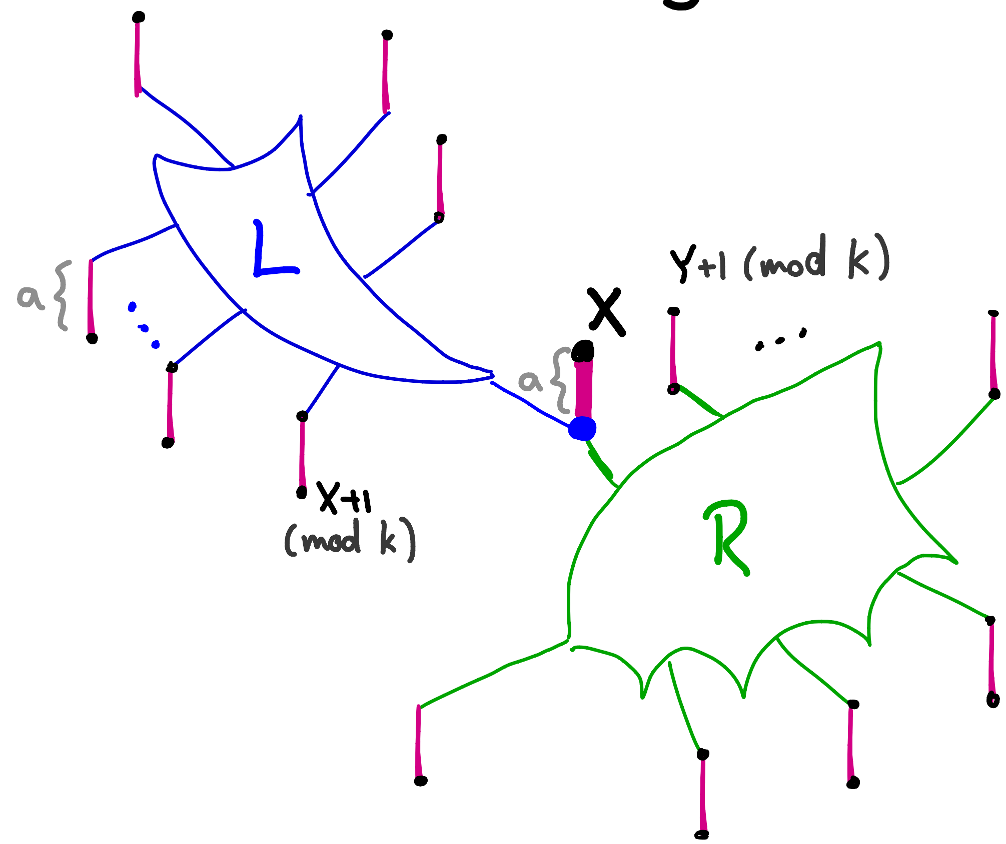

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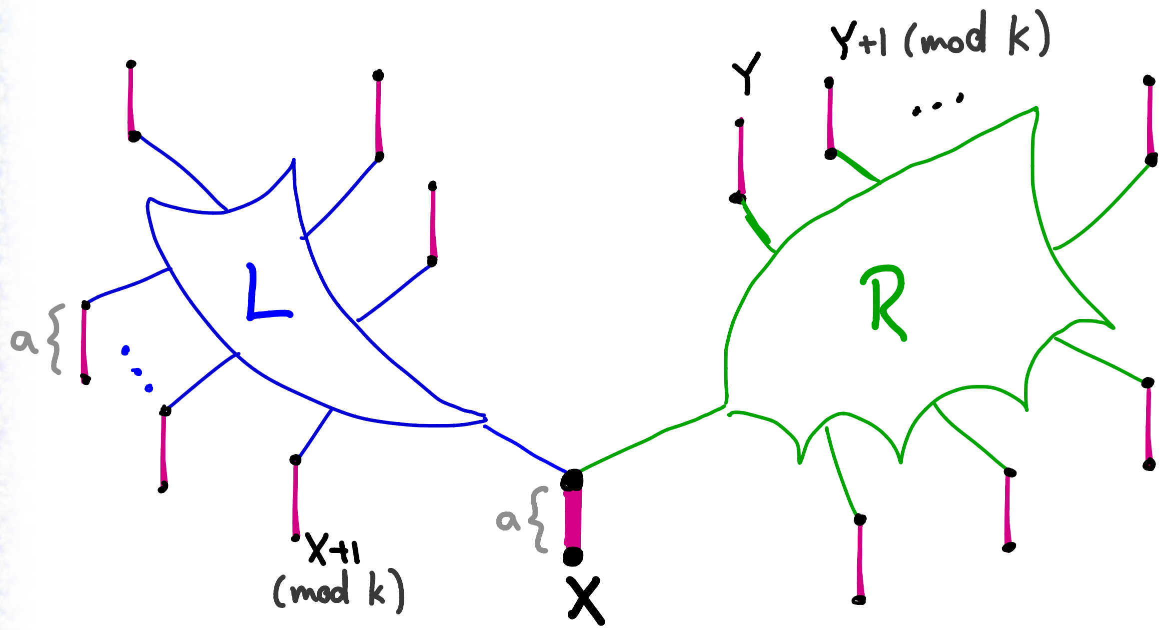


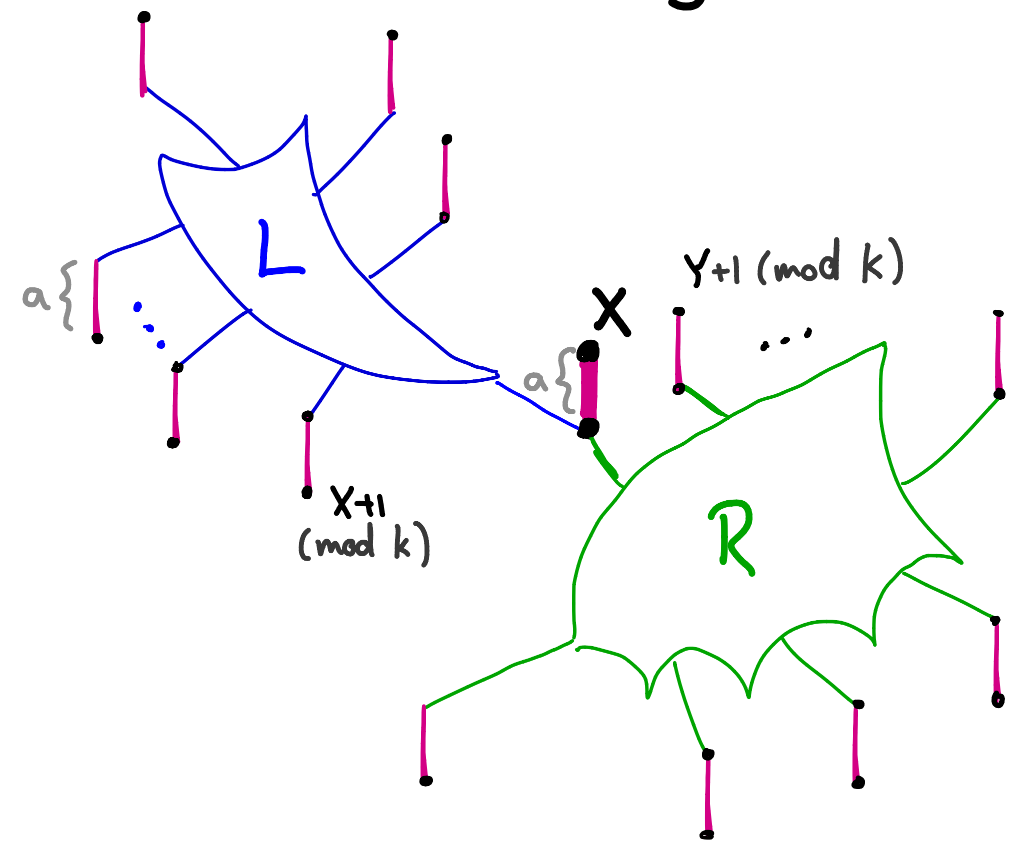


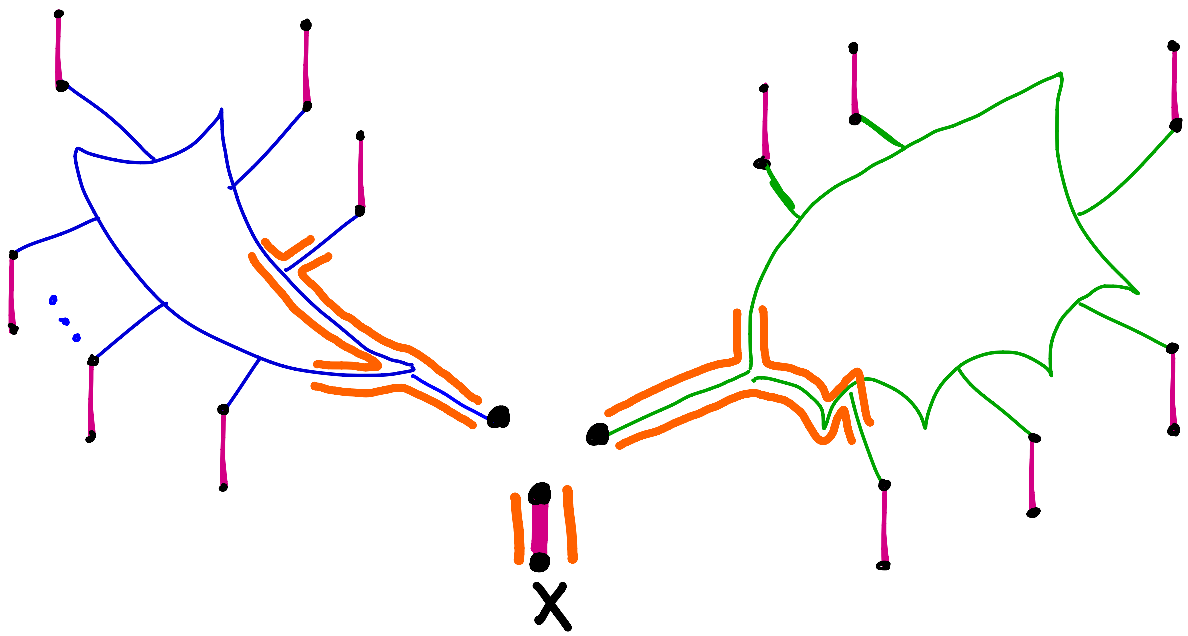

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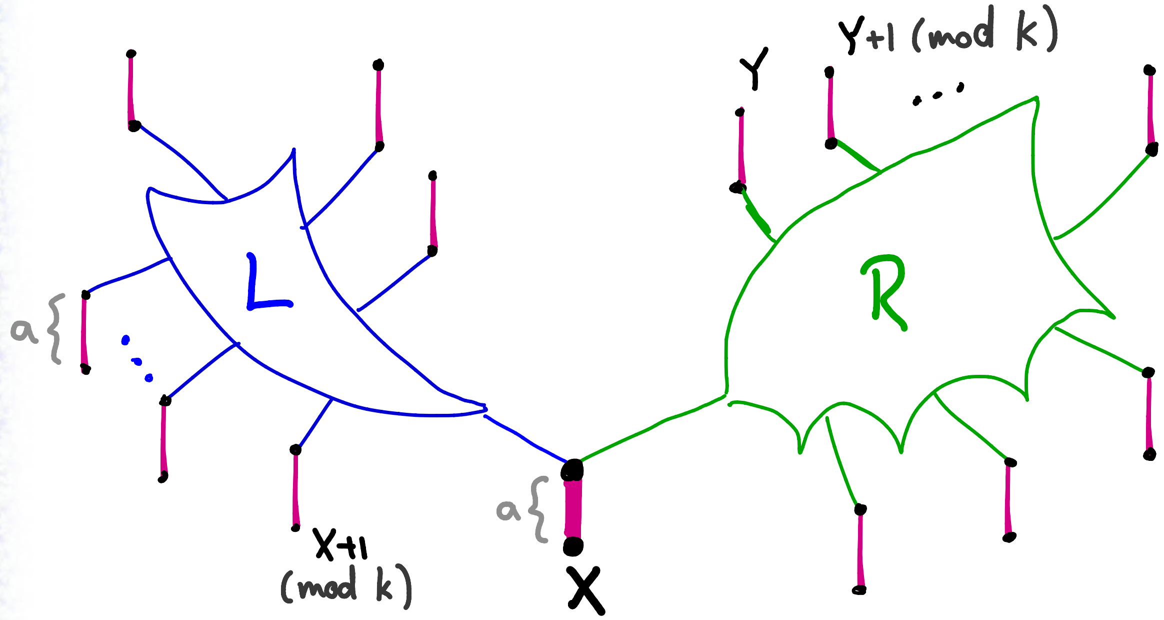


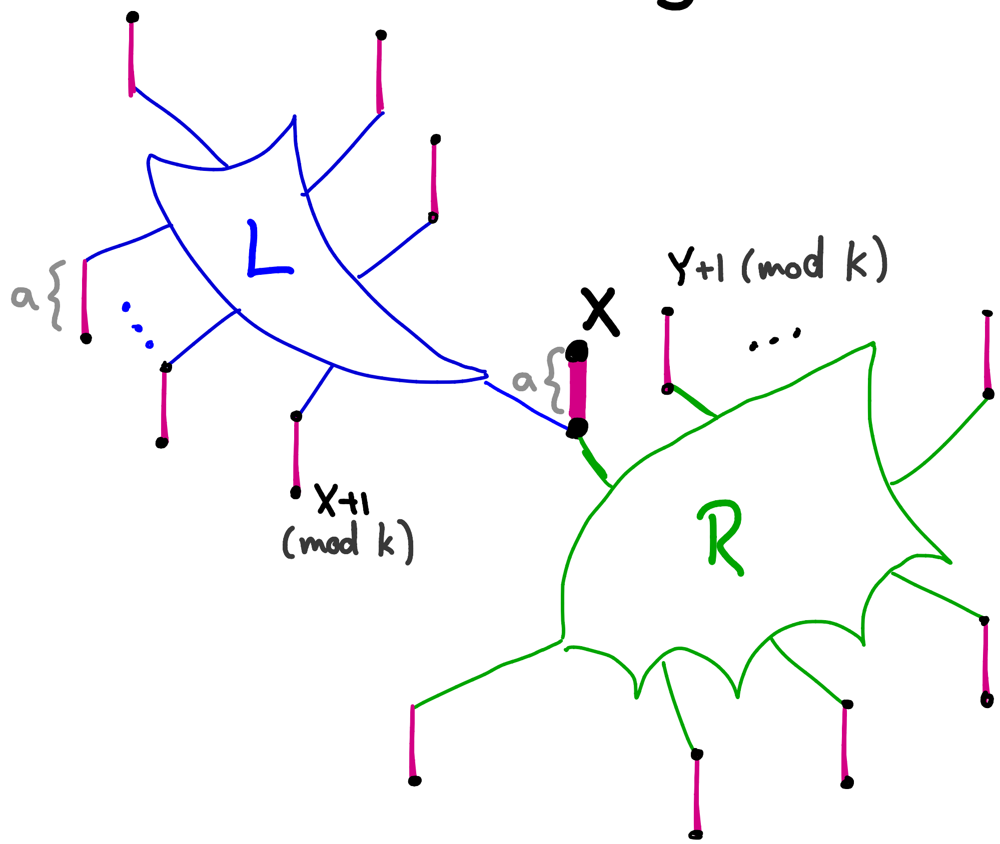


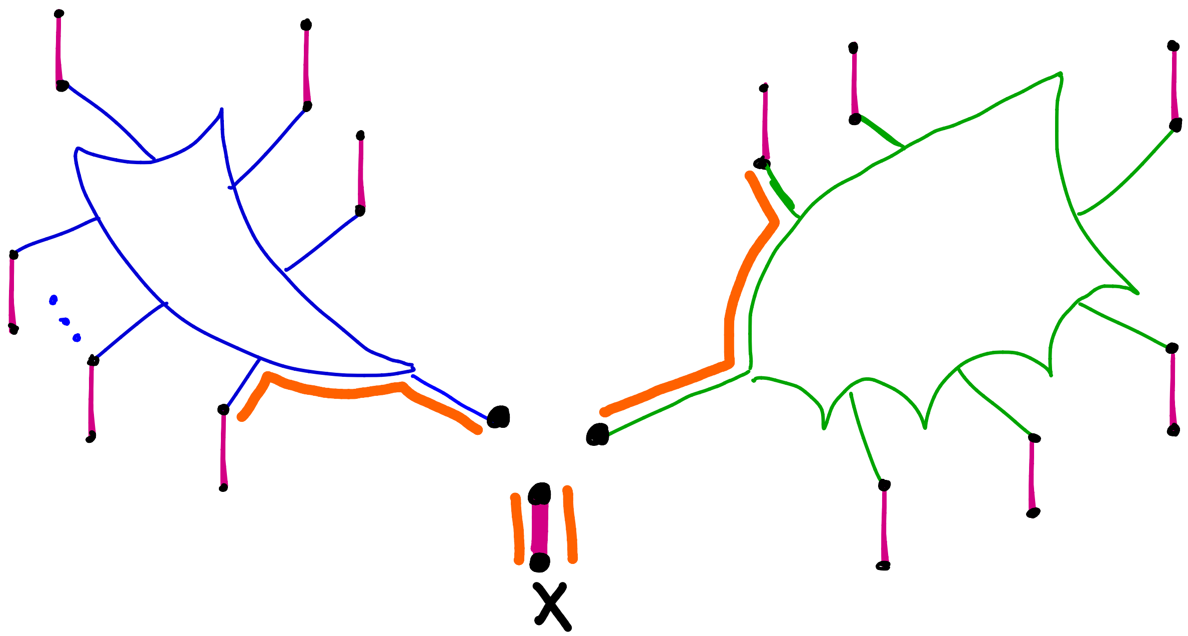

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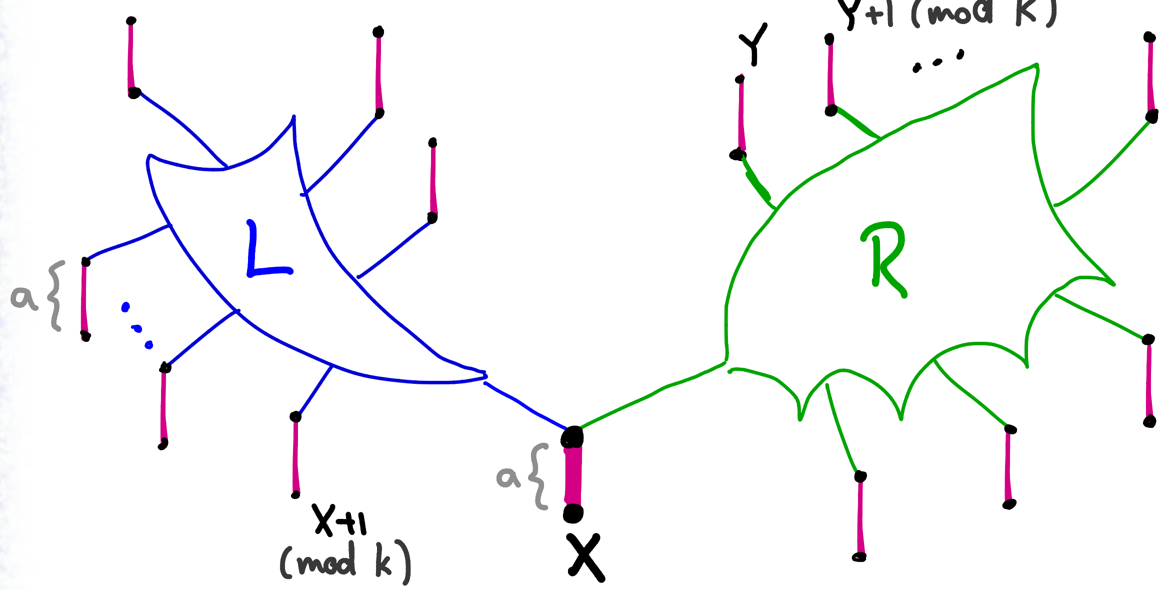
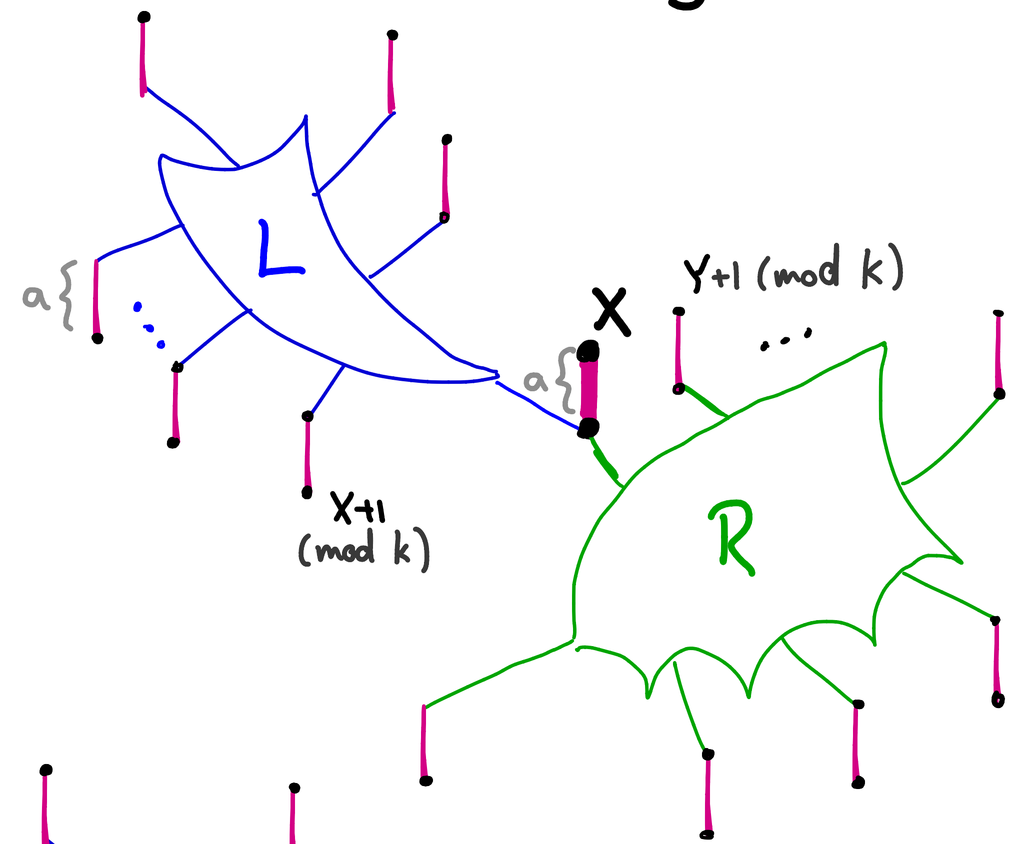
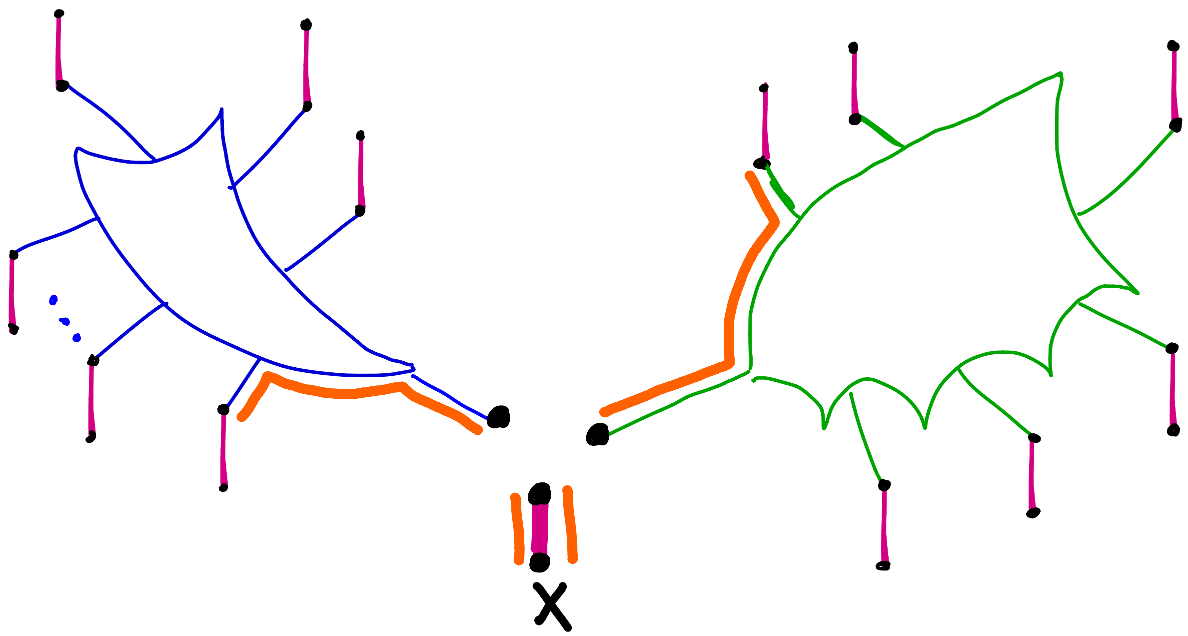
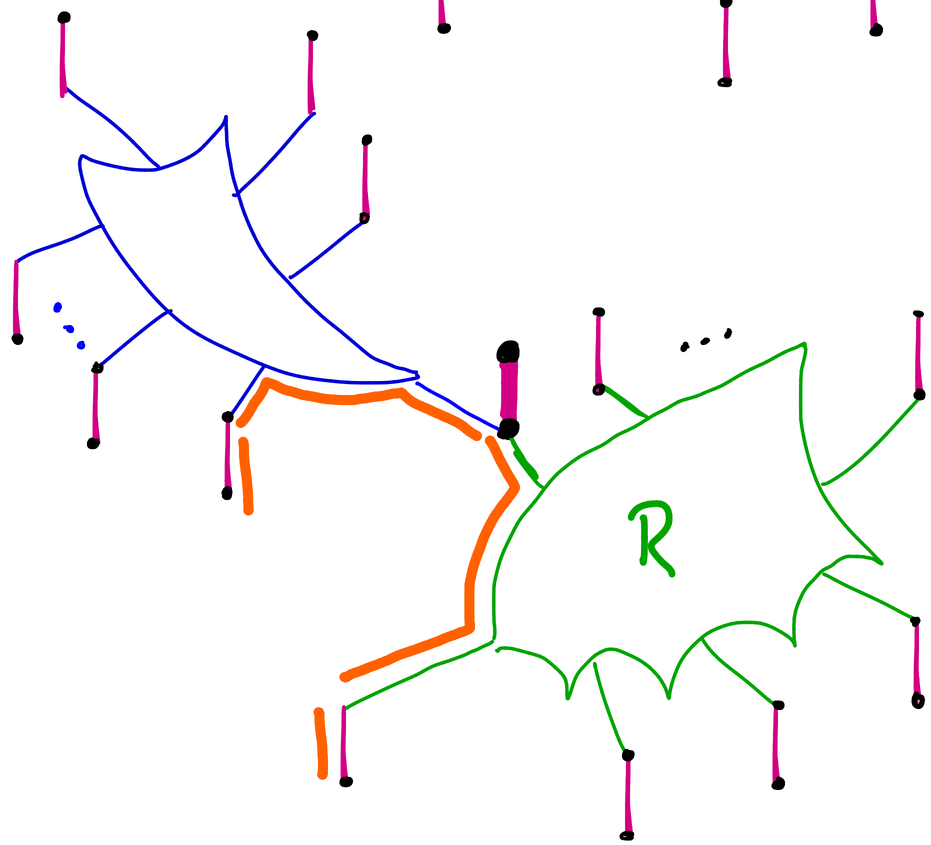
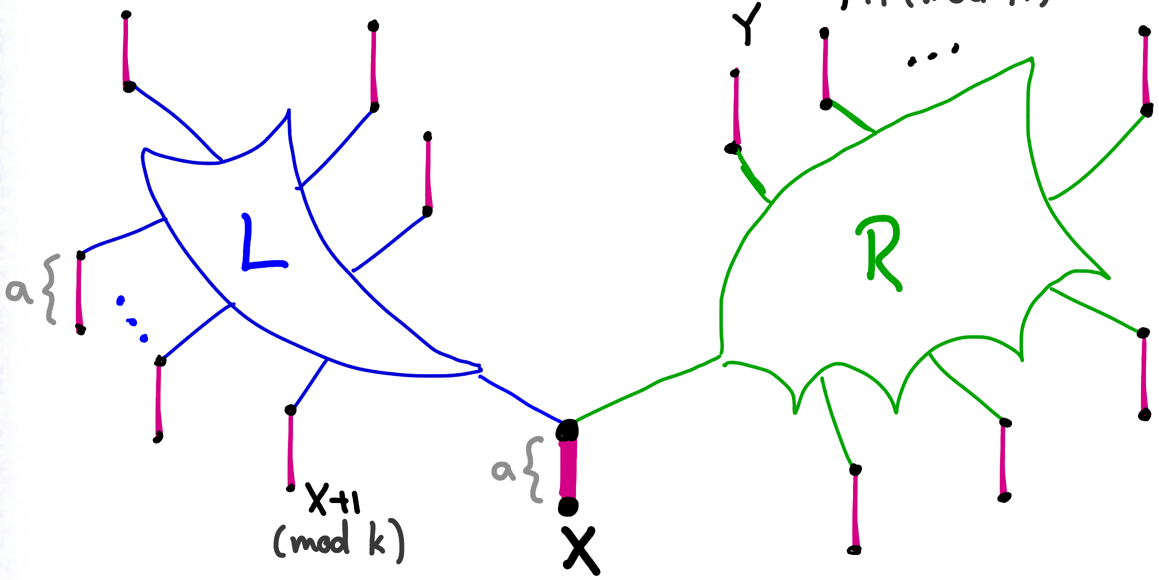


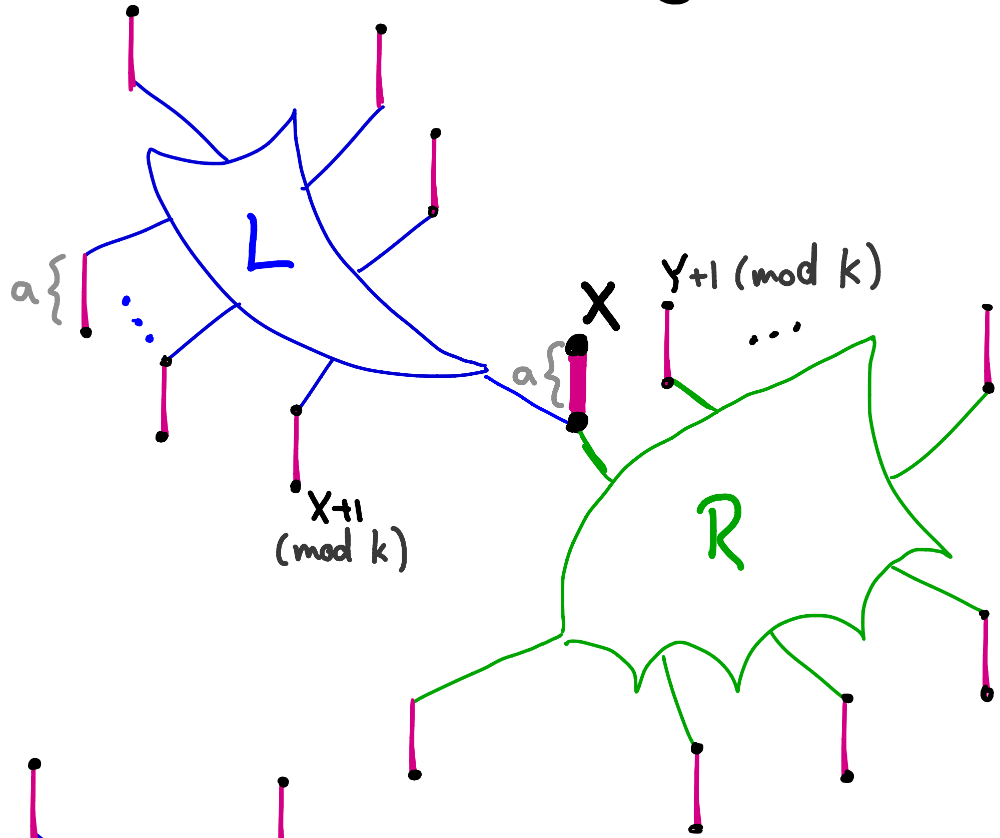



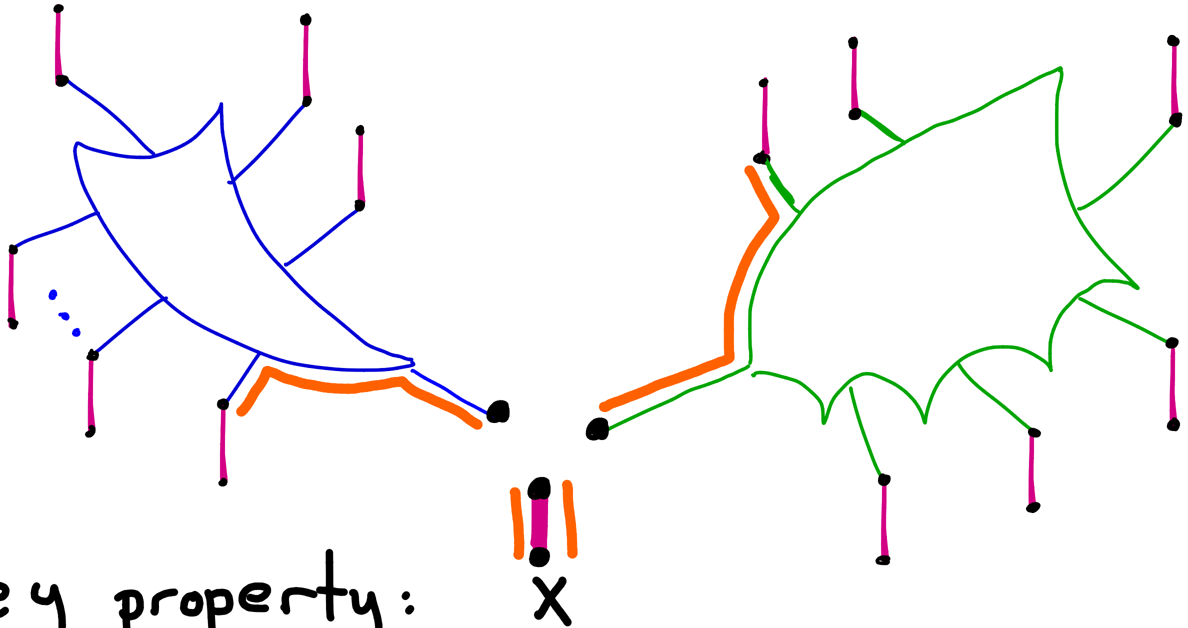



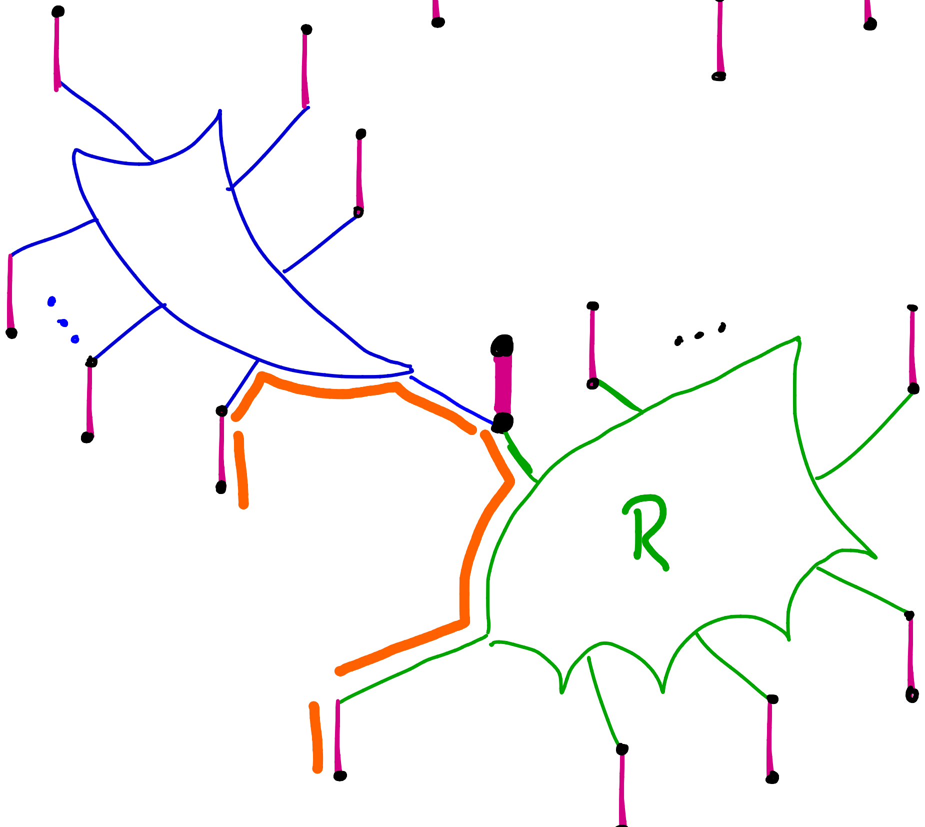



Proof contd. (general k): Always apply induction by removing shortest branch incident to a leaf.



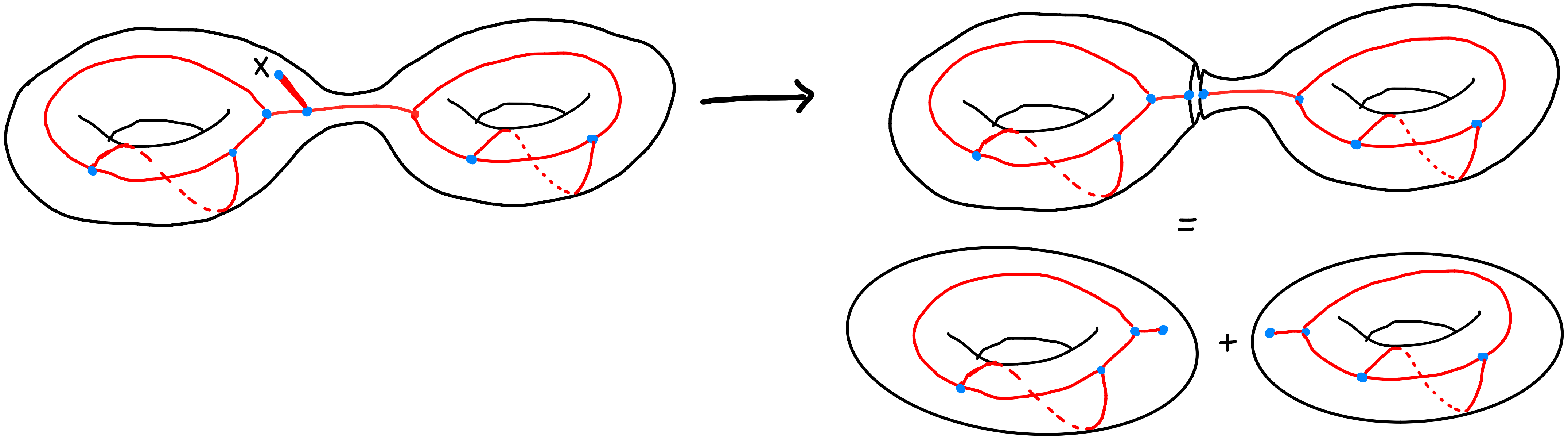




Key property: for induction Subtrees uniform binary trees with exponential edge lengths ∎

<u>General Surfaces</u>: Still remove shortest branch incident to a leaf.

**General Surfaces**: Still remove shortest branch incident to a leaf.
This can:

- Disconnect the map (& so the surface)



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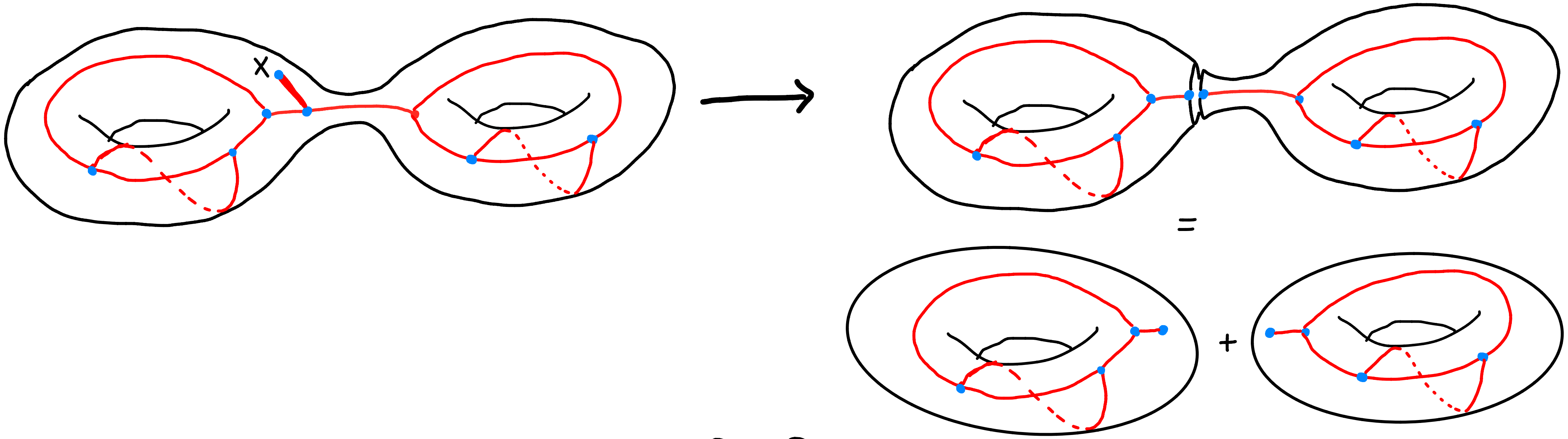
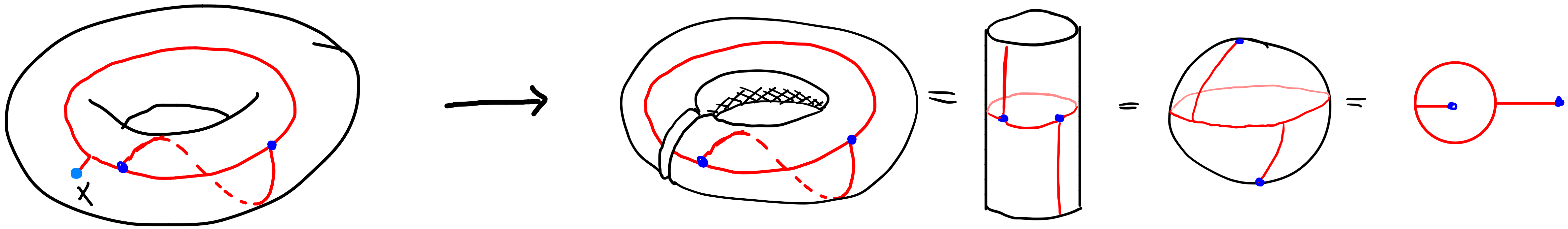



- Increase the number of faces without disconnecting the surface.



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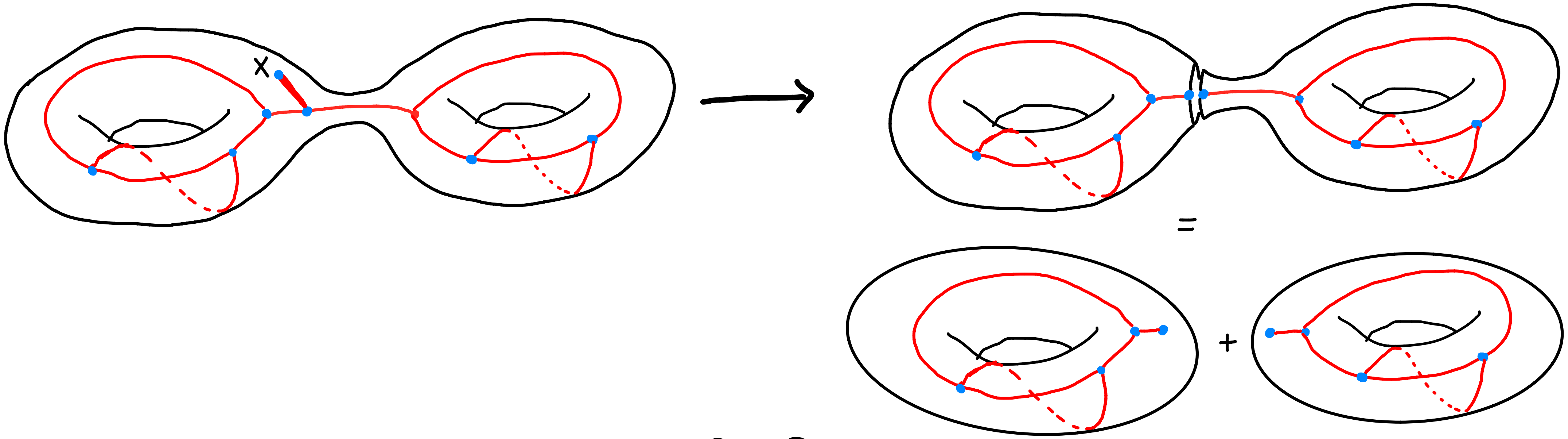


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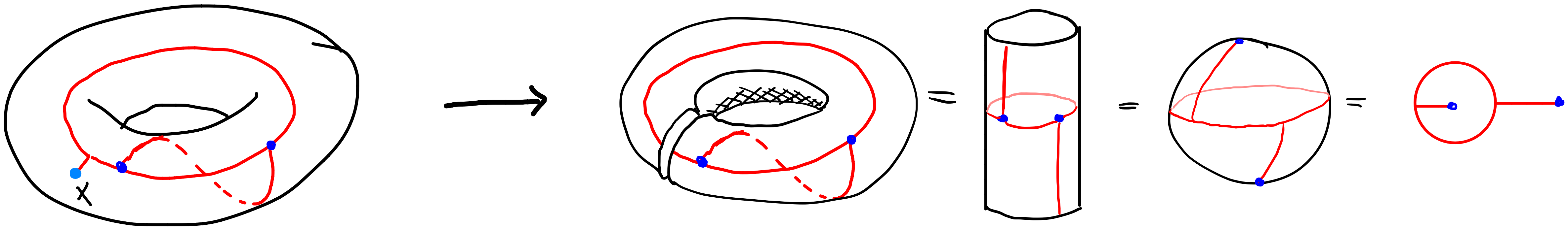


Forces us to work with maps with more than one face (but always a bounded # of faces).

**Main Theorem:** Equidistribution between Voronoi cells and intervals, "face-by-face", for general continuum random maps.

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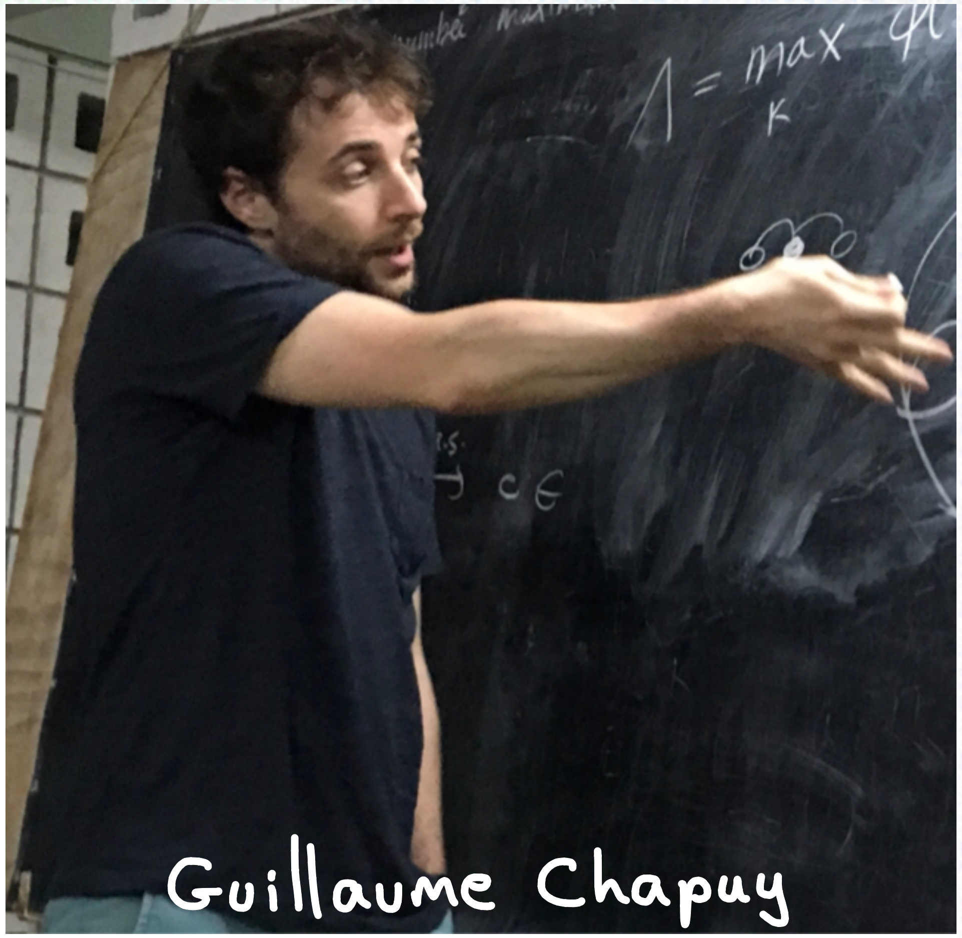
Remaining details omitted!


Thank you !
IMPACASTORINA
IMPA·NWL - NET

# Joint work


Christina Goldschmidt


Éric Fusy


Guillaume Chapuy

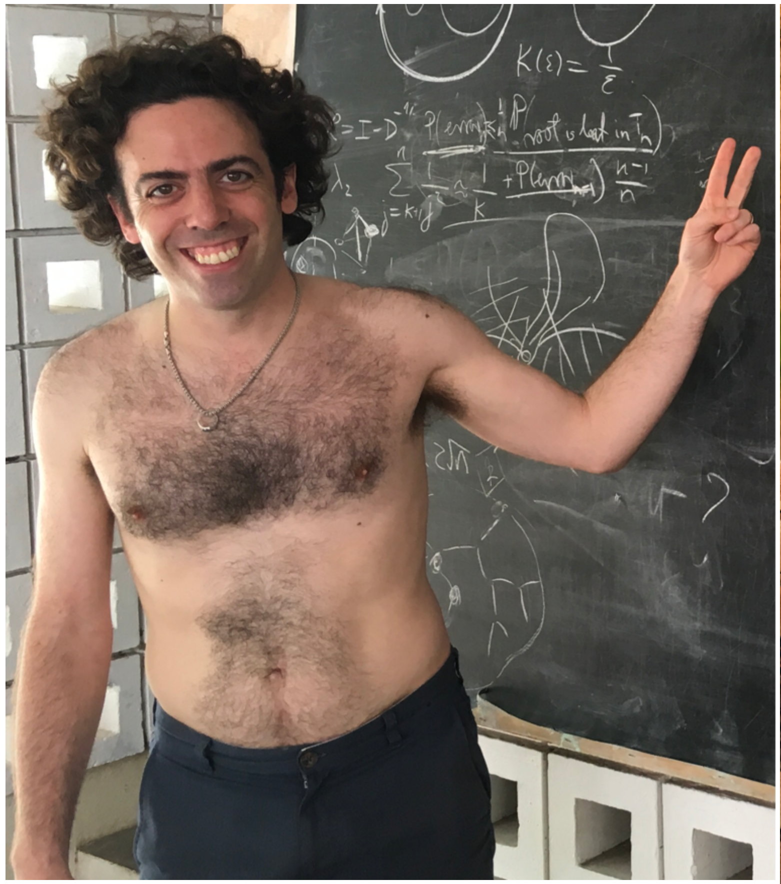

Omer Angel

# Joint work


Christina Goldschmidt


Guillaume Chapuy


Omer
Angel




Éric Fusy