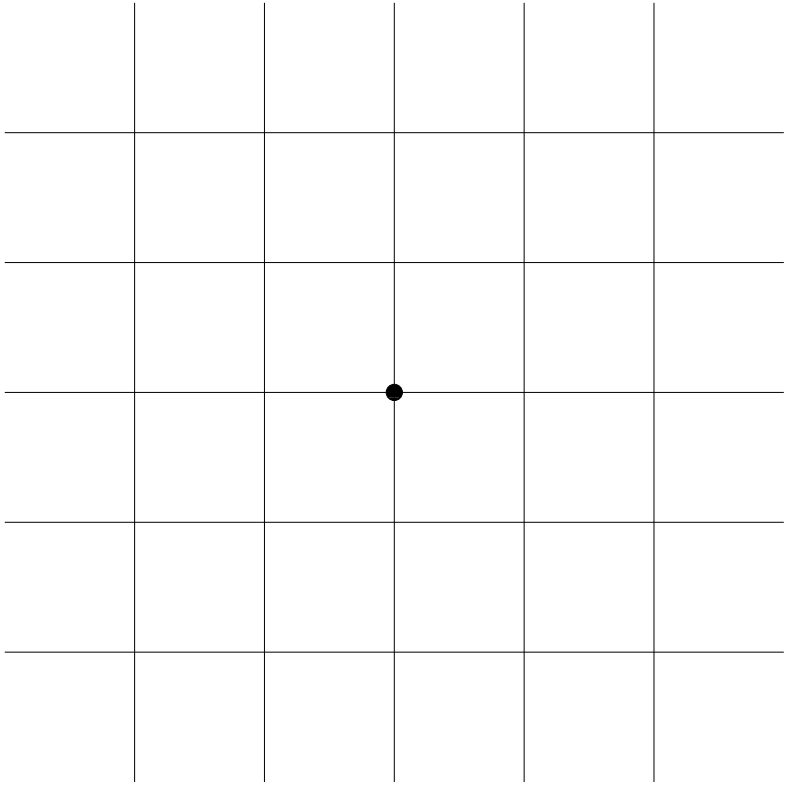
Science Atlantic
Wolfville

Random graphs and random trees

Louigi Addario-Berry (McGill)

I. Prehistory. A) Euclidean percolation

- Example: Nearest neighbour graph on $\mathbb{Z}^2$.



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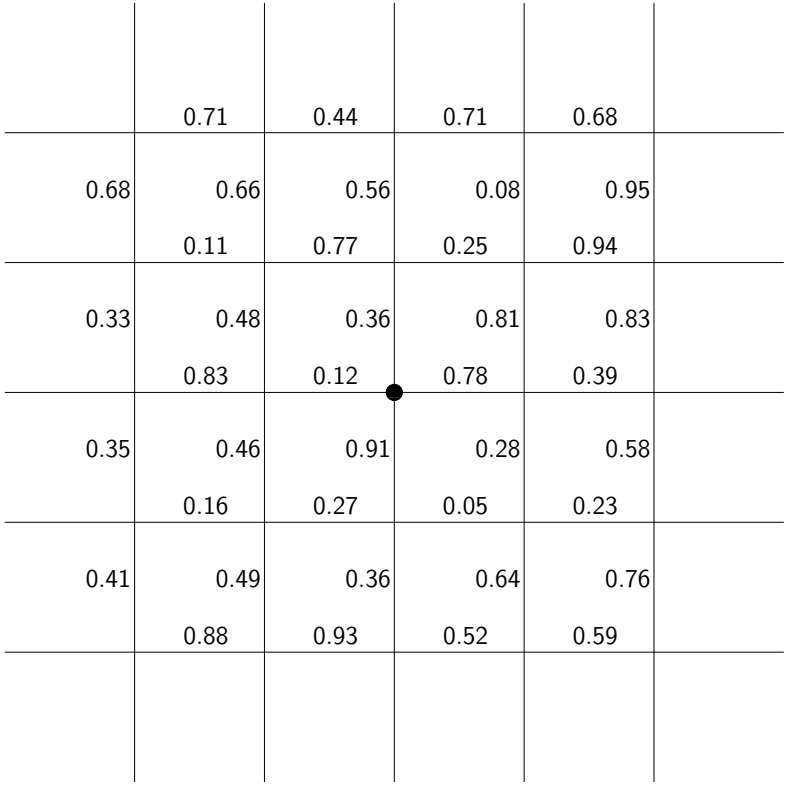

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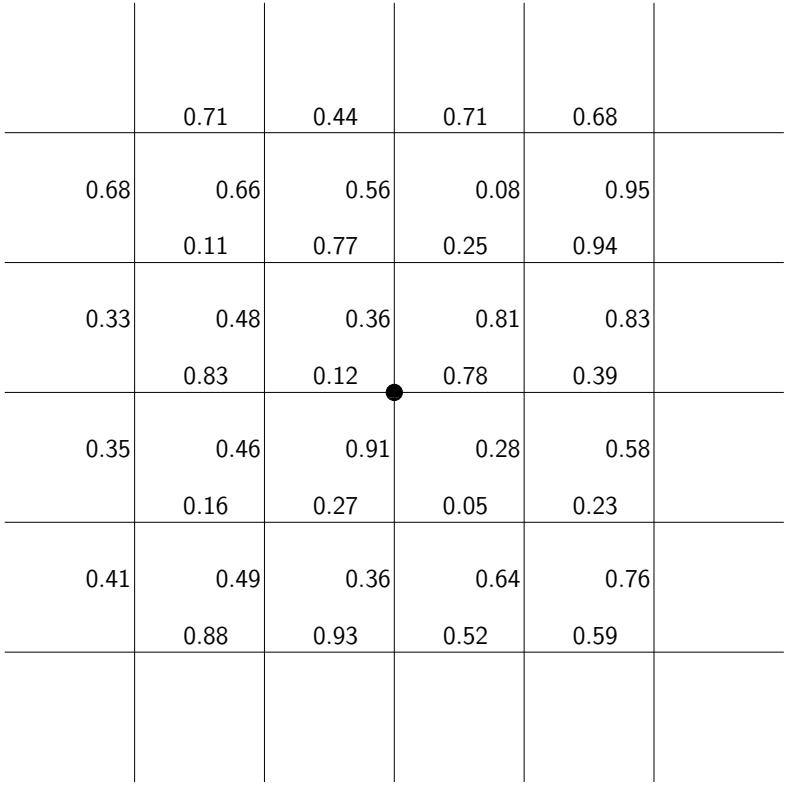

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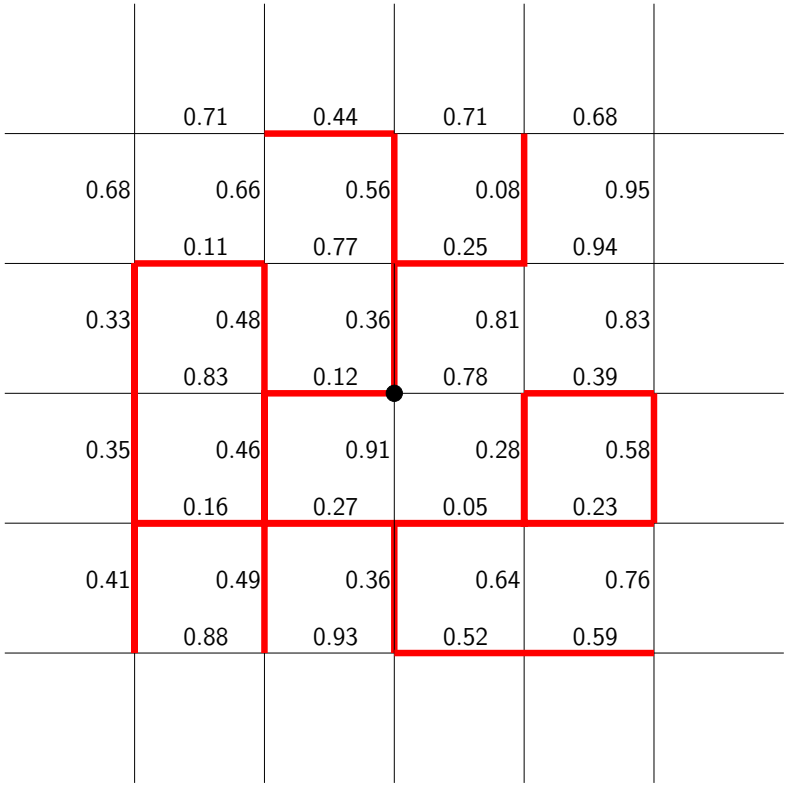

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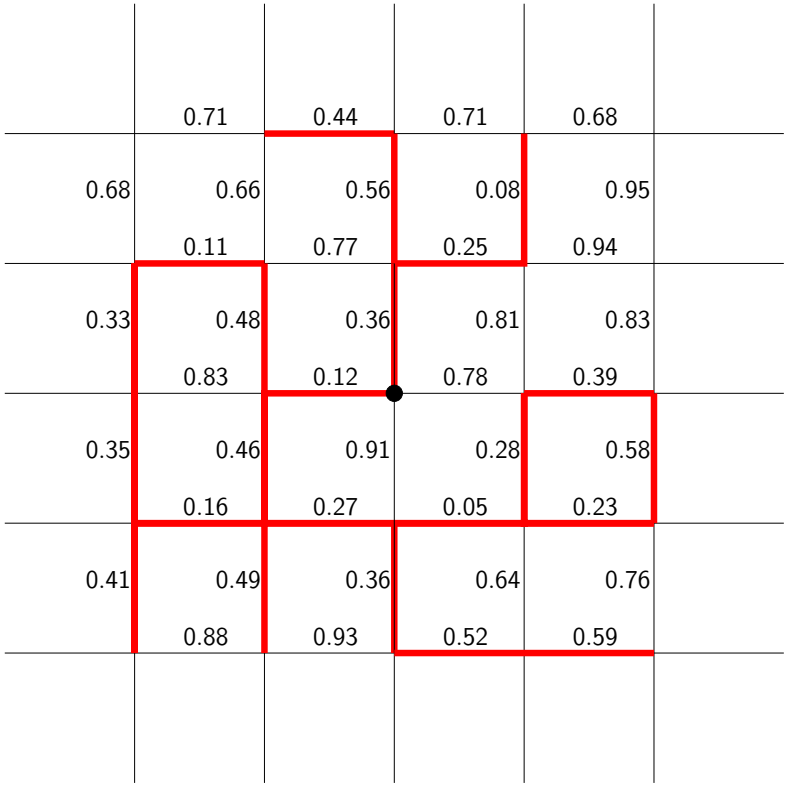
- Here $p = 0.6$.



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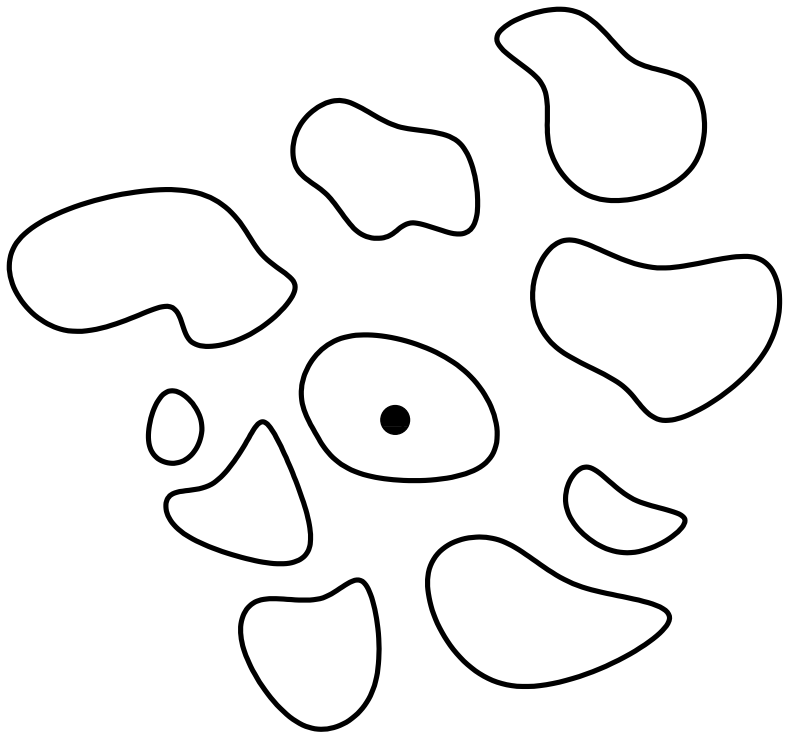

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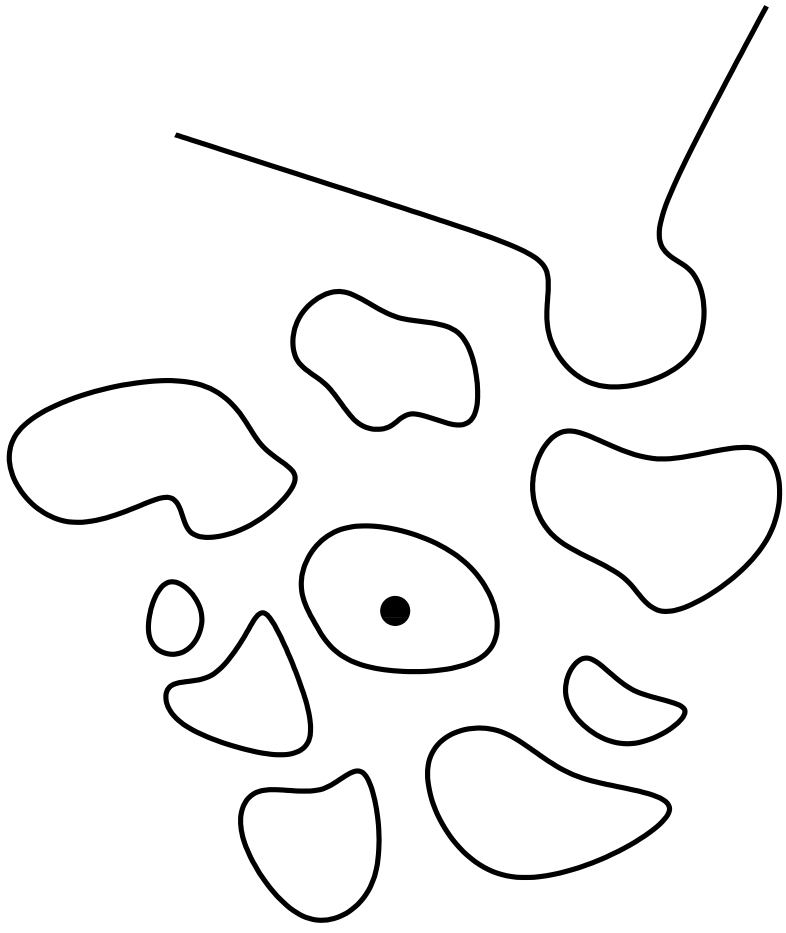

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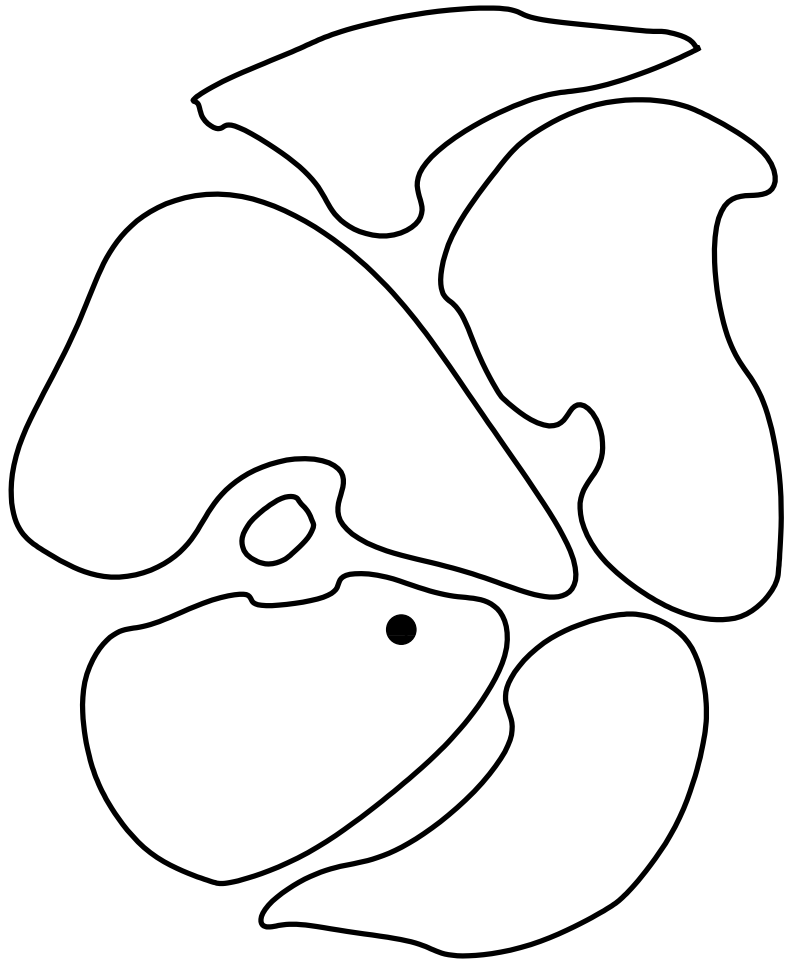

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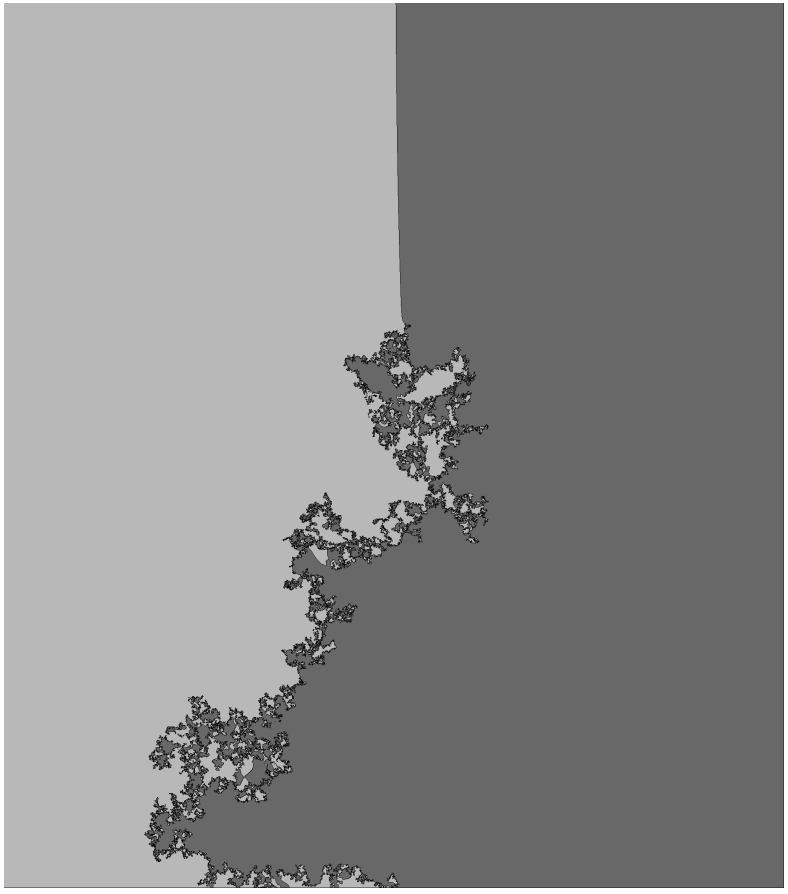

Image by Vincent Beffara

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Animation by Jason Miller

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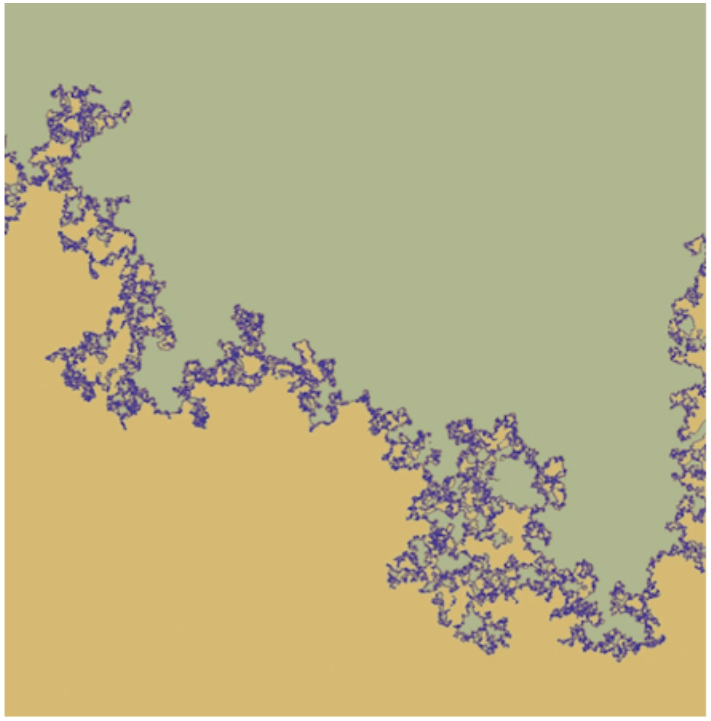

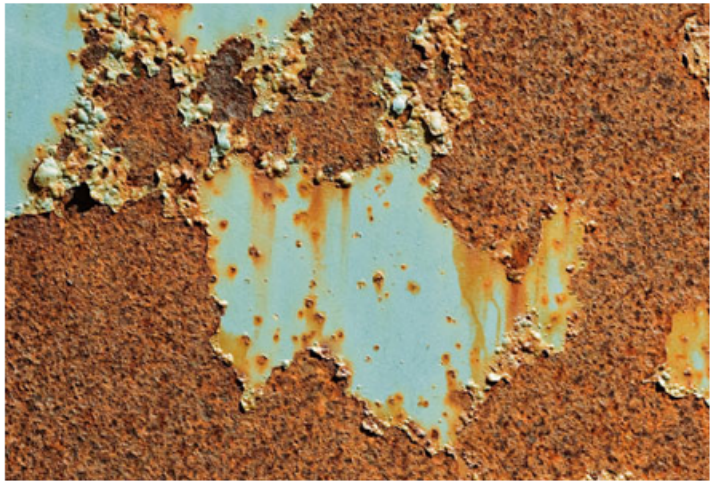

Image from `http://images.math.cnrs.fr/La-percolation-jeu-de-pavages-aleatoires.html`.

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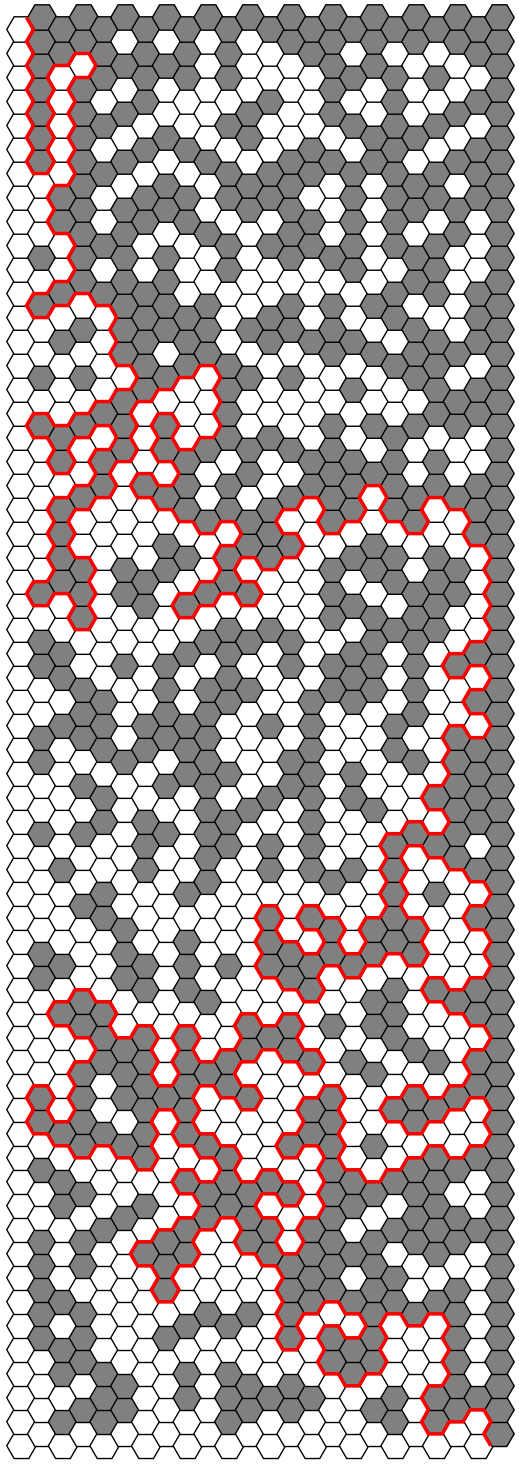
- Percolation interfaces described by SLE(6).
- Universality: choice of lattice should not matter
- But so far we only know the SLE(6) story for the hexagonal lattice, which has special "conformal" structure.



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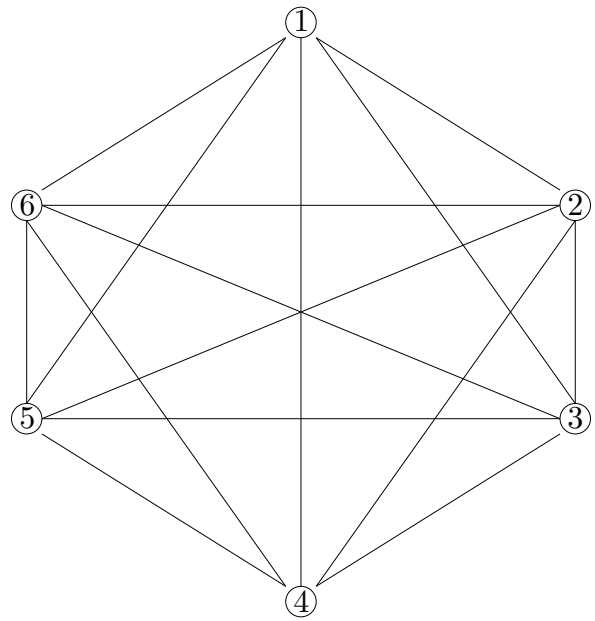
- Probably the most famous open problem in probability: **Prove (or disprove) that**

$$\mathbb{P}(\mathbb{Z}^d_{p_c} \text{ contains an infinite component}) = 0\,.$$

I. Prehistory. B) The Erdős–Rényi random graph

$G(n,p)$: Take vertices labelled $1, 2, \ldots, n$ and fix $p \in [0,1]$. For each pair of vertices, put an edge between them with probability p, independently for different pairs.

This is percolation on the complete graph at parameter p.



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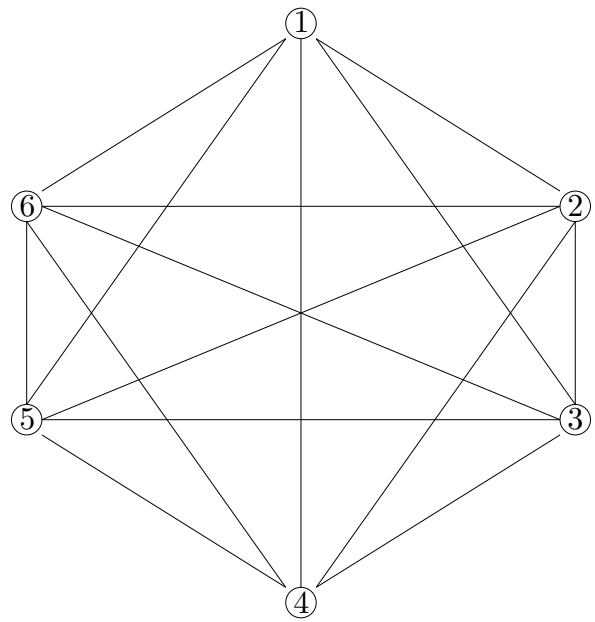
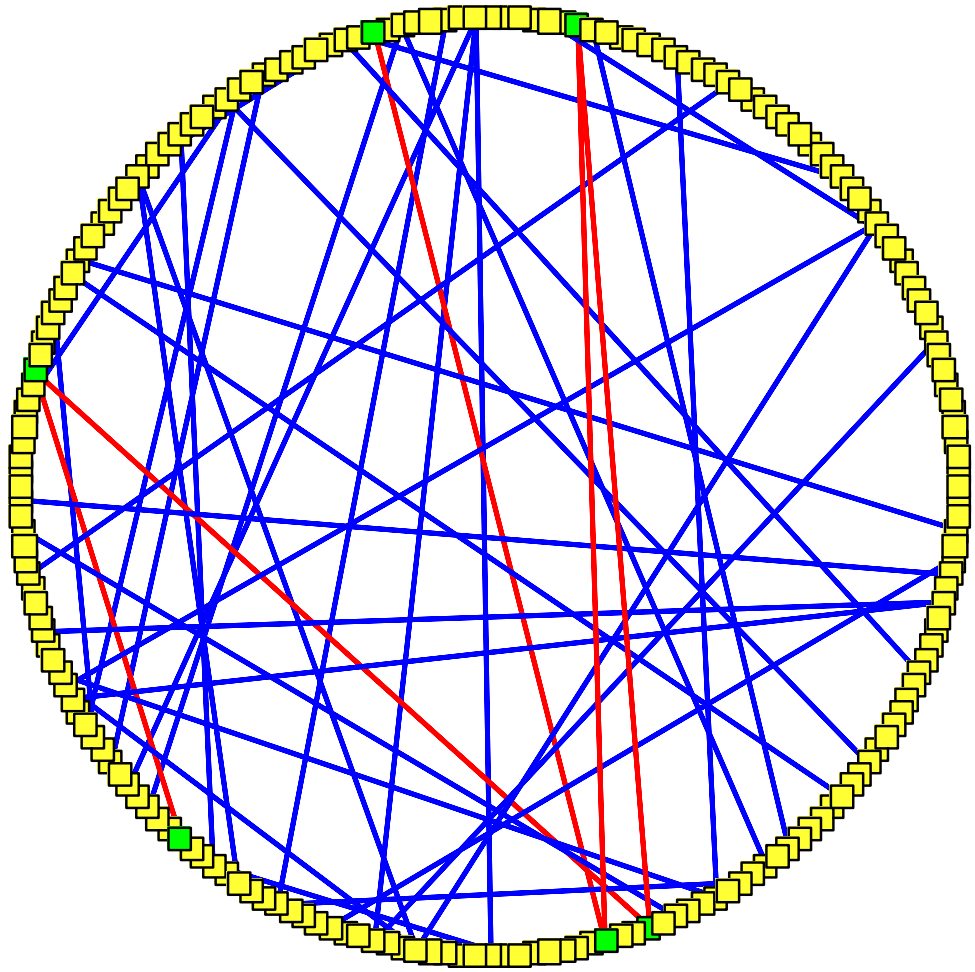


We will consider the case where $p = c/n$ for a constant $c > 0$, so that a single vertex has a Binomial$(n-1, c/n)$ number of neighbours, with mean approximately c for large n.

II. Erdős–Rényi random graph

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

$n = 200,\ c = 0.4$



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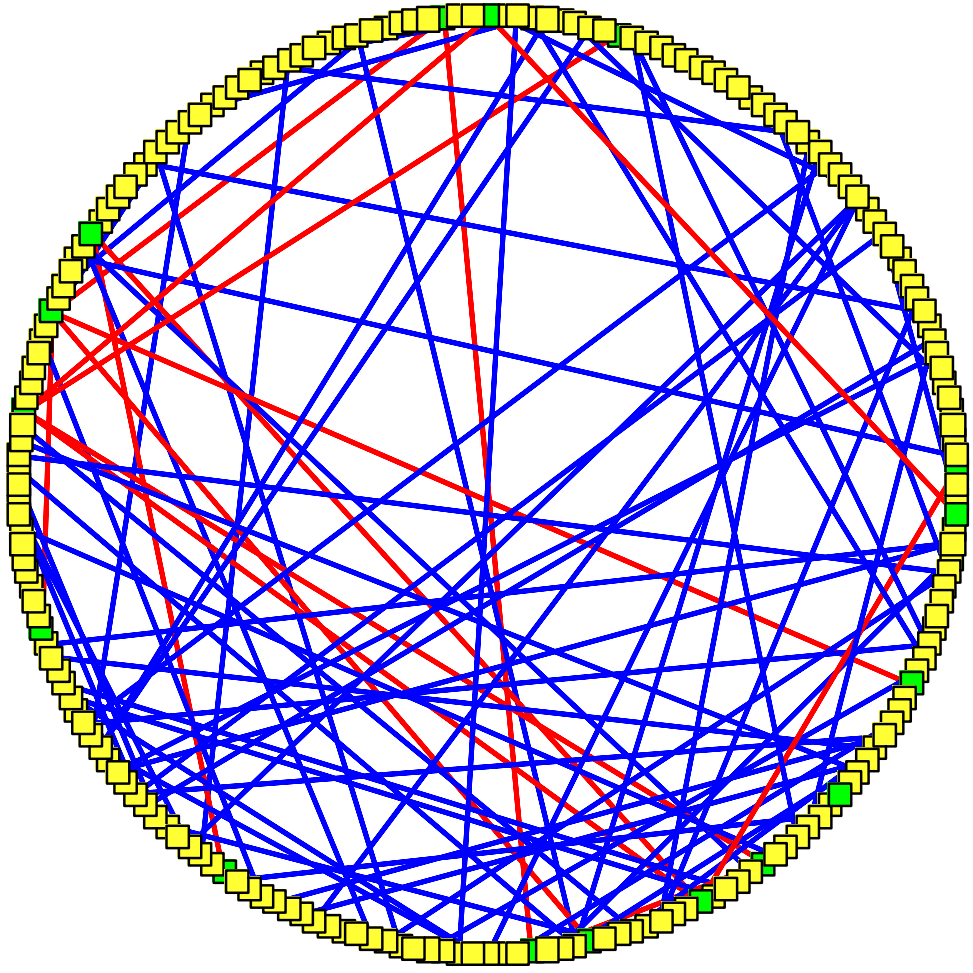

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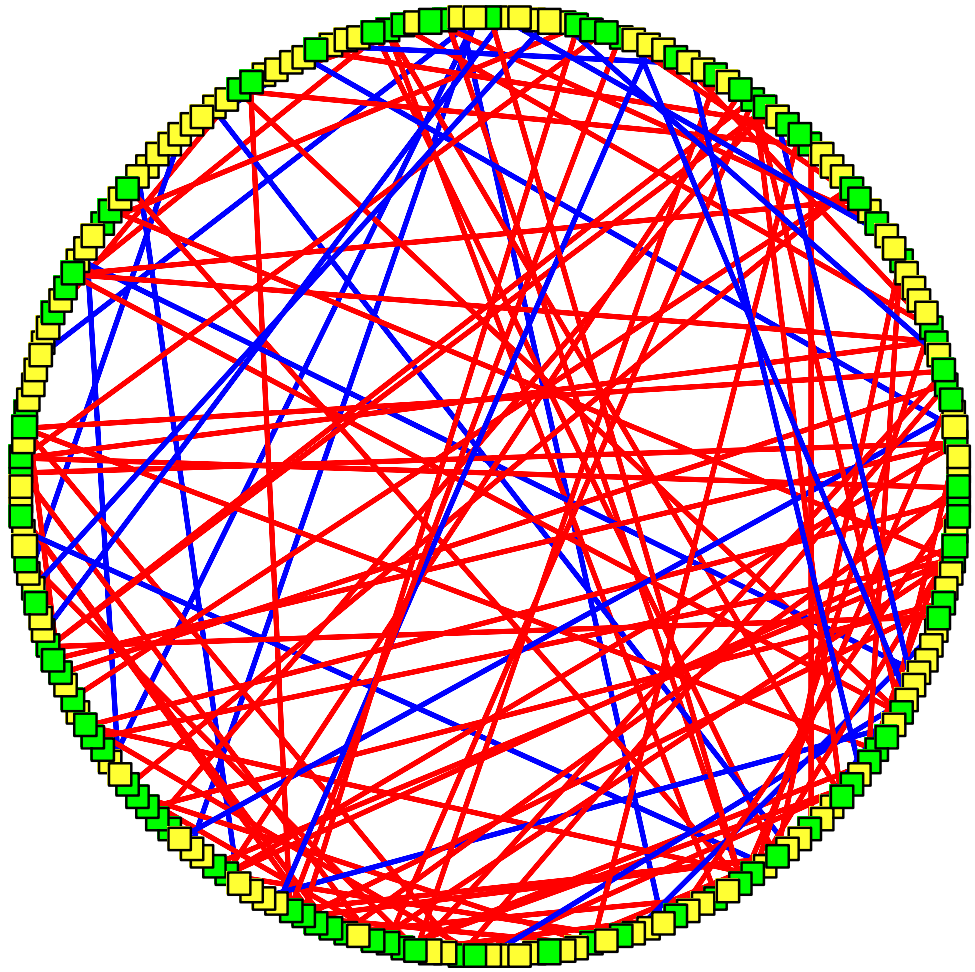

II. The phase transition

Theorem. (Erdős and Rényi (1960))
For $G(n, c/n)$ the following statements hold with probability tending to 1 as $n \to \infty$.

- If $c < 1$, the largest component contains $O(\log n)$ vertices.
- If $c > 1$, the largest component contains $\Theta(n)$ vertices and the rest contain $O(\log n)$ vertices each.

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In 2007 I started studying the structure of the critical random graph viewed as a *random metric space* (distance given by graph distance). Prototype question: what is the expected distance between two uniformly random vertices in the largest component?

III. Convergence of metric measure spaces

Let $(M_n, d_n, \mu_n)_{n\geq 1}$ be a sequence of (random) metric spaces (M_n, d_n) each endowed with a probability measure μ_n on (the Borel sets of) M_n.

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Idea: as we select more and more points, we get better and better (finite) approximations to the true space (M_n, d_n). If each sequence of finite approximations converges, we say that the spaces do.

III. Convergence of metric measure spaces

Fix $k \geq 2$ and let $U_1^n, U_2^n, \ldots, U_k^n$ be independent random variables, each with distribution μ_n. For $1 \leq i, j \leq k$, let

$$D_{i,j}^{n,k} = d_n(U_i^n, U_j^n)$$

and let $D^{n,k} = (D_{i,j}^{n,k})_{1 \leq i,j \leq k}$ be the pairwise distance matrix.

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in distribution as $n \to \infty$. **N.B.** If the metric spaces (M_n, d_n, μ_n) are random then the matrices $D^{n,k}$ have "two levels" of randomness – this requires care to set up formally.

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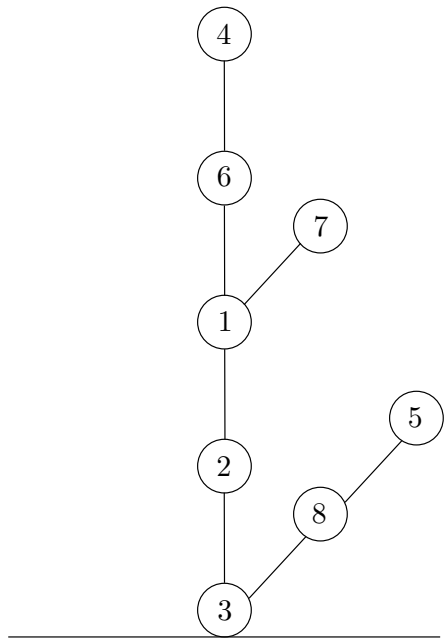

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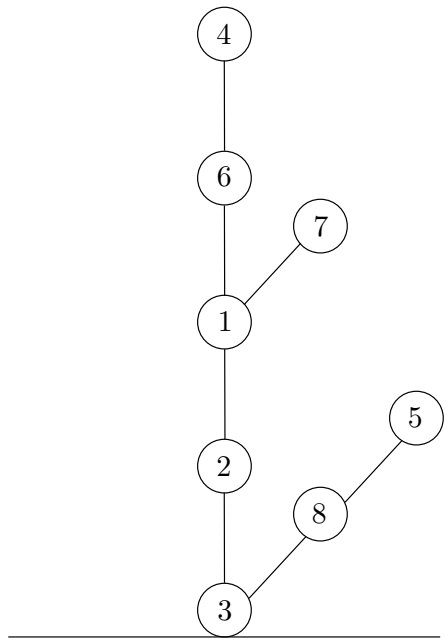


What does the tree look like as n gets large?

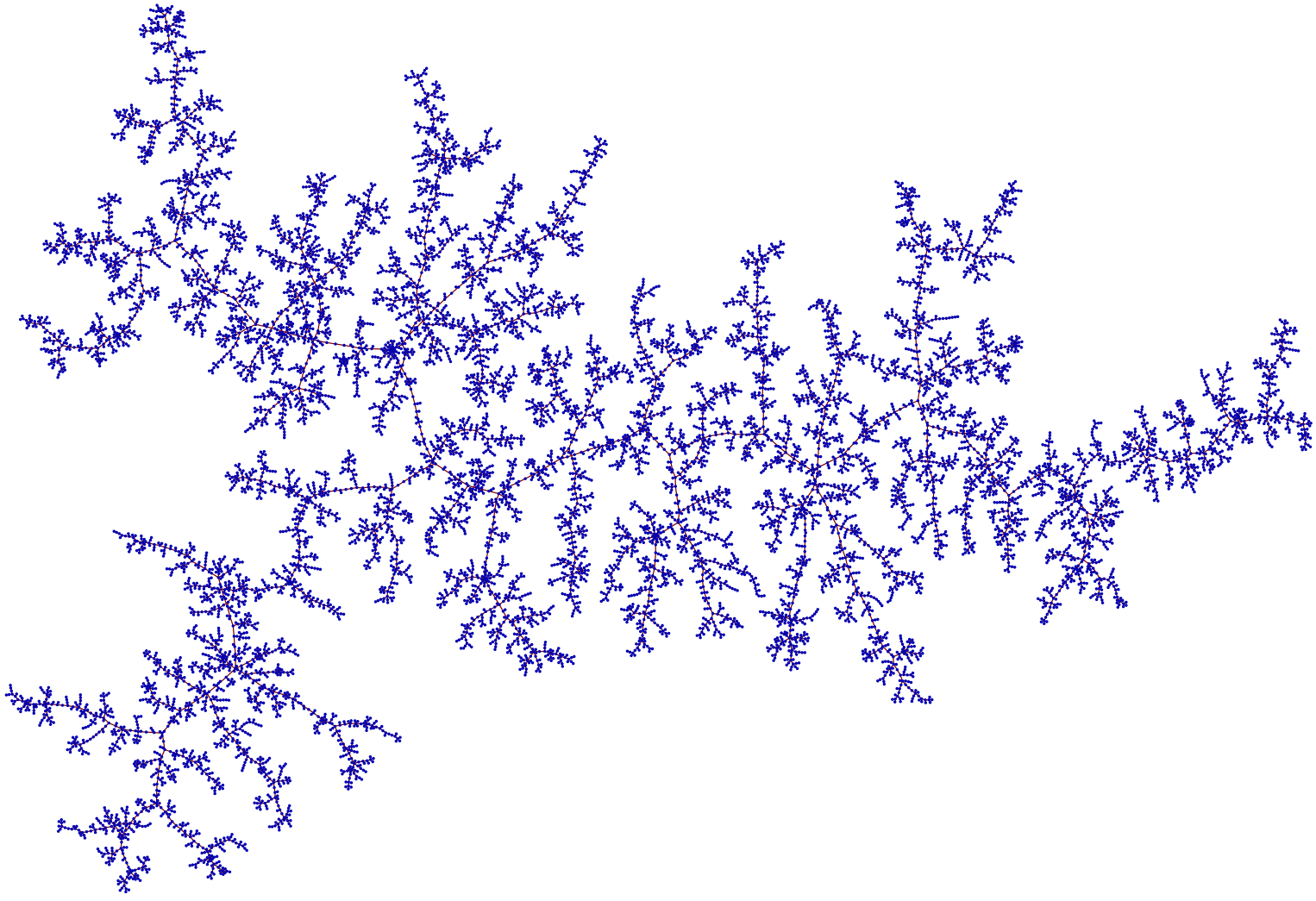
IV. Convergence of the uniform random tree

Theorem. (Aldous (1991))
As $n \to \infty$,

$$\frac{1}{\sqrt{n}} T_n \xrightarrow{d} \mathcal{T}$$

where $\mathcal{T}$ is the Brownian continuum random tree (CRT).

IV. The Brownian continuum random tree



[Picture by Igor Kortchemski]

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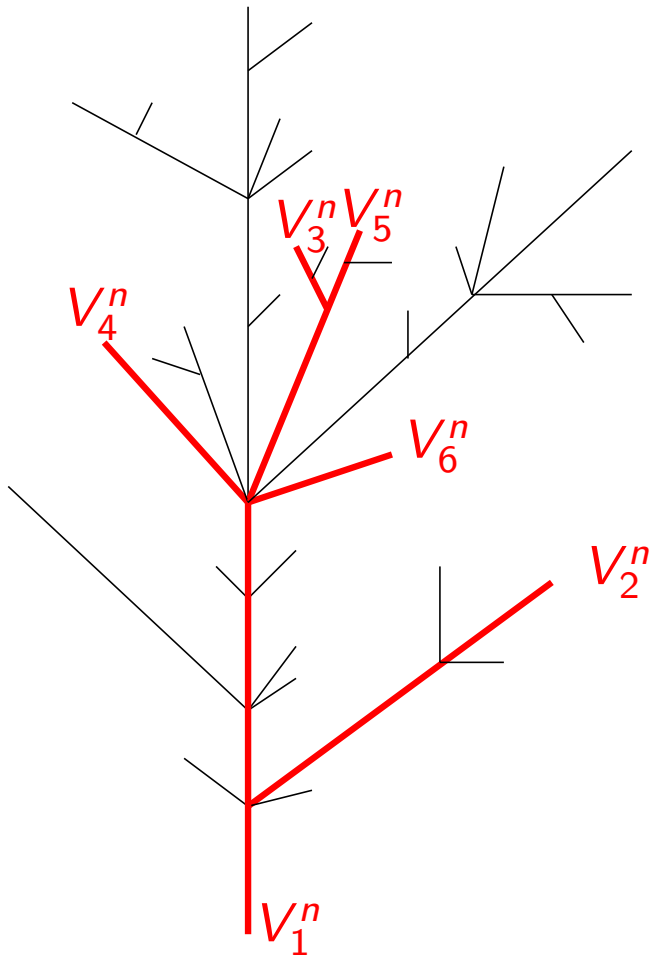
Fix $k \geq 2$, take the uniform random tree T_n and pick k independent uniform random vertices, $V_1^n, V_2^n, \ldots, V_k^n$.

$k = 6$:



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The matrix of pairwise distances between these independent uniform points is

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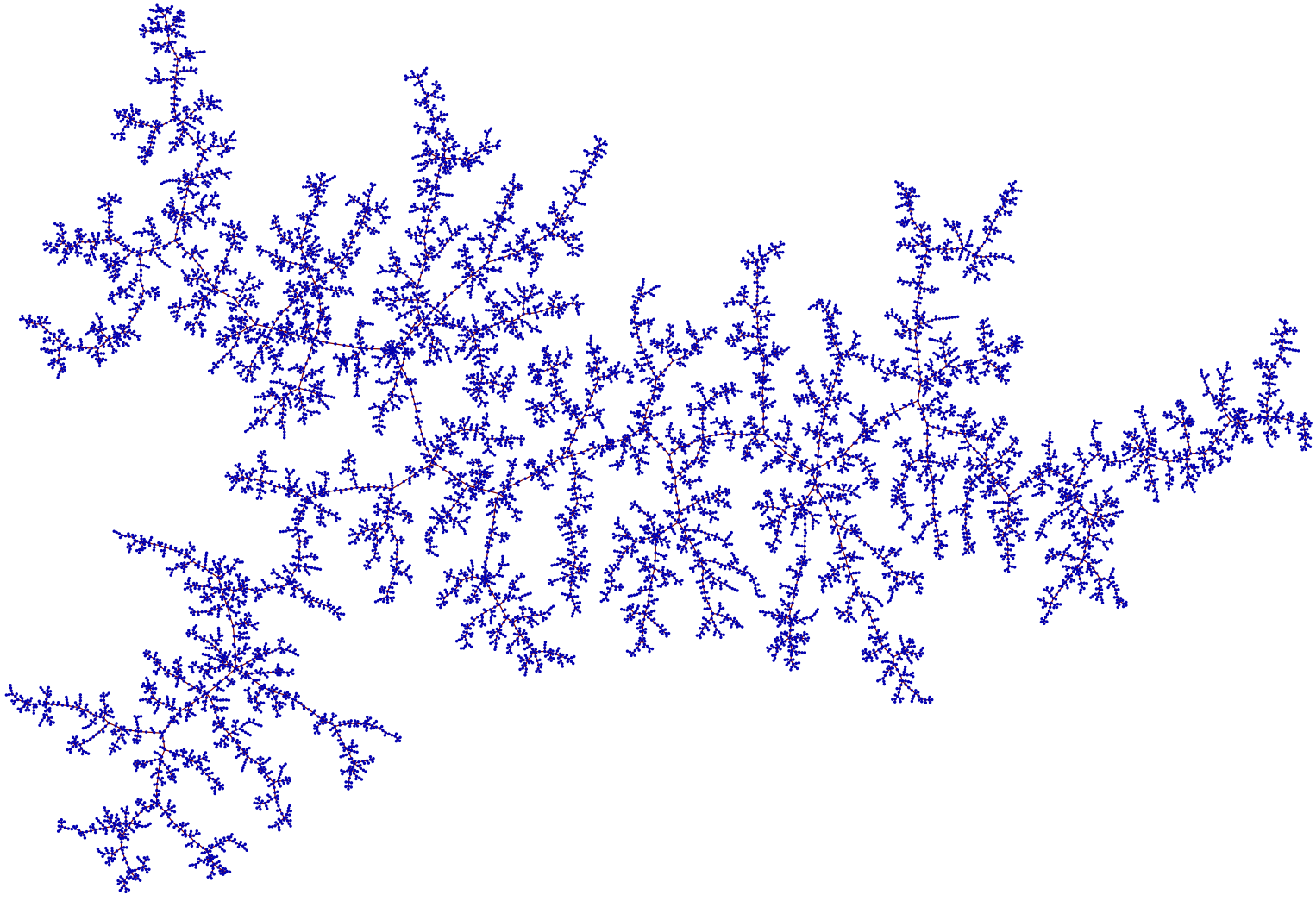
The $1/\sqrt{n}$ scaling says T_n is in some sense 2-dimensional: n points, diameter of order $\sqrt{n}$.

IV. The Brownian continuum random tree



[Picture by Igor Kortchemski]

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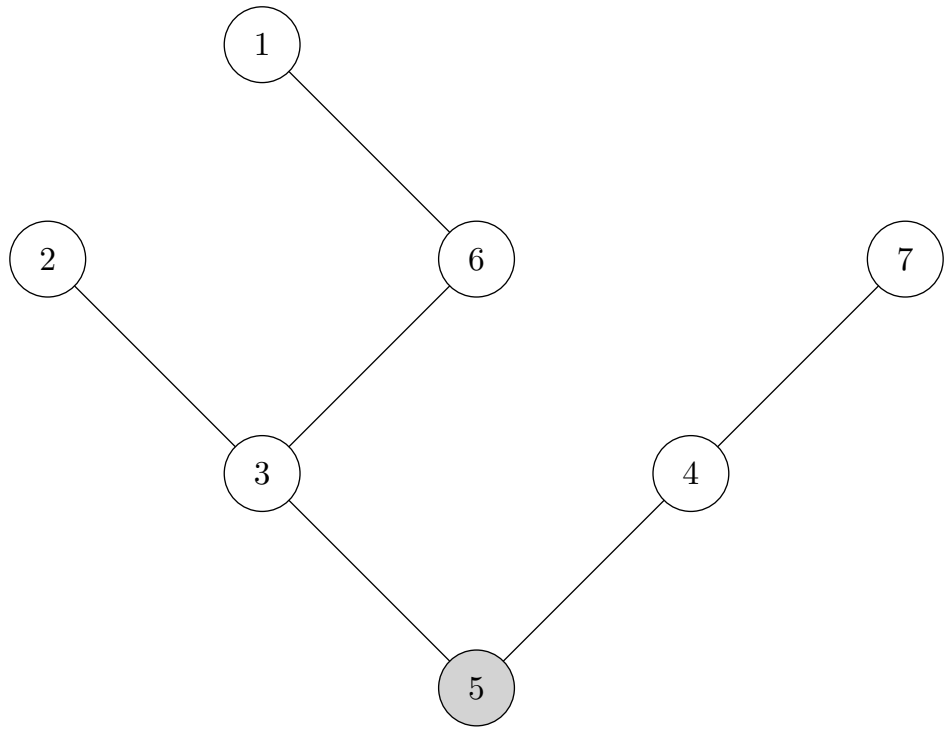

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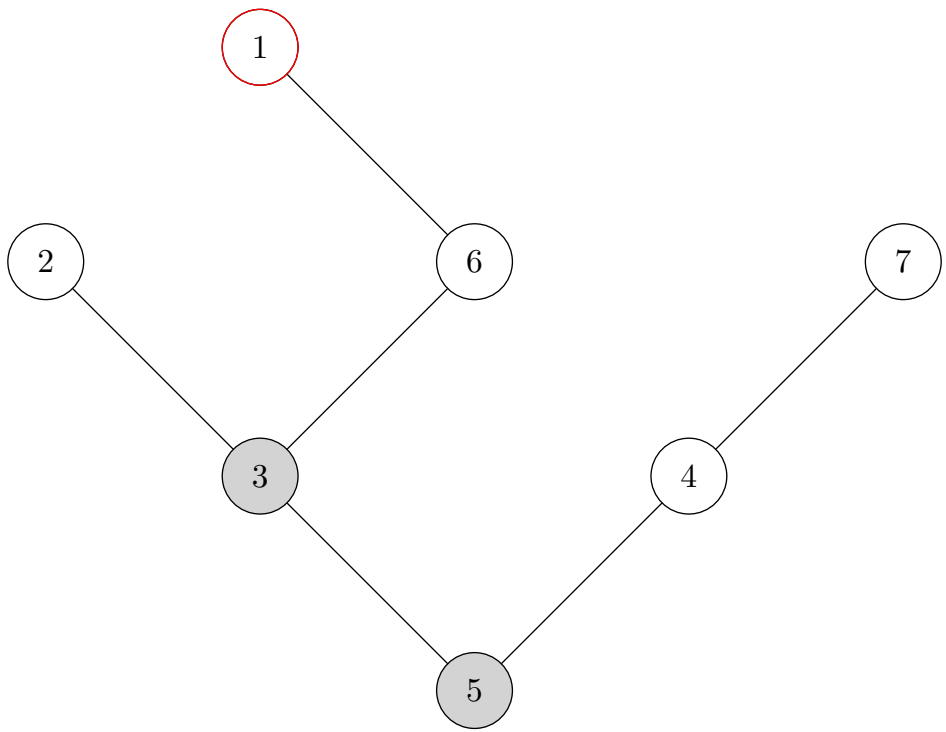

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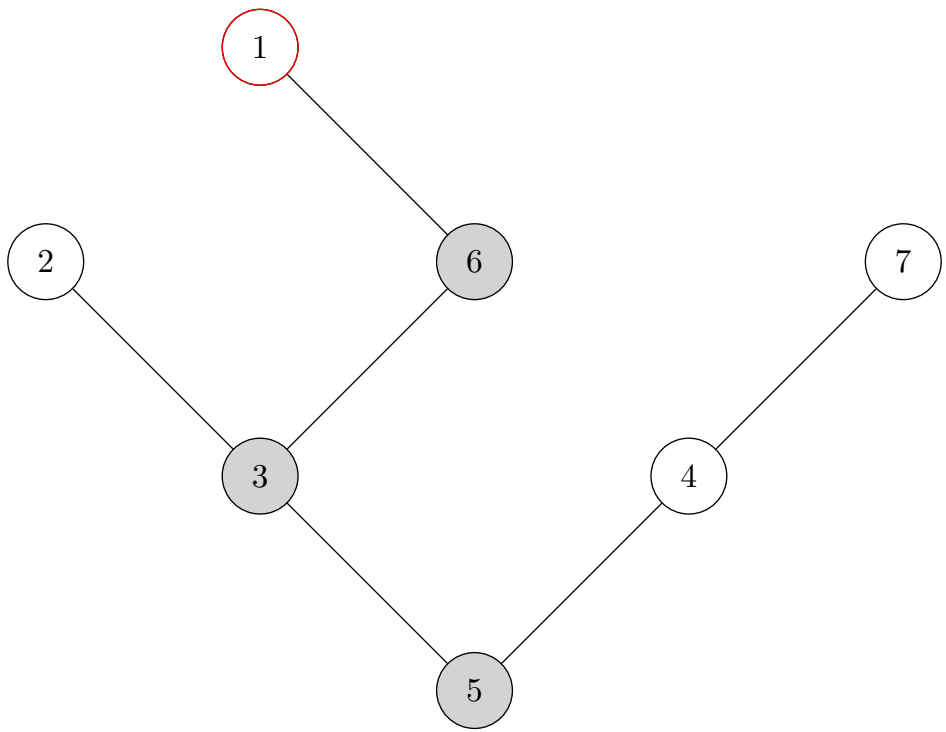

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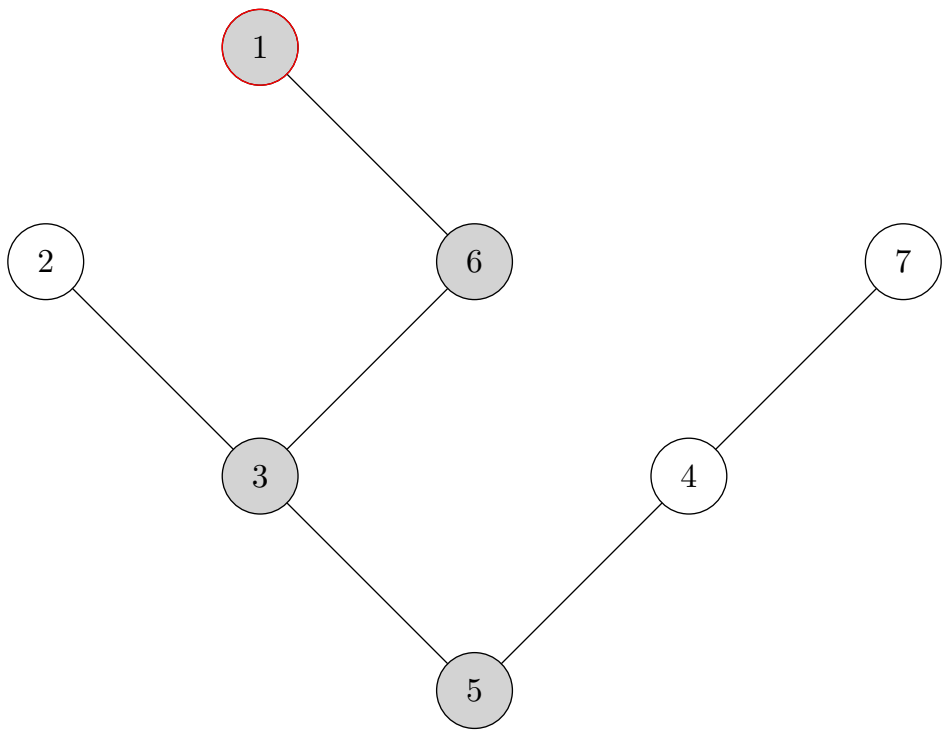

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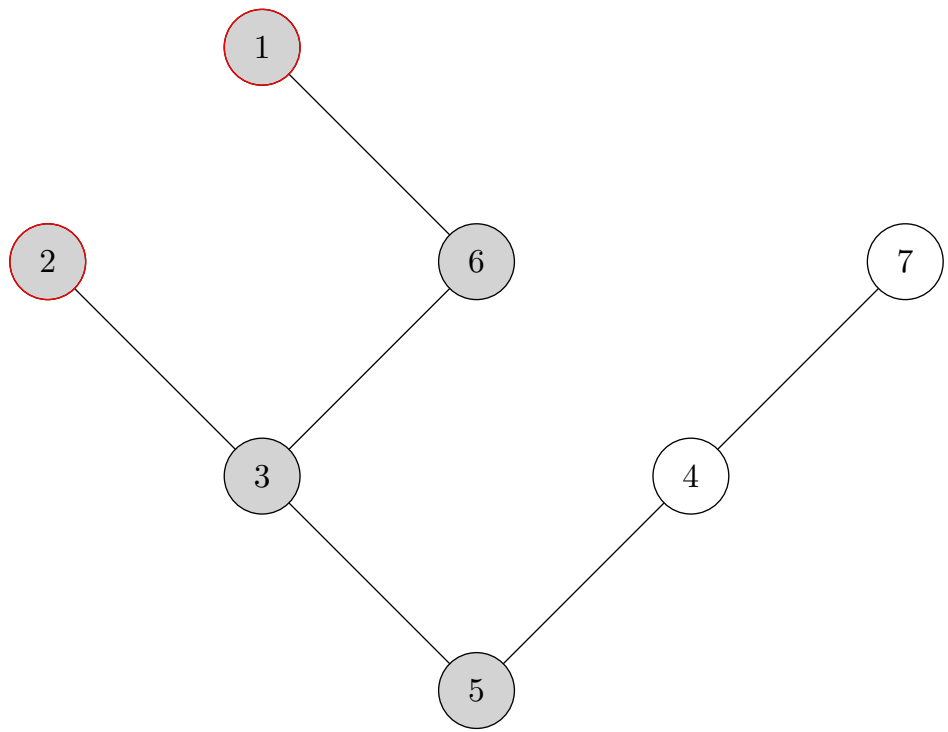

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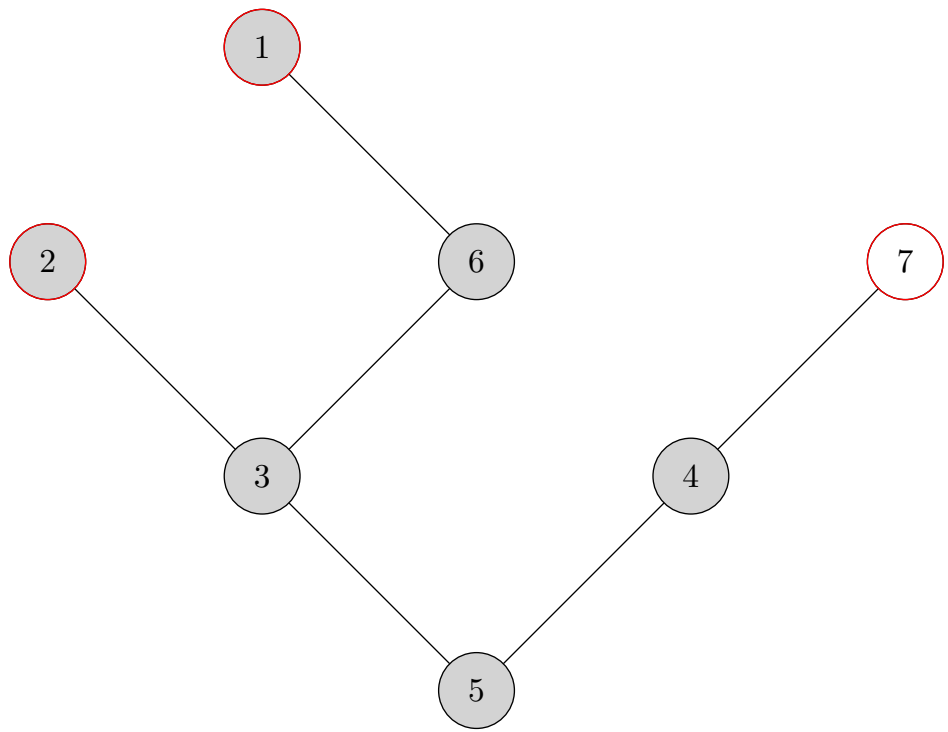

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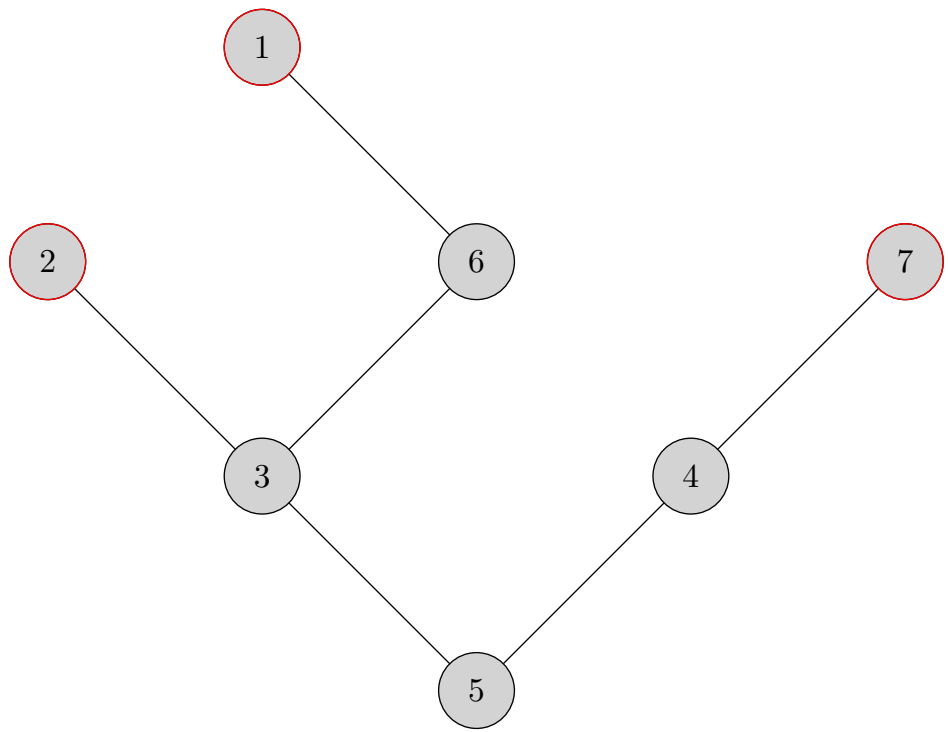
$(5, 3, 6, 3, 5, 4)$



V. Random finite-dimensional distributions from Prüfer codes

We generate a labelled, unordered tree, with vertex labels 1,2,...,n, by sampling a uniformly random element $(S_1, S_2, \ldots, S_{n-1})$ from the set $[n]^{n-1} = \{(s_1, \ldots, s_{n-1}) : s_i \in [n] \text{ for } 1 \leq i \leq n-1\}$, and *inverting* the above encoding. The resulting tree is rooted at the vertex with label S_1. We build the rest of the tree step-by-step.

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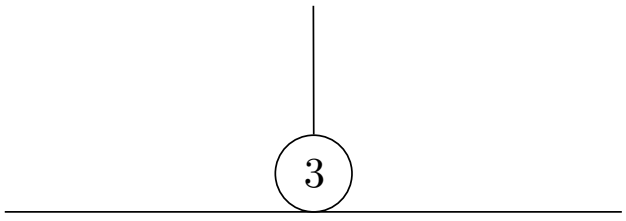
Finish by adding a vertex corresponding to the final leftover label.

V. Random finite-dimensional distributions from Prüfer codes

$(3, 2, 1, 6, 3, 8, 1)$



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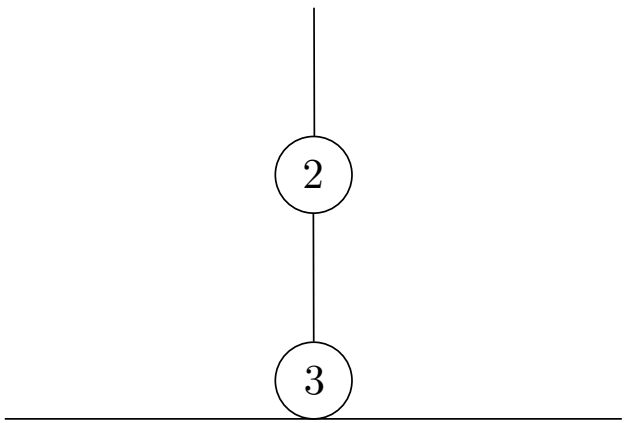

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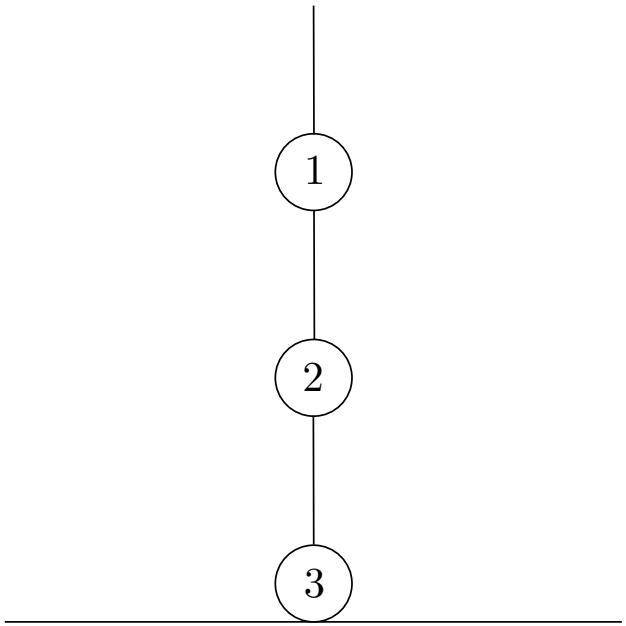

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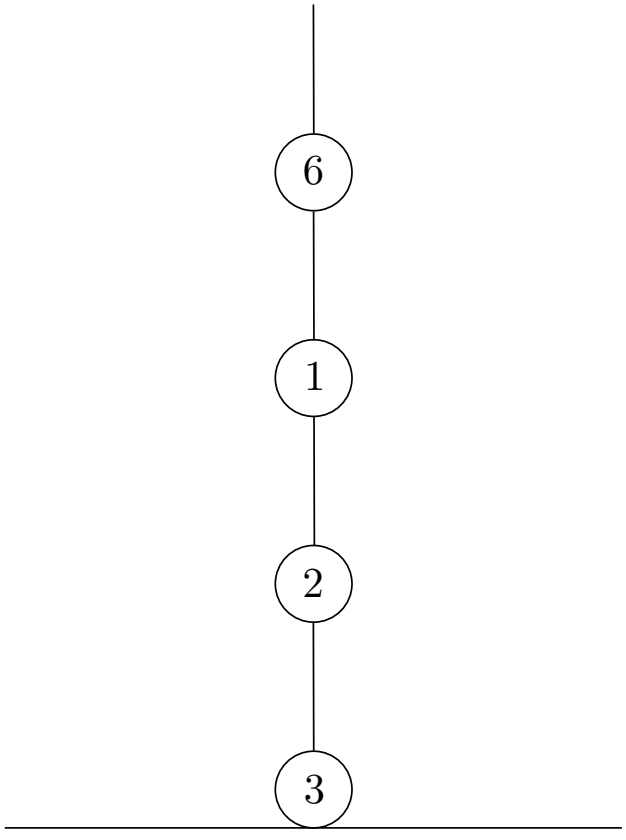

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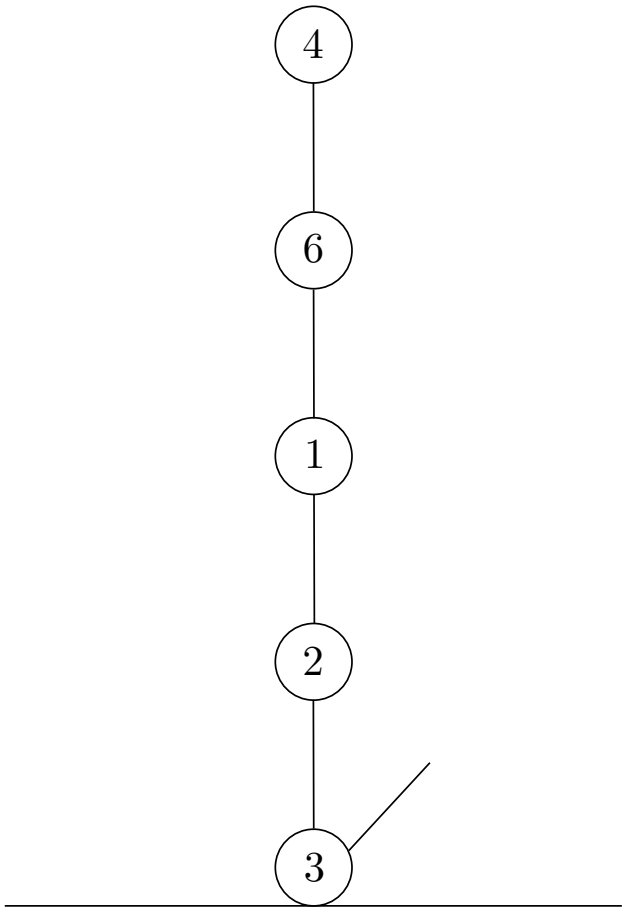

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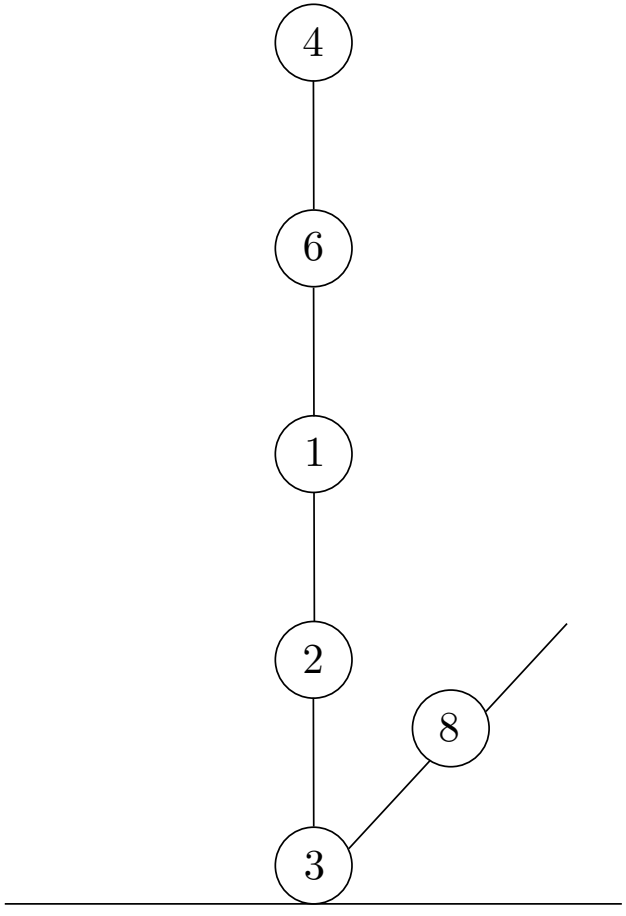

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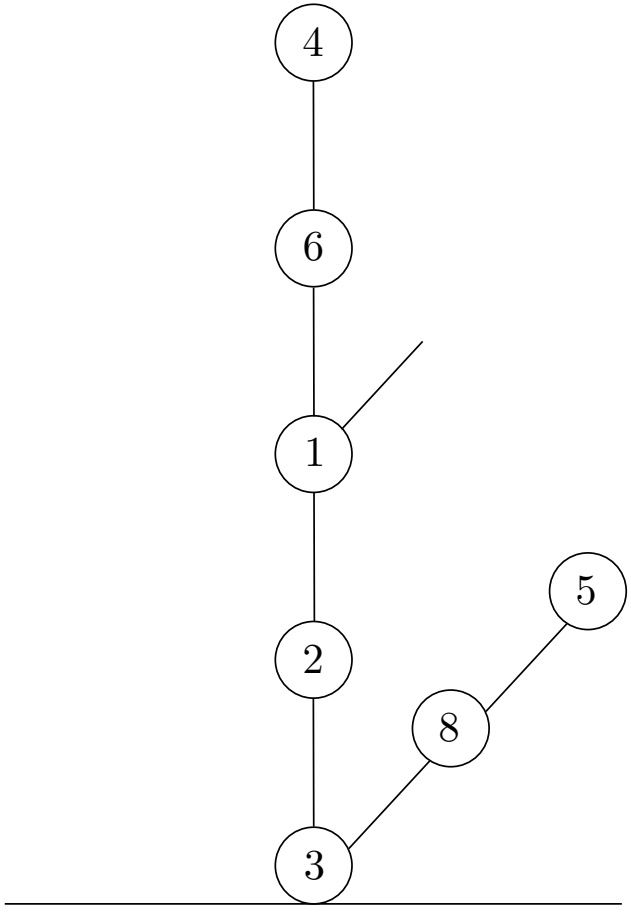
$(3, 2, 1, 6, 3, 8, 1)$ Stick 1: 32164; Stick 2: 385



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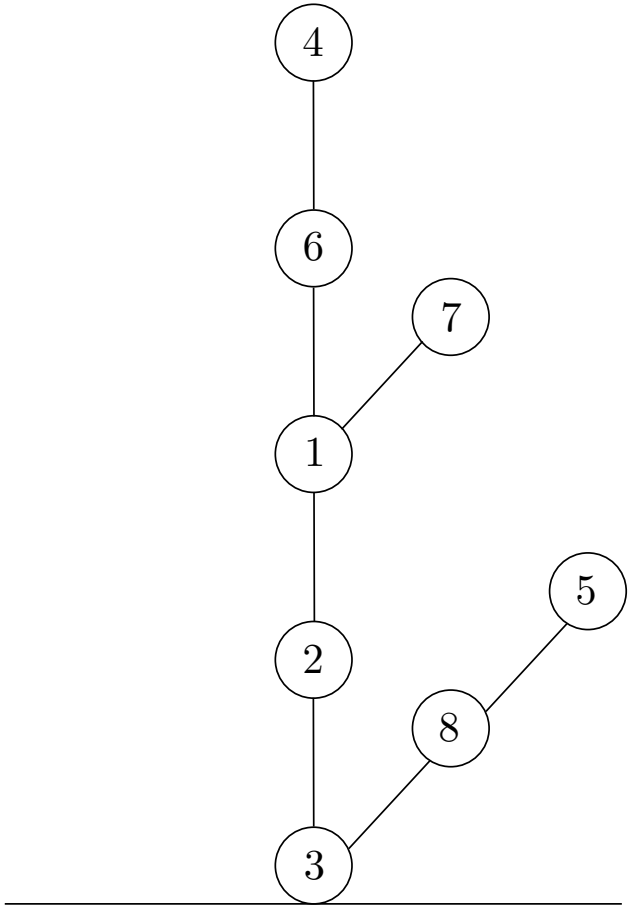

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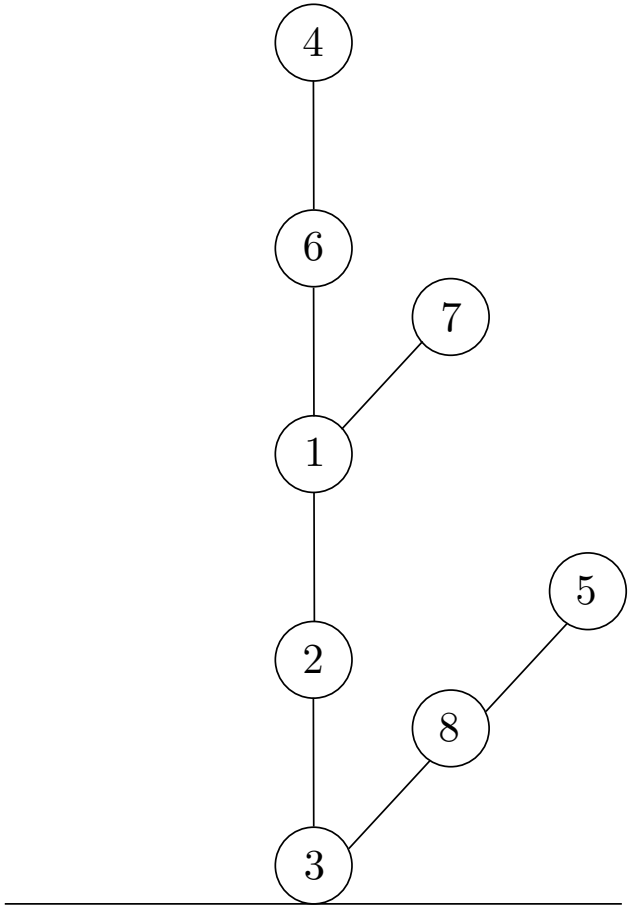

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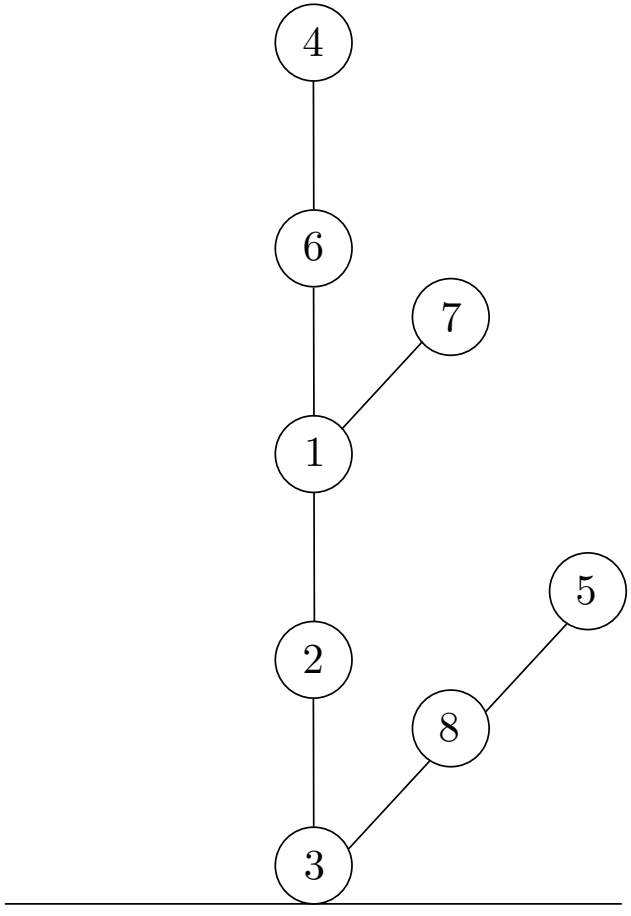
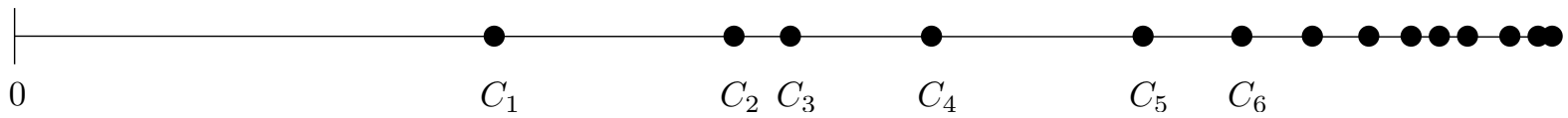


- Sticks get shorter on average as the process goes along since repetitions become more likely.
- Birthday paradox: for fixed k, for large n, the first k sticks have length of order $n^{1/2}$.

V. Continuous version of Prüfer encoding: Line-breaking

Take $E_1, E_2, \ldots, E_k$ independent and identically distributed Exponential(1/2) random variables, i.e. $\mathbb{P}(E_1 > x) = e^{-x/2}$. For $j \geq 1$, let $C_j = \sqrt{\sum_{i=1}^{j} E_i}$.



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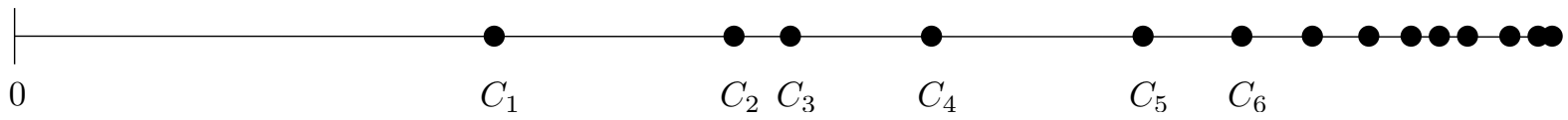

Consider the line-segments $[0, C_1), [C_1, C_2), \ldots, [C_{k-1}, C_k)$.

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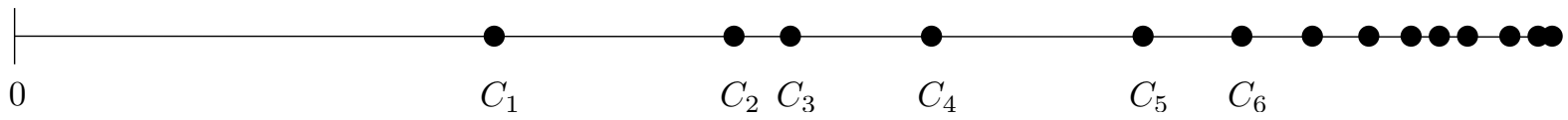

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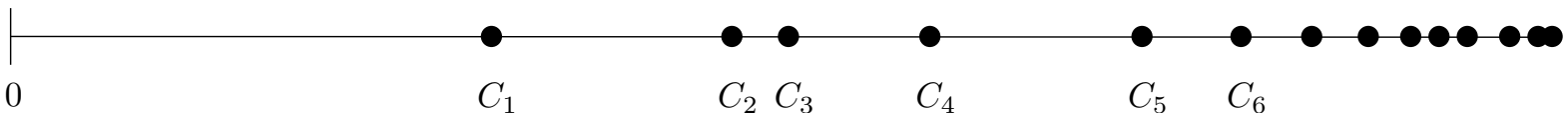

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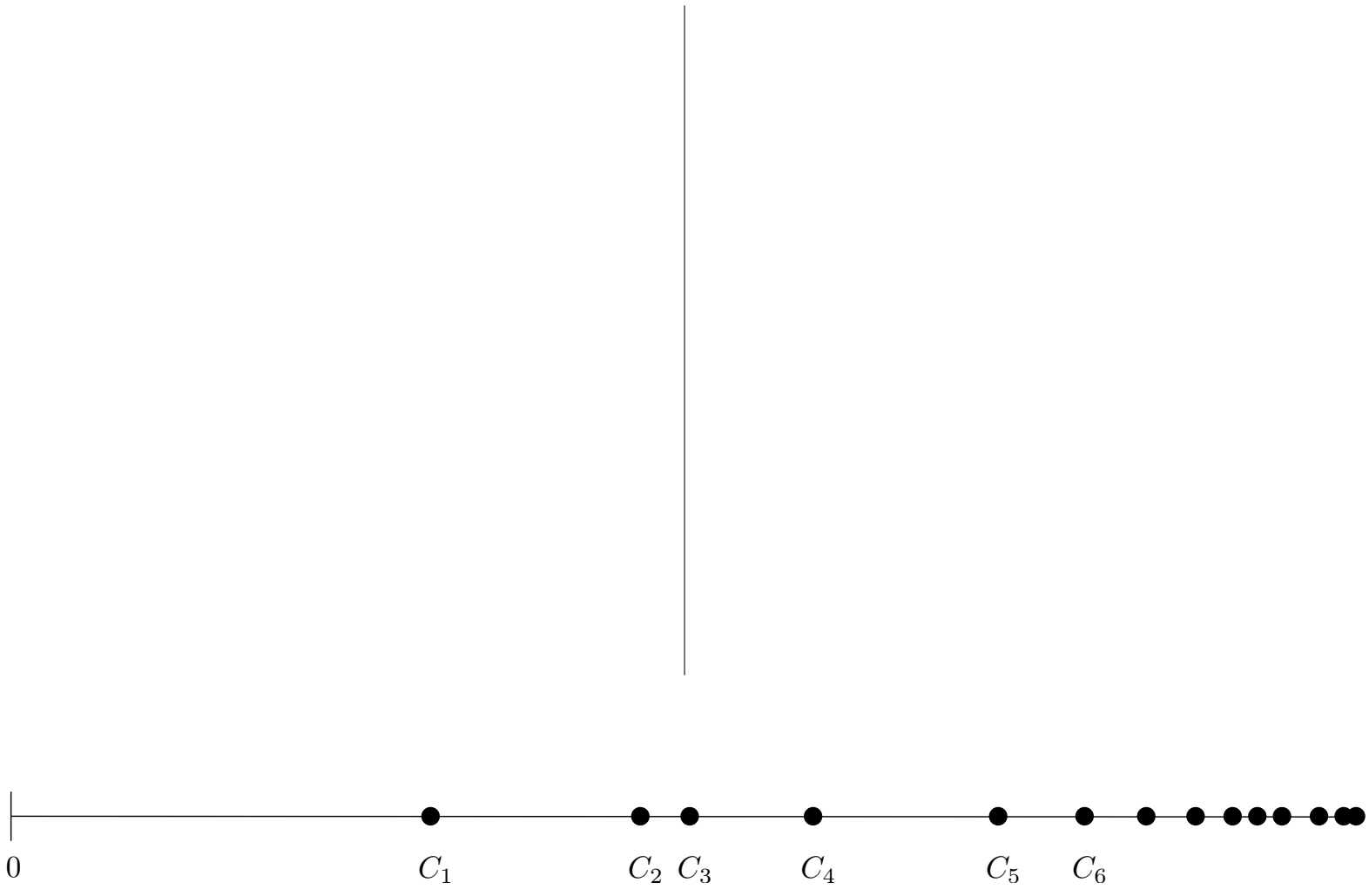
For $2 \leq i \leq k$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.

V. Line-breaking



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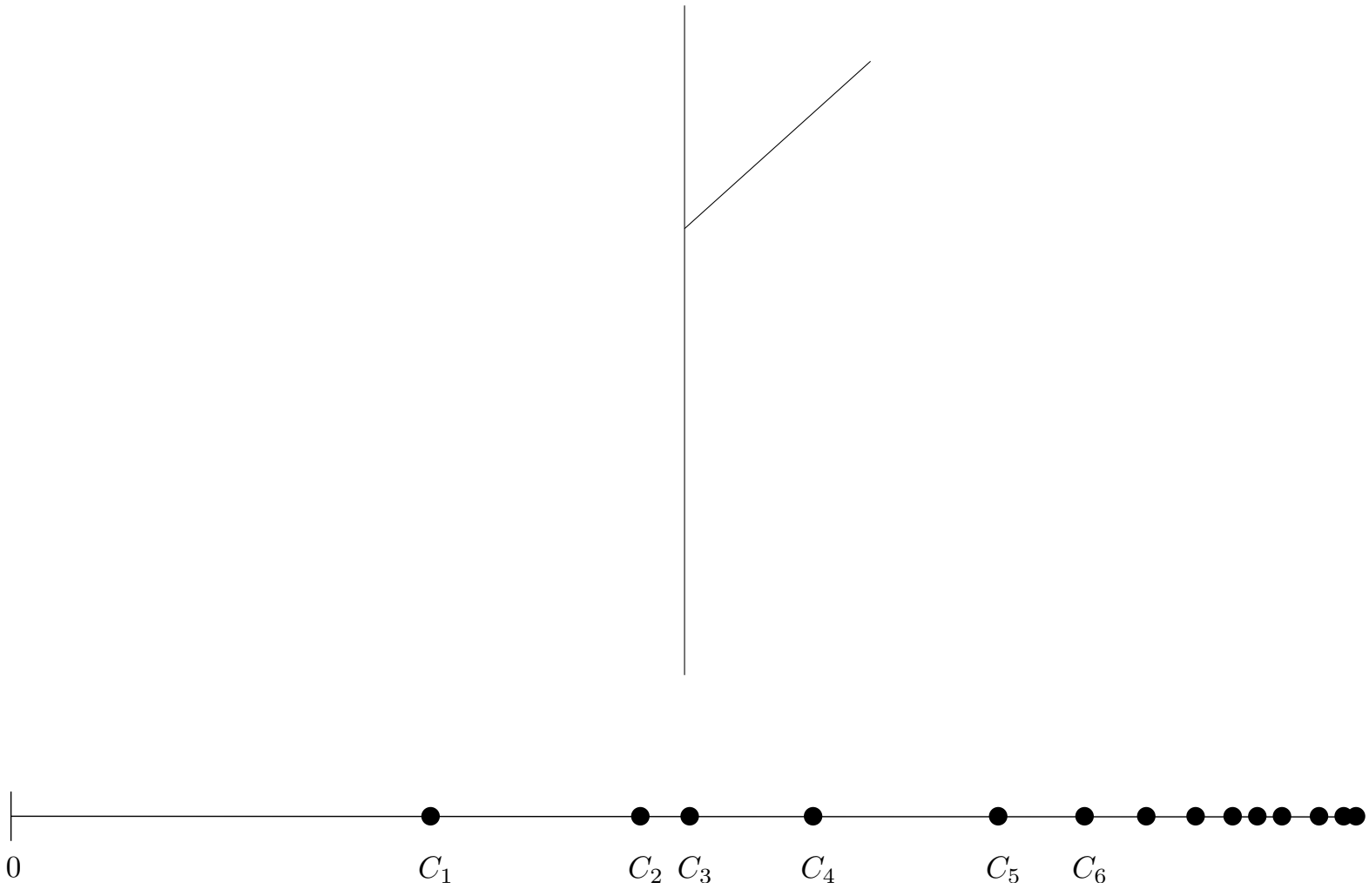

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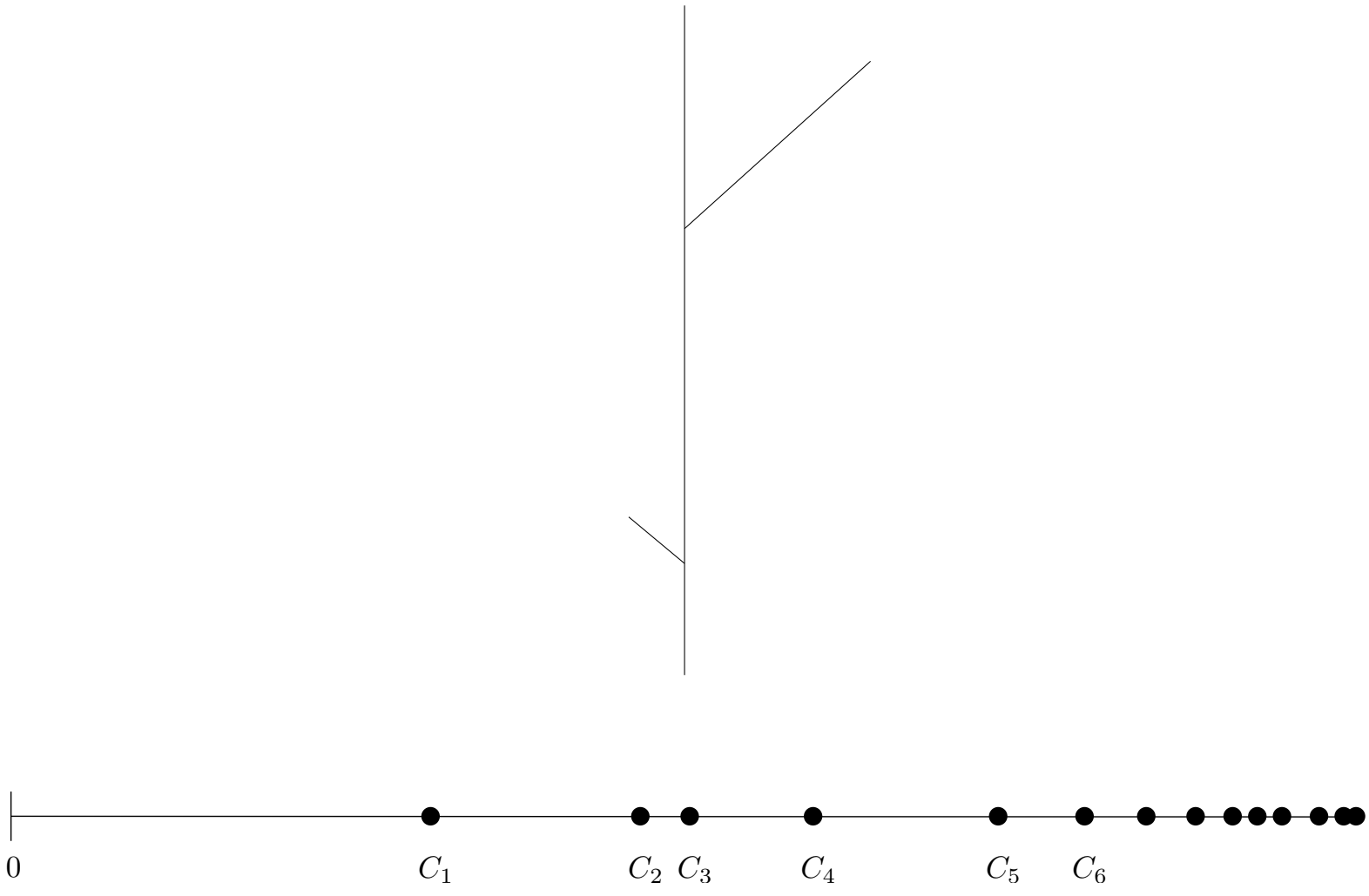

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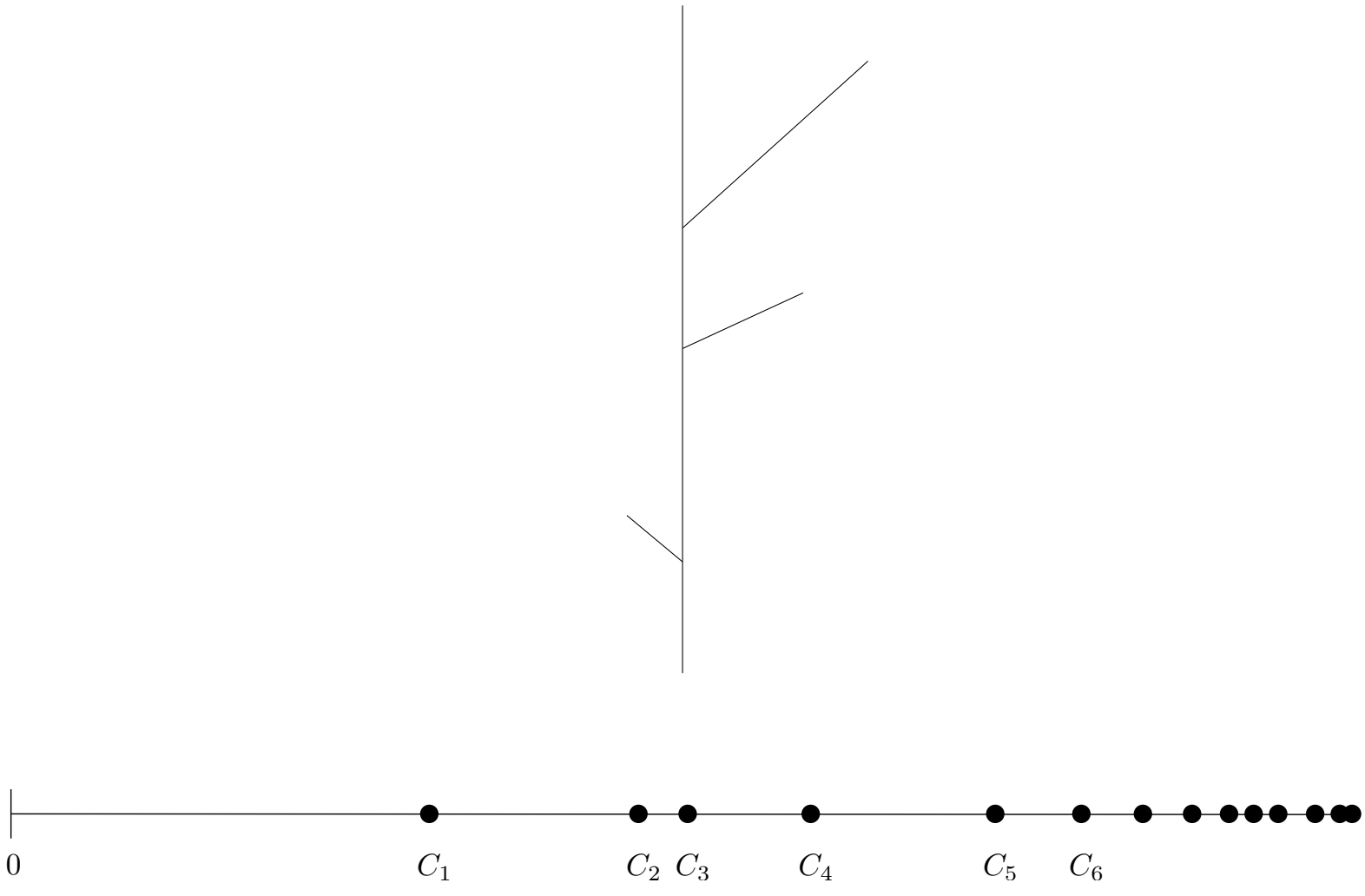

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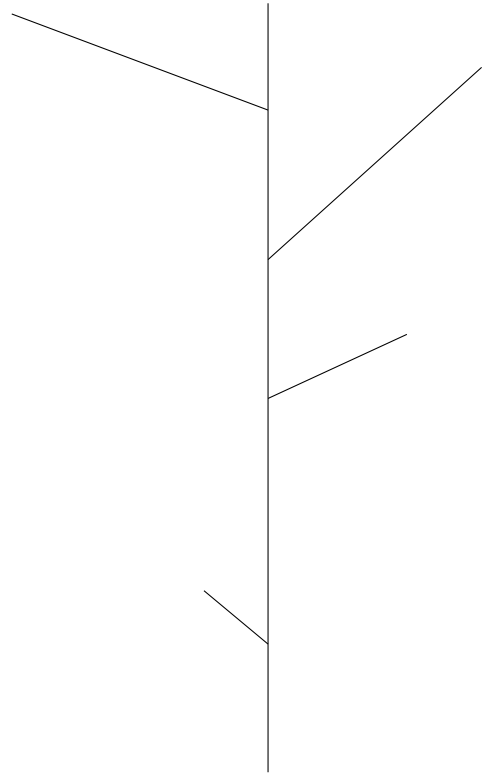

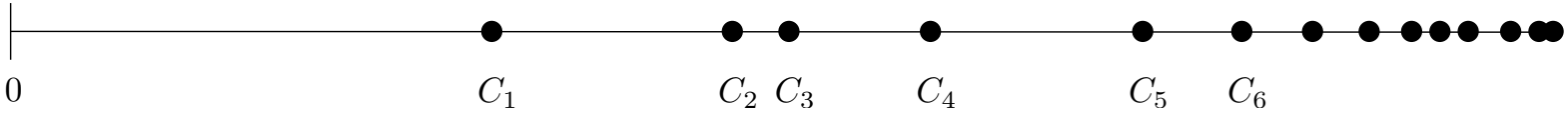

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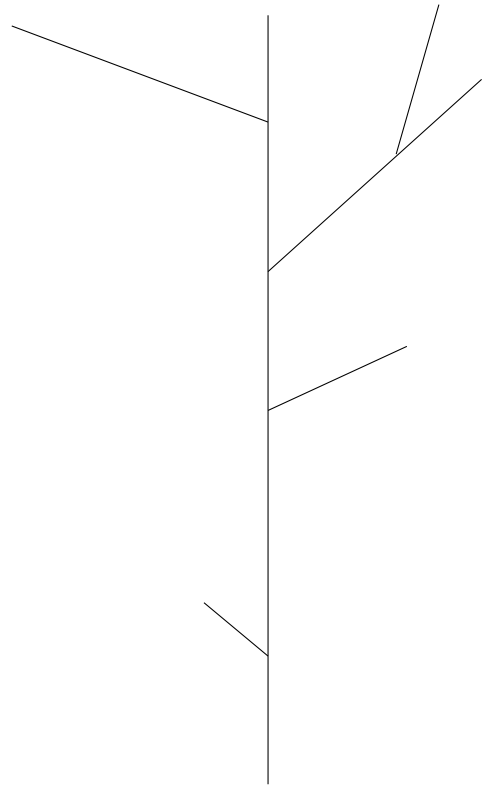

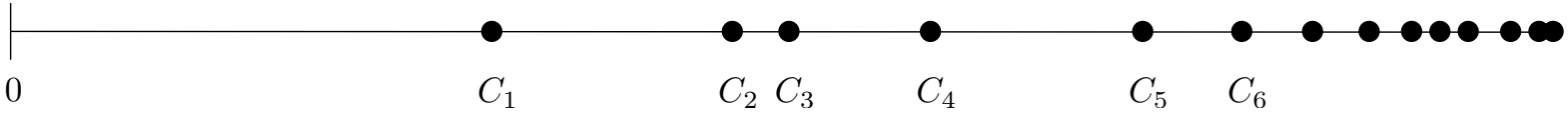

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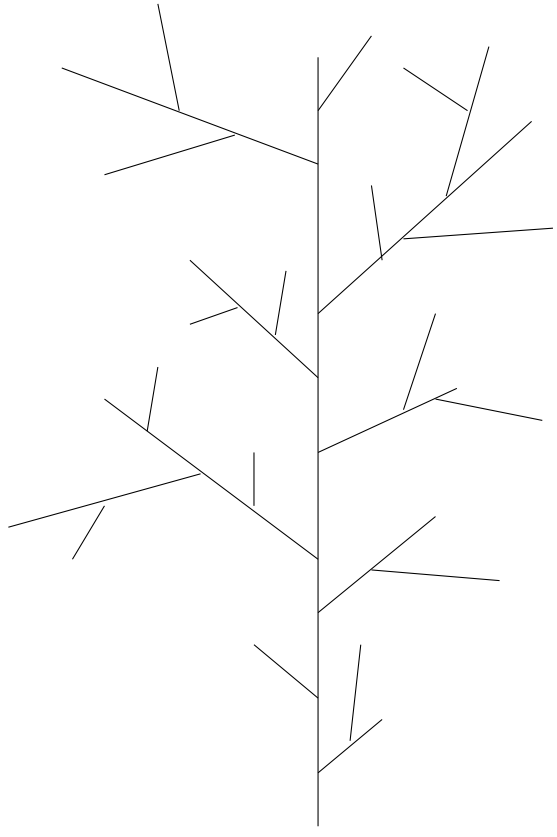

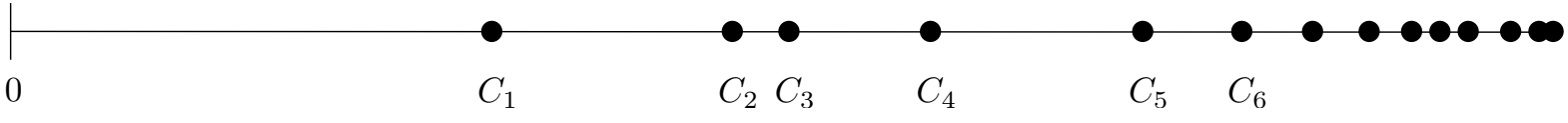

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D^k is the matrix of pairwise distances between the first k leaves in the line-breaking construction.

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For increasing k, the line-breaking construction gives a sequence of better and better approximations to the Brownian Continuum Random Tree. One way to *define* the Brownian CRT is then as the metric space completion of the increasing limit as $k \to \infty$ of the sequence of approximations.

Aside: Universality

The convergence result is, in fact, much more general, in that (up to a constant scaling factor) very many random trees have the Brownian CRT as their scaling limit.

- uniform unordered unlabelled rooted trees
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- Branching process trees with offspring mean 1 and finite offspring variance, conditioned to have size n
- random trees with a prescribed degree sequence satisfying certain conditions
- random dissections of a polygon.
- random graphs from "subcritical combinatorial classes".

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In particular, all these objects are in some sense "2-dimensional".

Critical random graphs in Oxford

In 2007, when I was a postdoc, two friends and I realized we could apply metric space convergence theory to the setting of the critical Erdős–Rényi random graph. (Left: Christina Goldschmidt, Oxford. Right: Nicolas Broutin, Sorbonne.)

VI. The critical Erdős–Rényi random graph

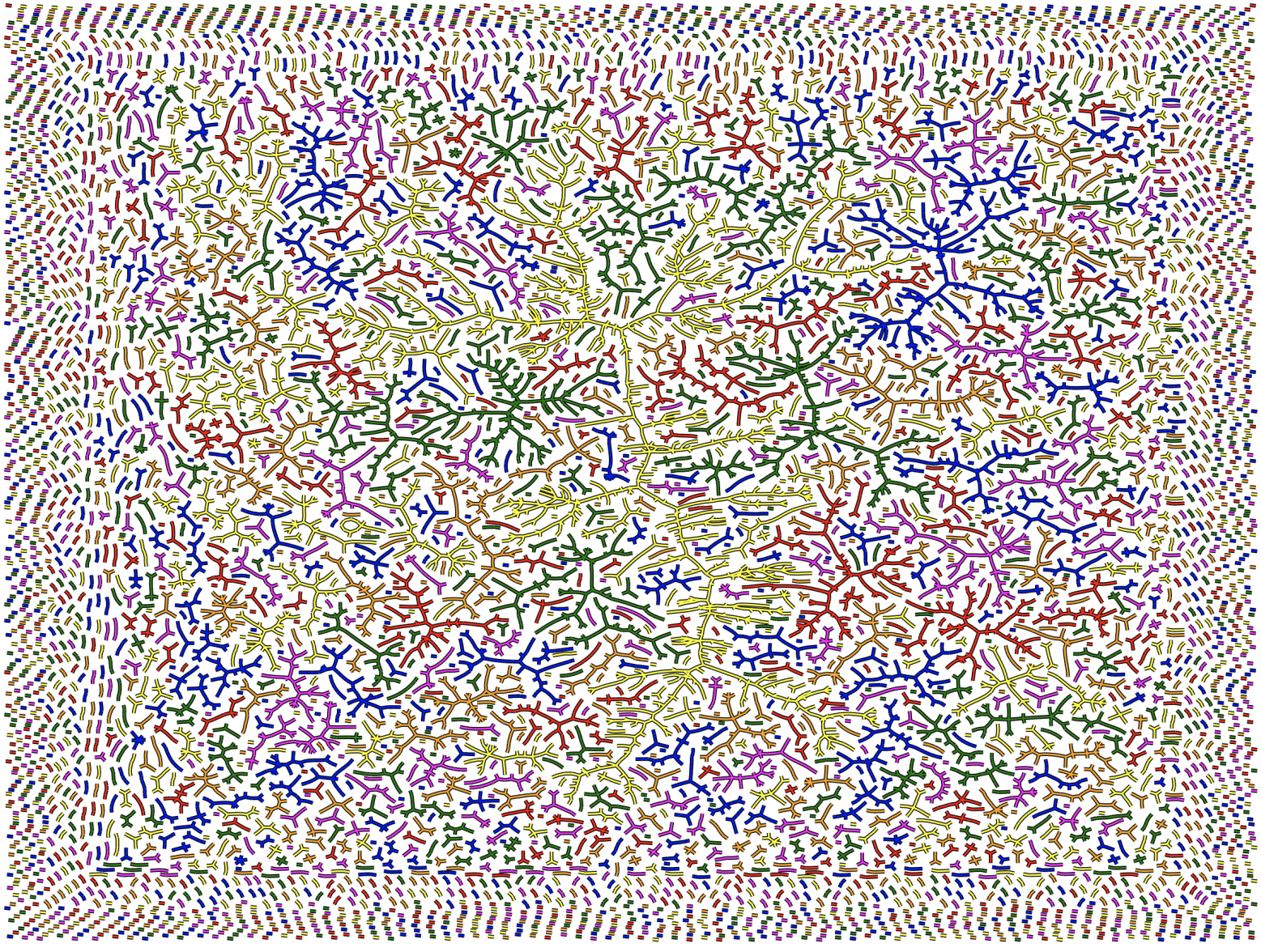

Image by Nicolas Broutin.

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We saw earlier that the Erdős–Rényi random graph $G(n, p)$ undergoes a phase transition when $p = 1/n$. What happens exactly at the critical point?

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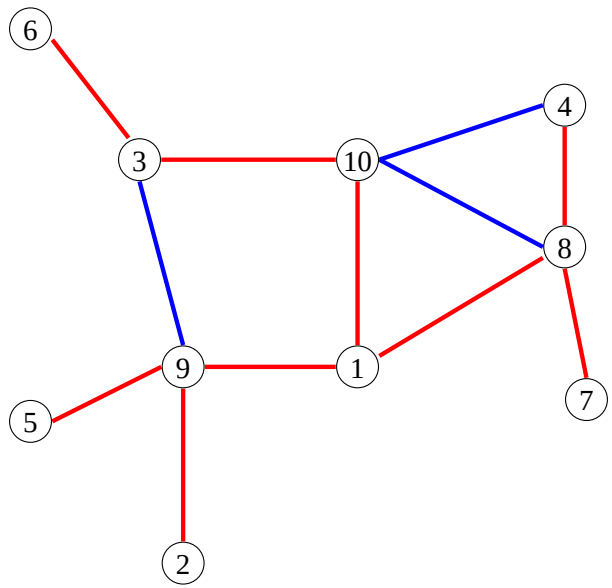
We saw earlier that the Erdős–Rényi random graph $G(n, p)$ undergoes a phase transition when $p = 1/n$. What happens exactly at the critical point?

We start by considering the sizes of the components. We will also be interested in the surplus of a component, that is the number of edges more than a tree that it has.

A component with surplus 3:



VI. The critical Erdős–Rényi random graph

Let $C_1^n, C_2^n, \ldots$ be the ordered component sizes of $G(n, 1/n)$.
Let $S_1^n, S_2^n, \ldots$ be the corresponding surpluses.

Theorem. (Aldous (1997))
As $n \to \infty$, we have jointly that

$$\frac{1}{n^{2/3}}(C_1^n, C_2^n, \ldots) \xrightarrow{d} (C_1, C_2, \ldots)$$

and

$$(S_1^n, S_2^n, \ldots) \xrightarrow{d} (S_1, S_2, \ldots),$$

where $C_1, C_2, \ldots \in \mathbb{R}_+$ and $S_1, S_2, \ldots \in \mathbb{Z}_+$ are random variables with an explicit (but complicated) distribution, such that $\mathbb{P}(S_k = 0) \to 1$ as $k \to \infty$.

Component convergence

This theorem tells us the limiting rescaled sizes of the components, and that they are quite close to being trees. Indeed, most of them really are trees!

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Suppose, for the moment, we know that the vertices $v_1, v_2, \ldots, v_m$ form a tree-component of $G(n, 1/n)$. Then that tree is uniform its vertex-set (which we may as well think of as $\{1, 2, \ldots, m\}$).

Component convergence

This theorem tells us the limiting rescaled sizes of the components, and that they are quite close to being trees. Indeed, most of them really are trees!

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Since components have sizes on the order of $n^{2/3}$, we should rescale by $n^{-1/3}$.

Component convergence

What about components which aren't trees? Again, if we know that $v_1, v_2, \ldots, v_m$ form a component of $G(n, 1/n)$ with surplus $s \geq 1$, that component is uniform among the possibilities.

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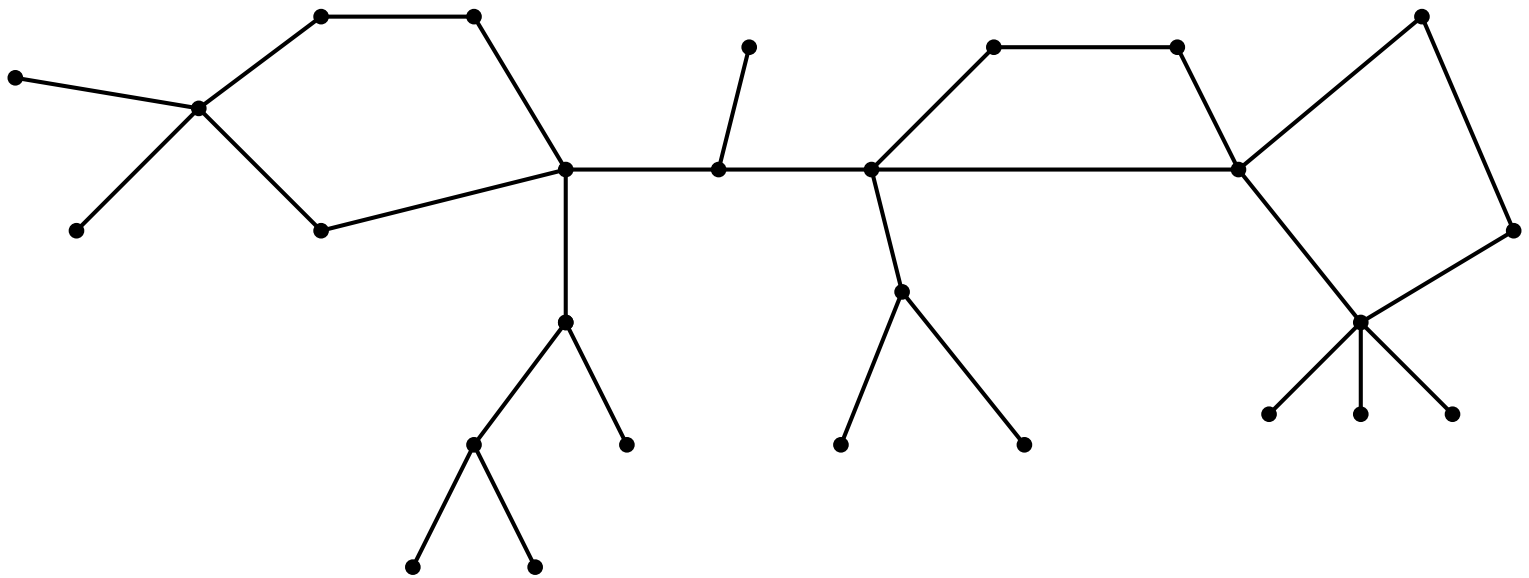
So we need to understand the large-m behaviour of a uniform random connected graph on m vertices with surplus s.

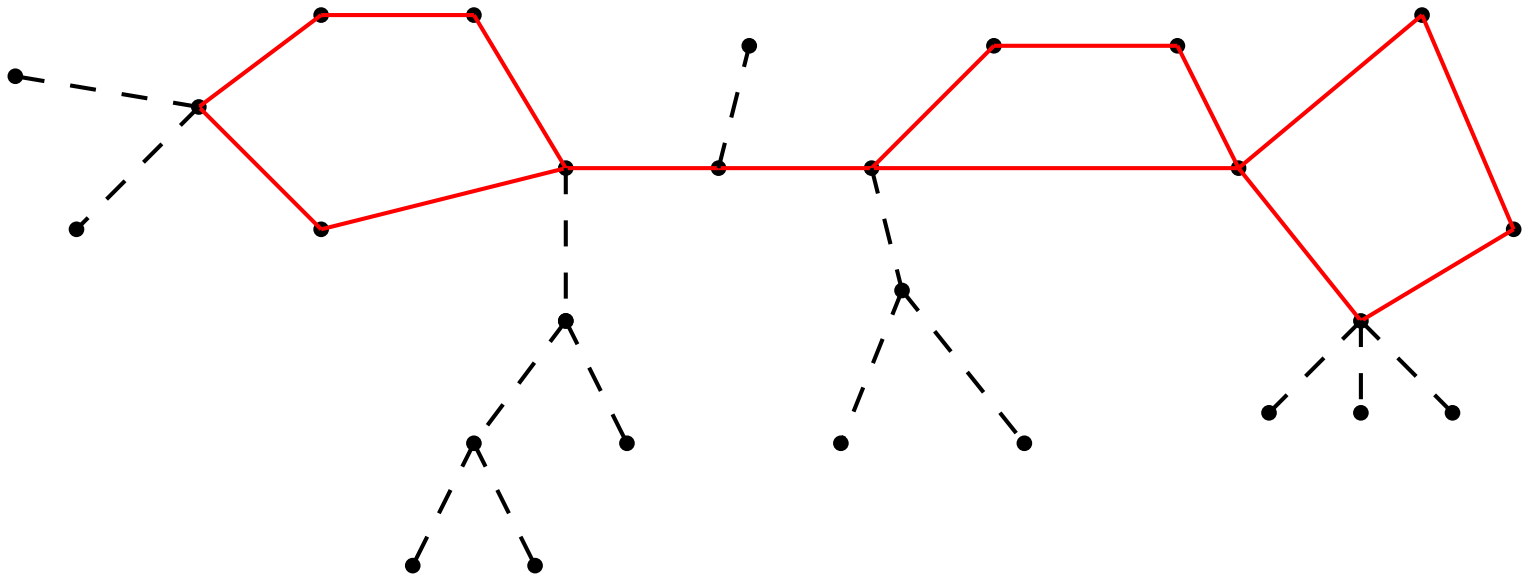
Cycle structure

Let's first look into the cycle structure.



Cycle structure

Core $C(G)$



Cycle structure

The kernel $K(G)$ is the multigraph which gives the "shape of the core": take the vertices of the core of degree 3 or more; contract the paths between them to a single edge.

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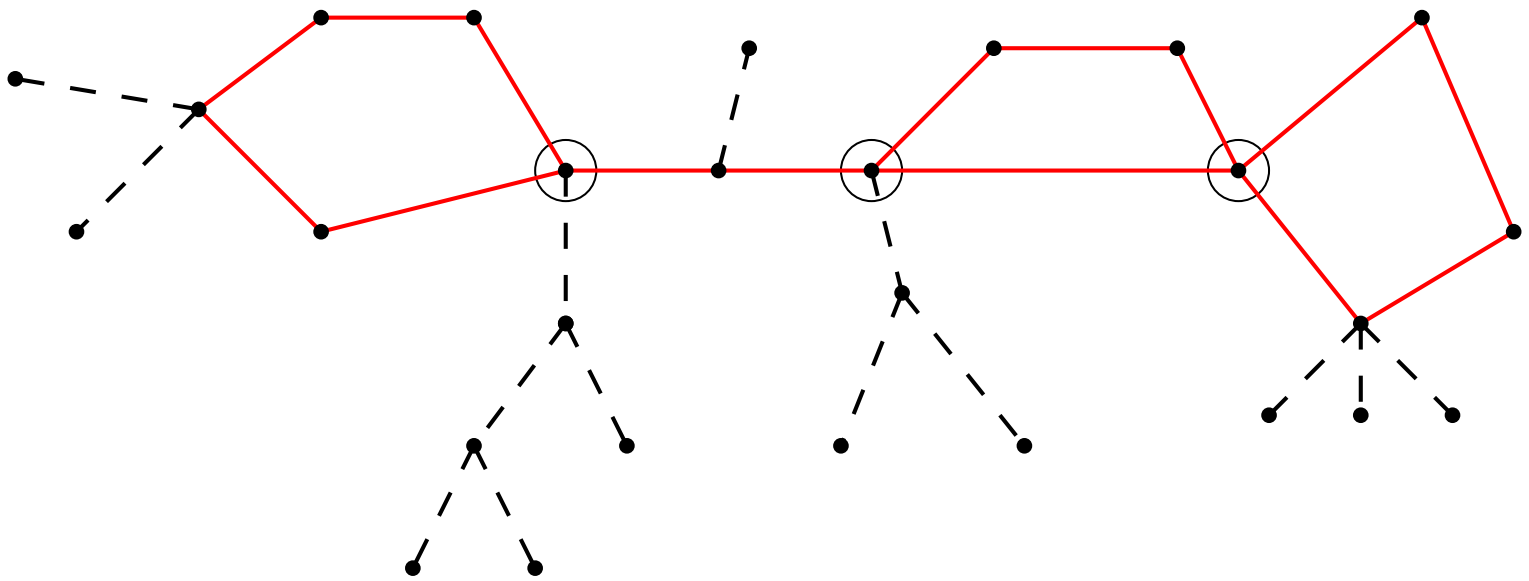
(By convention, the kernel of a tree or unicyclic component is empty.)

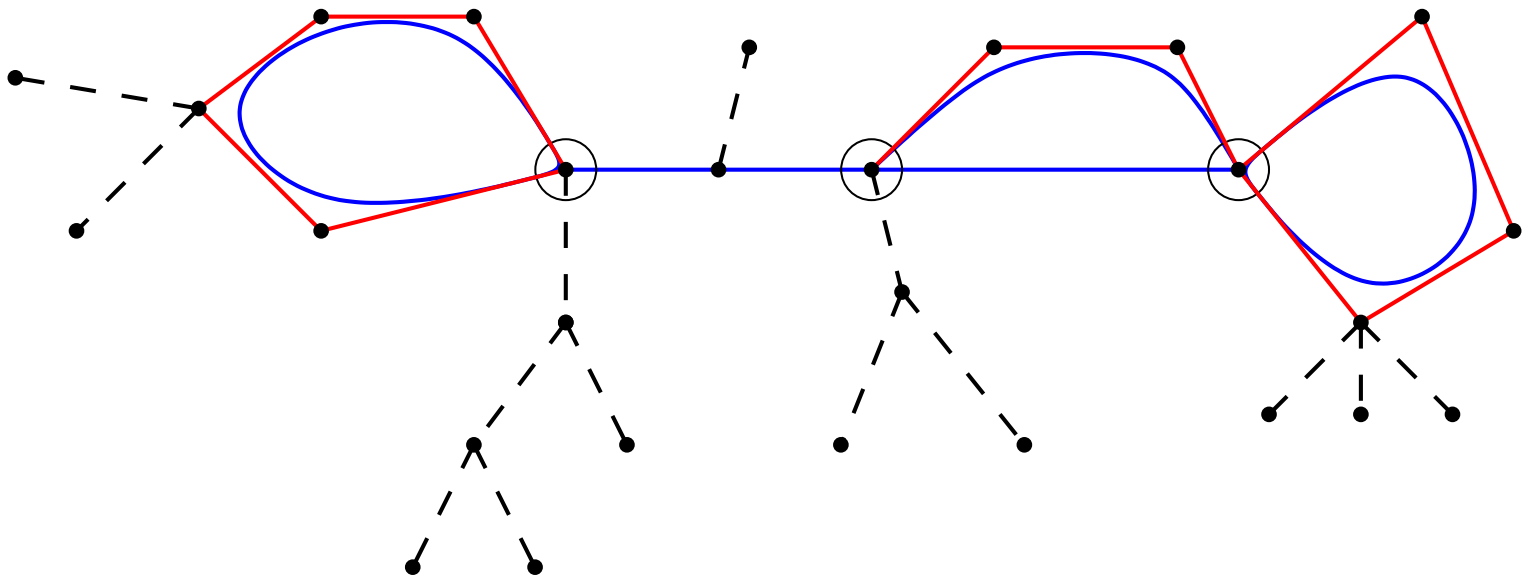
Cycle structure

Vertices of degree at least 3 in the core



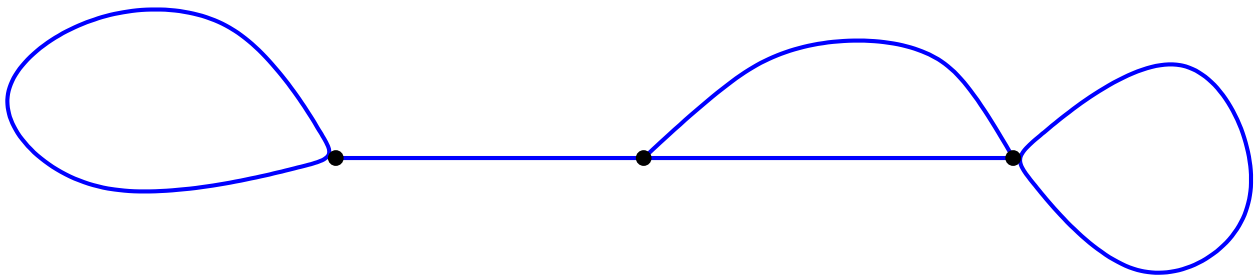
Cycle structure

Contract paths between them



Cycle structure

Kernel $K(G)$



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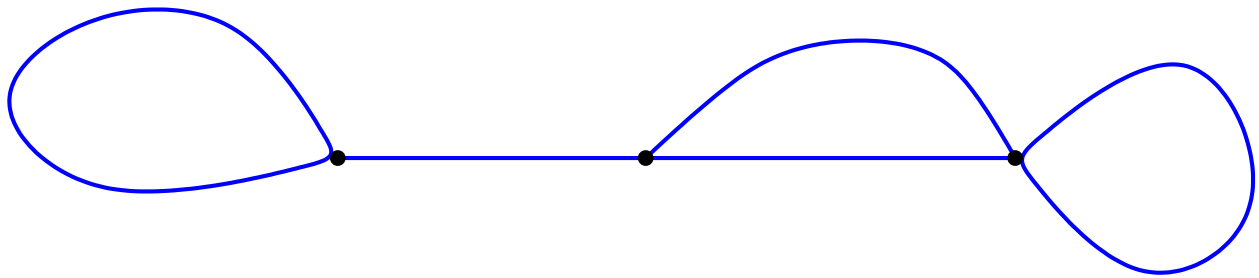

Note that the kernel has the same surplus as the original graph.

Scaling limits for uniform connected graphs

Theorem. (A.-B., Broutin and Goldschmidt. (2010, 2012))
Let U_m^s be a uniform connected graph with vertices labelled by $1, 2, \ldots, m$ and surplus $s \geq 1$. Then

$$\frac{1}{\sqrt{m}} U_m^s \xrightarrow{d} \mathcal{U}^s,$$

for a random limit object we will construct in a moment.

Scaling limits for uniform connected graphs

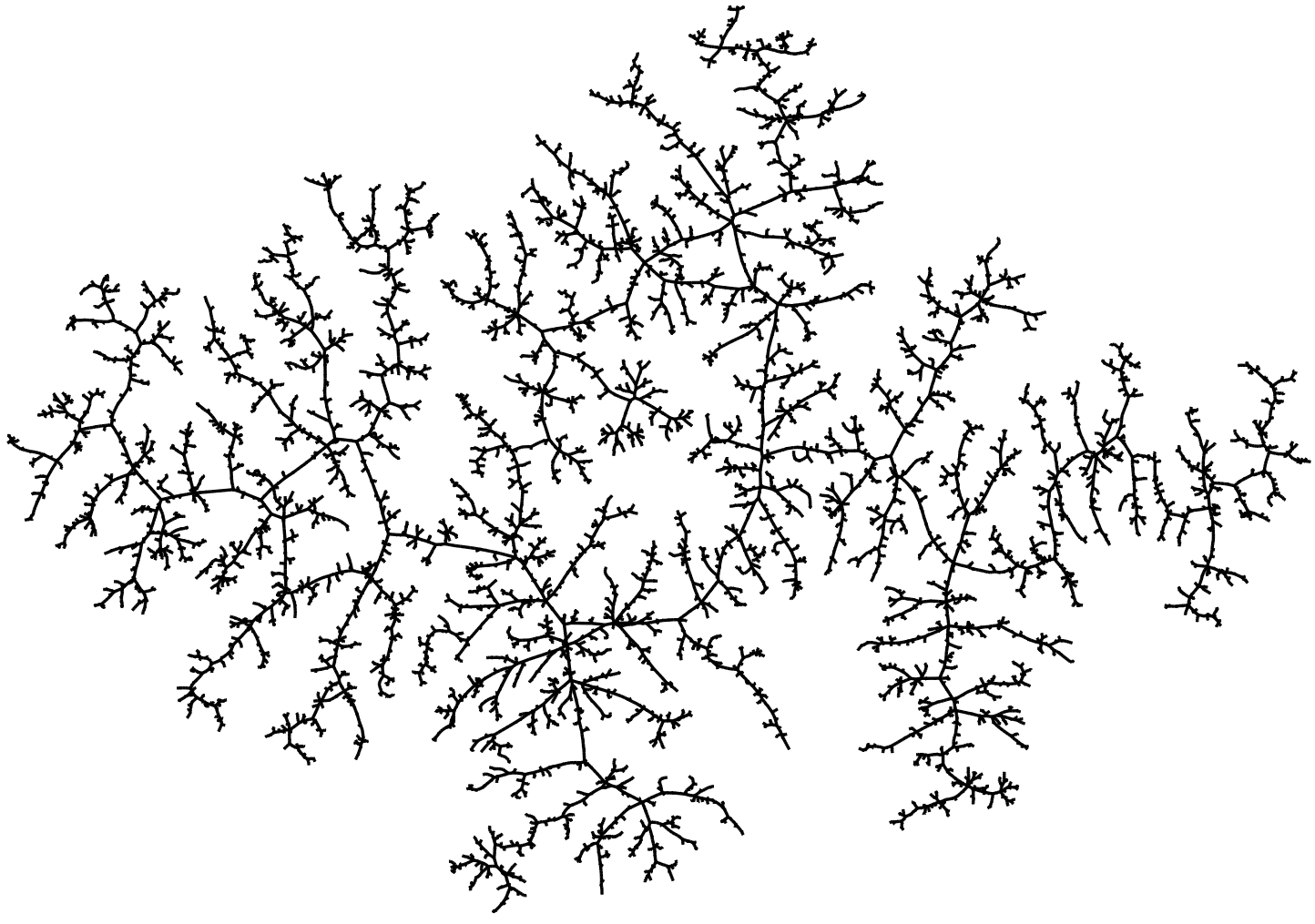

Image by Nicolas Broutin.

Scaling limits for uniform connected graphs

Again, this convergence means that if we sample k uniform random points and look at the matrix of pairwise distances between them, that matrix converges on rescaling by $m^{-1/2}$.

Construction of $\mathcal{U}^s$ for $s \geq 2$. Step 1: The core.

- Take $2s-2$ labelled vertices and assign each one 3 half-edges.

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- Take $2s - 2$ labelled vertices and assign each one 3 half-edges.
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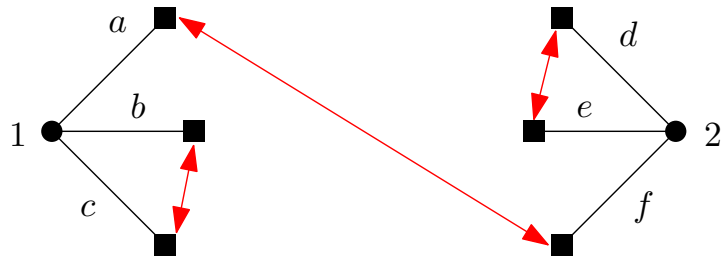

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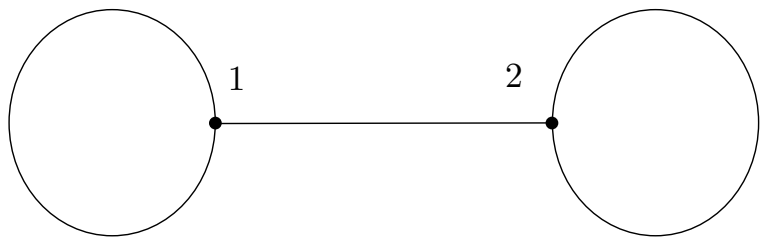

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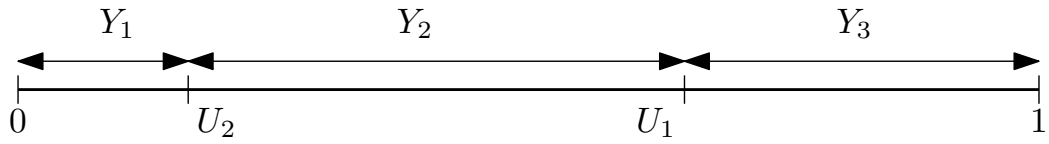

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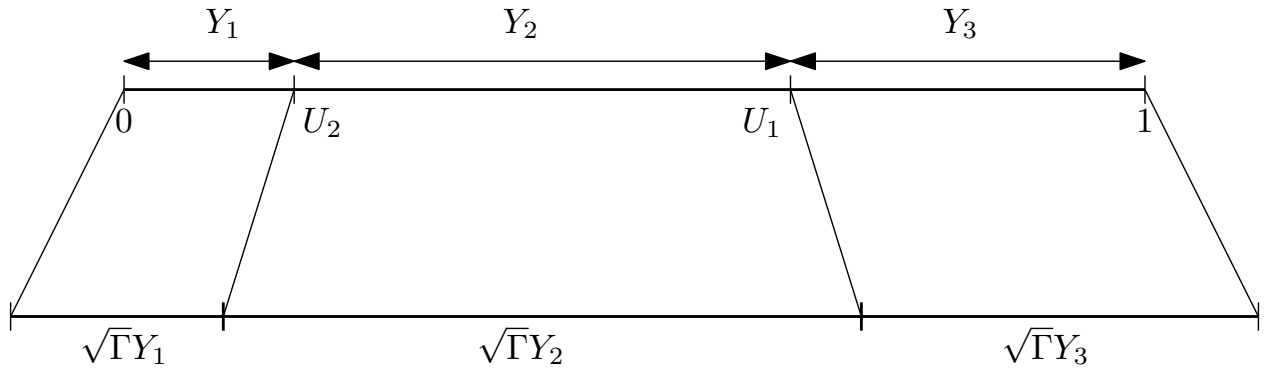

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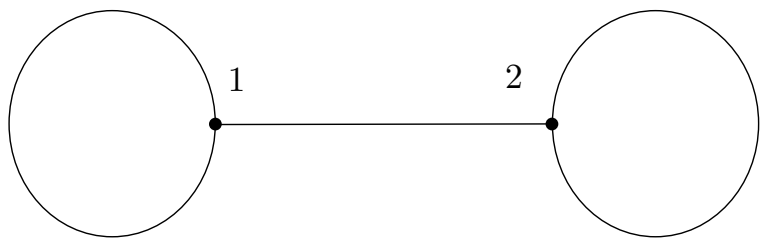

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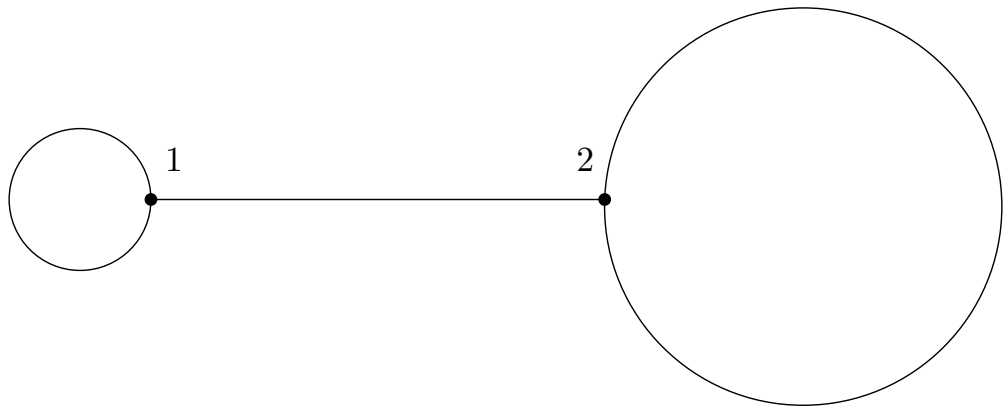

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- Sample $E_1, E_2, \ldots$ independent Exp(1/2) random variables. Let $C_0 = \sqrt{\Gamma}$ and, for $j \geq 1$, let $C_j = \sqrt{\Gamma + \sum_{i=1}^{j} E_i}$.
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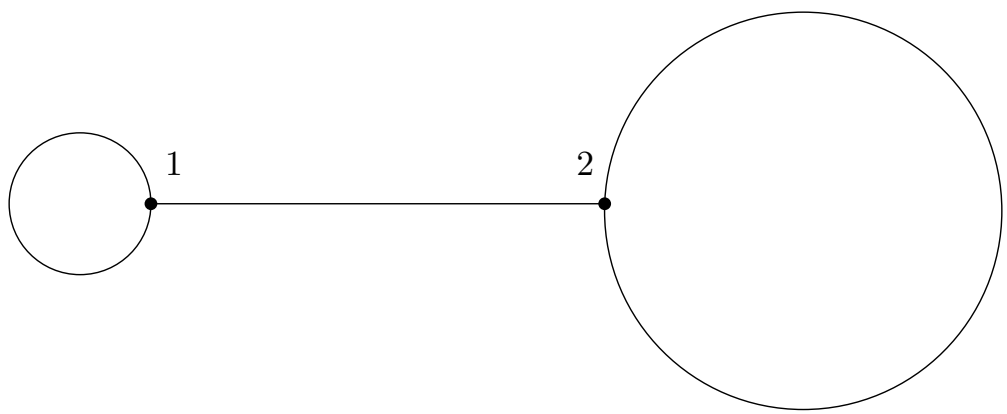

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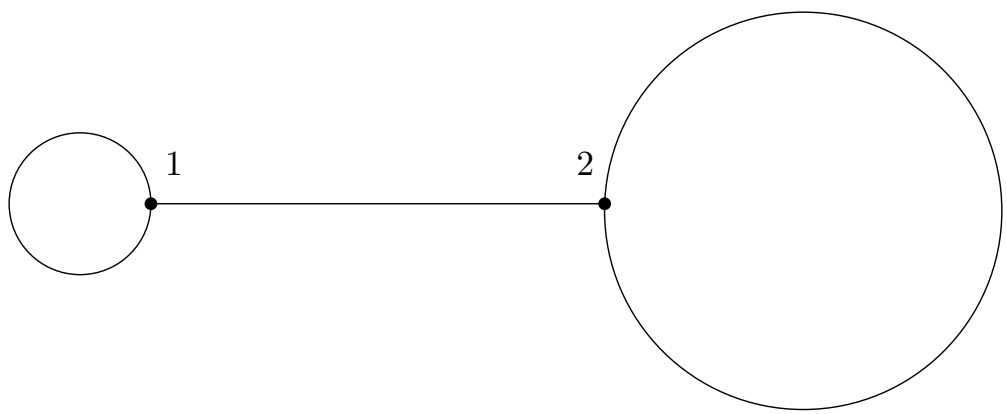

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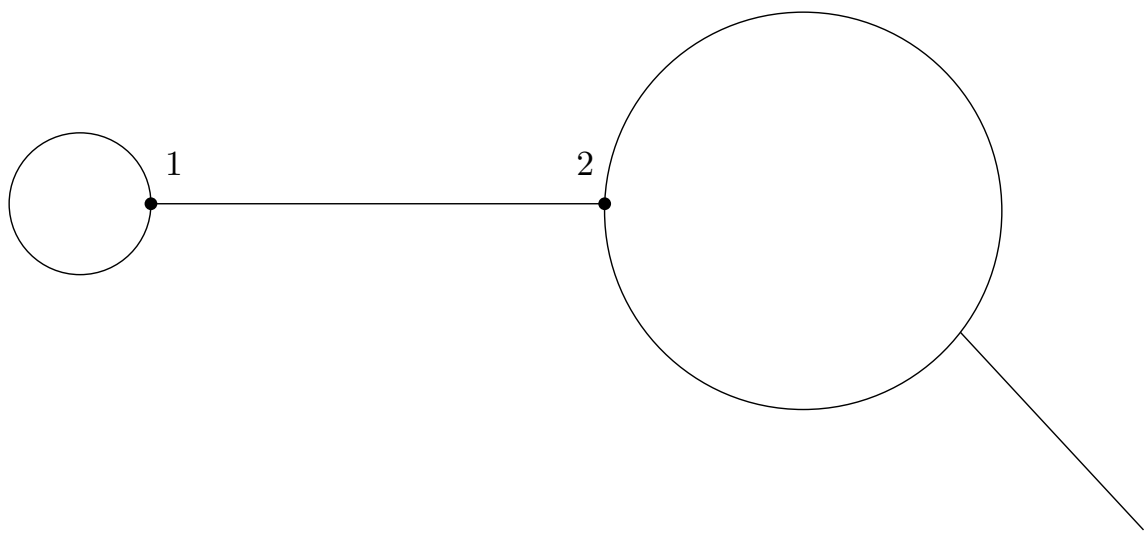

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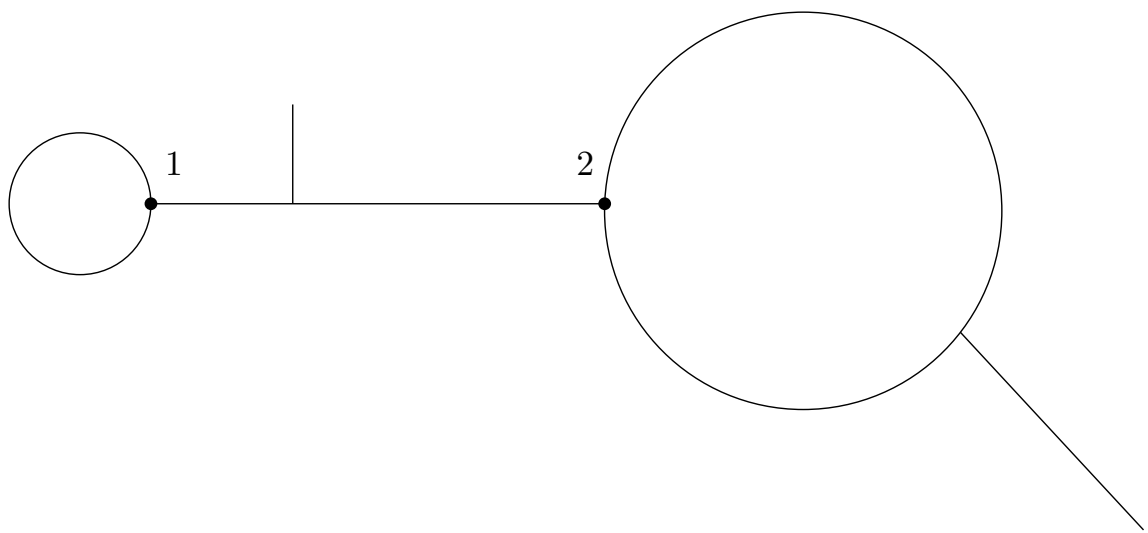

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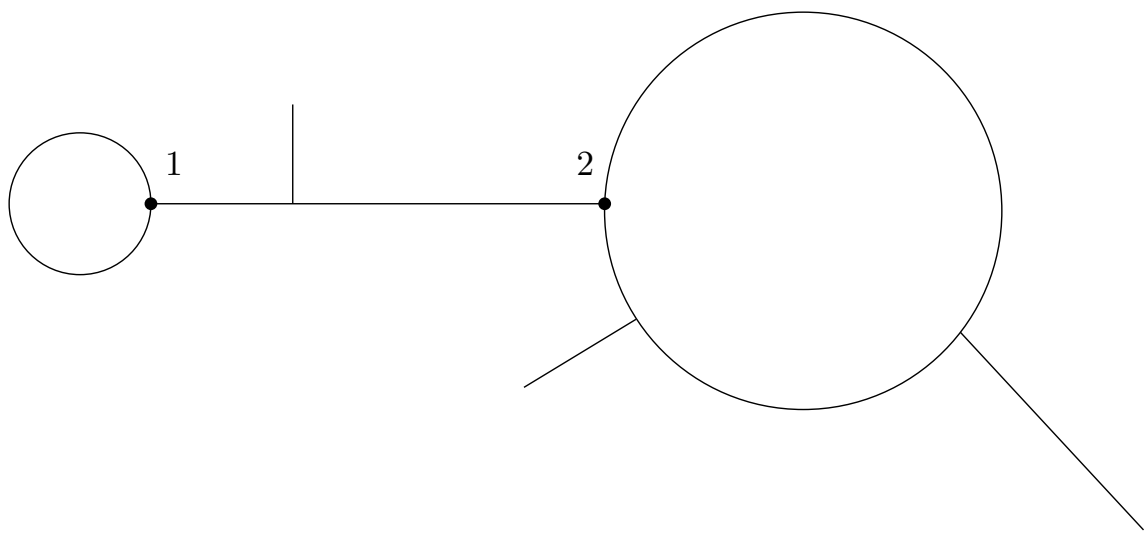

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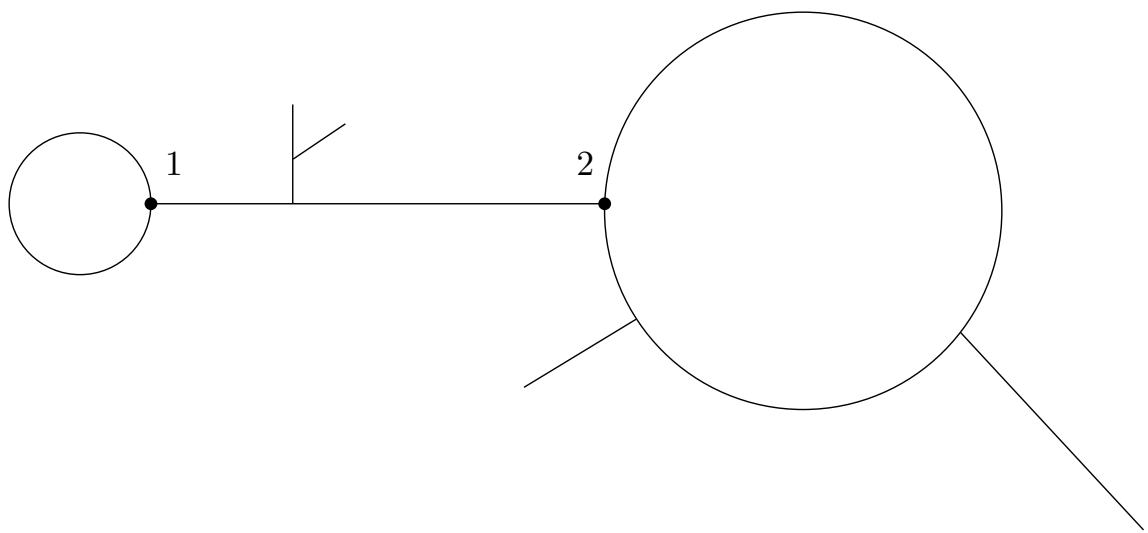

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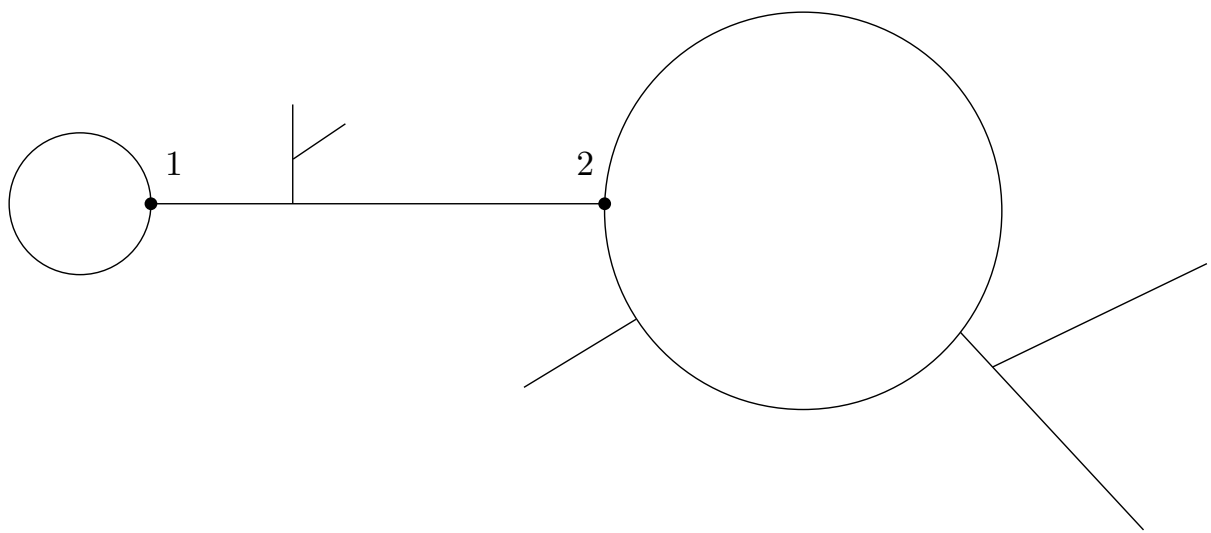

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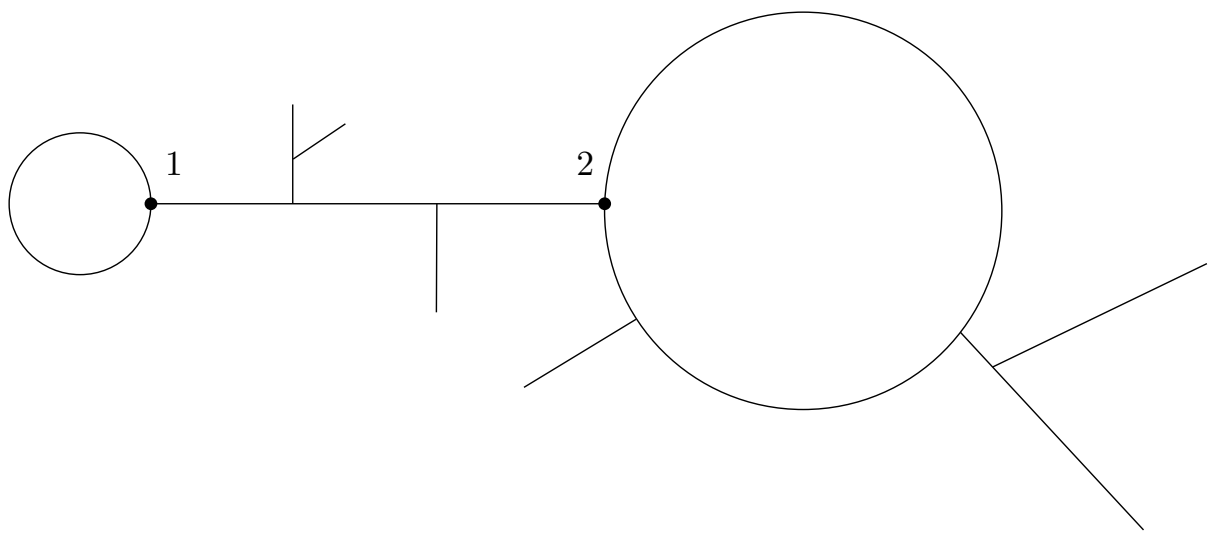

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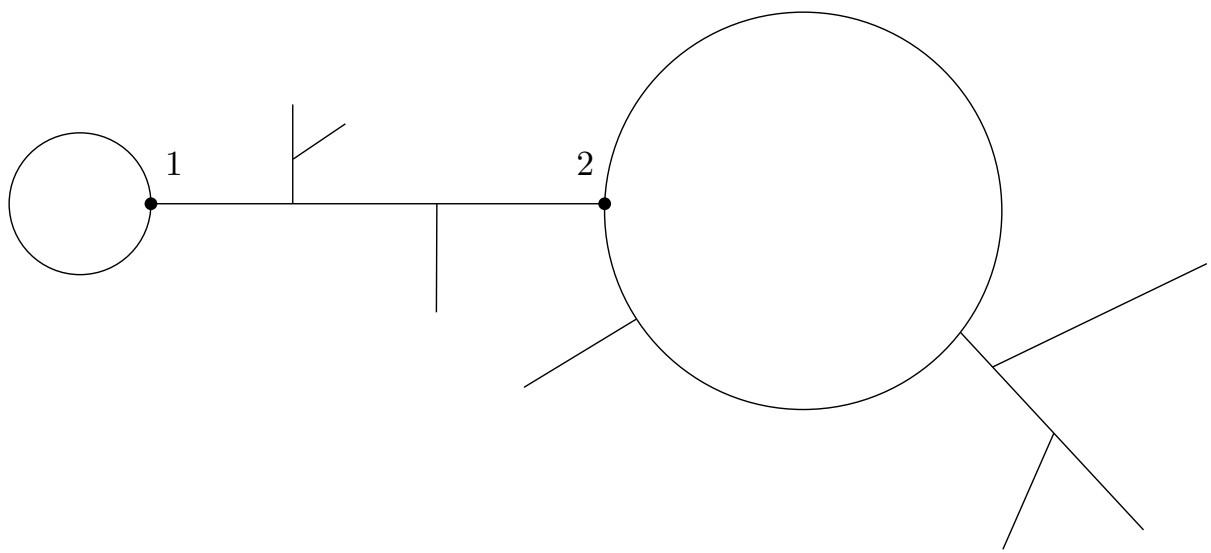

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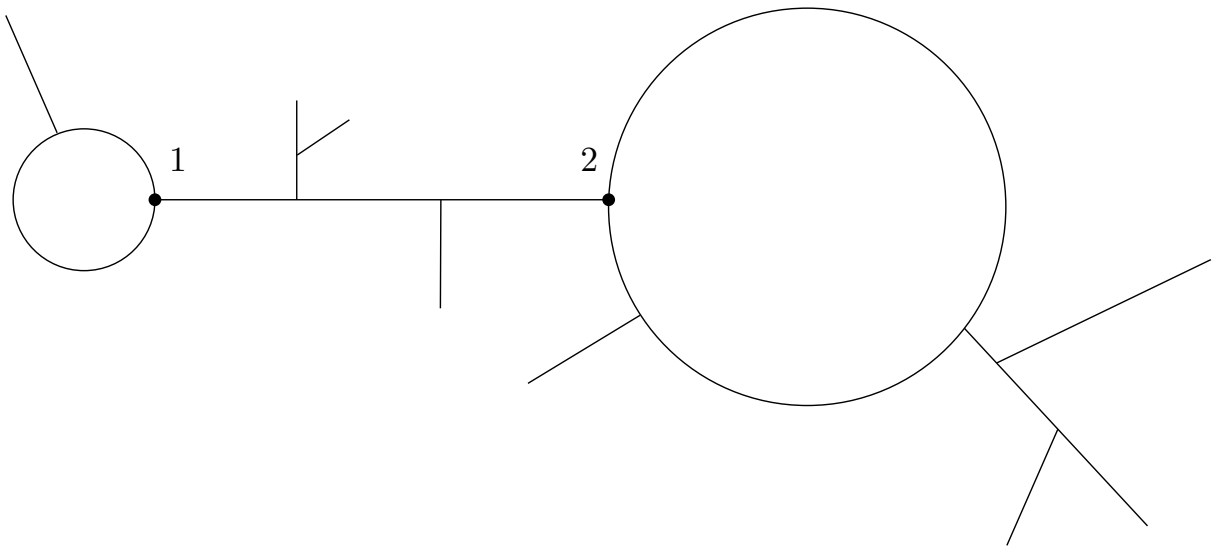

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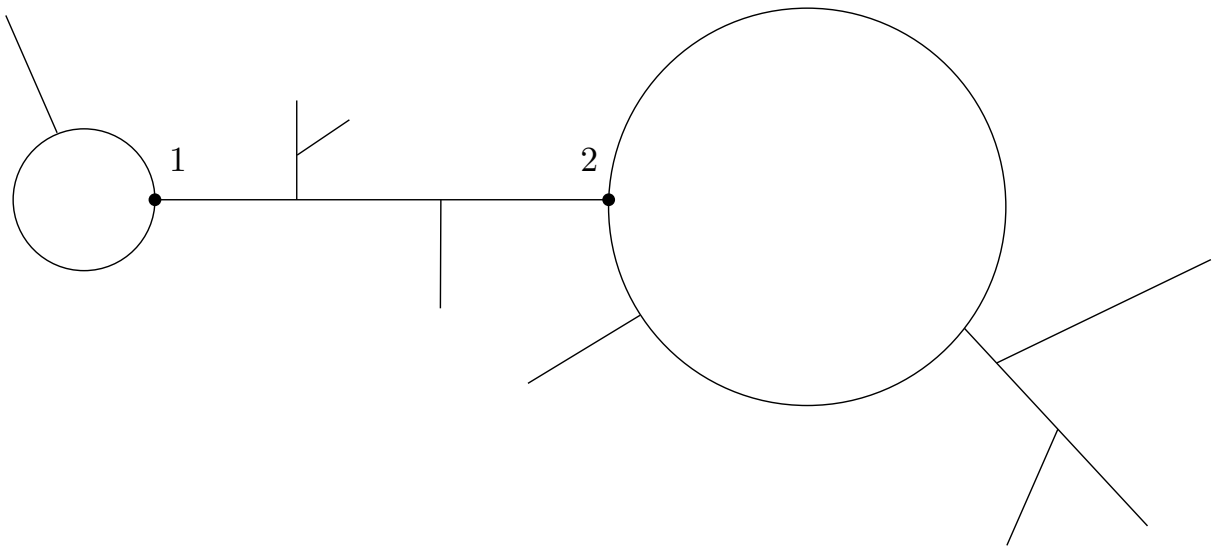
- Now perform line-breaking starting from the core, and recursively gluing line-segments $[C_j, C_{j+1}]$ uniformly over the structure built so far.
- Do this forever, then take the completion of the limit; the result is $\mathcal{U}^s$.



Construction of $\mathcal{U}_s$ for $s = 1$.

Needs to be handled separately because the kernel is empty.

Step 1: The core

- Start from a lollipop shape  of total length $\sqrt{\Gamma}$ where $\Gamma \sim \text{Gamma}(\frac{3}{2}, \frac{1}{2})$.
- Split that length uniformly between the cycle and the stick.

Step 2: Line-breaking

- This step works exactly as before.

Scaling limit

Putting this together with Aldous' theorem yields a scaling limit for the whole critical Erdős–Rényi random graph.

Let $G_1^n, G_2^n, \ldots$ be the components of $G(n, 1/n)$ in decreasing order of size.

Theorem. (A.-B., Broutin and G. (2010,2012))
As $n \to \infty$,

$$\frac{1}{n^{1/3}}(G_1^n, G_2^n, \ldots) \xrightarrow{d} (\mathcal{G}_1, \mathcal{G}_2, \ldots)$$

where $\mathcal{G}_1, \mathcal{G}_2, \ldots$ are conditionally independent given the sizes $(C_1, C_2, \ldots)$ and surpluses $(S_1, S_2, \ldots)$ with $\mathcal{G}_i \stackrel{d}{=} \sqrt{C_i}\, \mathcal{U}^{S_i}$.

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The continuum limit of critical random graphs

L. Addario-Berry · N. Broutin · C. Goldschmidt

Received: 30 November 2009 / Revised: 30 June 2010 / Published online: 12 October 2010

Abstract We consider the Erdős–Rényi random graph $G(n, p)$ inside the critical window, that is when $p = 1/n + \lambda n^{-4/3}$, for some fixed $\lambda \in \mathbb{R}$. We prove that the sequence of connected components of $G(n, p)$, considered as metric spaces using the graph distance rescaled by $n^{-1/3}$, converges towards a sequence of continuous compact metric spaces. The result relies on a bijection between graphs and certain marked random walks, and the theory of continuum random trees. Our result gives access to the answers to a great many questions about distances in critical random graphs. In particular, we deduce that the diameter of $G(n, p)$ rescaled by $n^{-1/3}$ converges in distribution to an absolutely continuous random variable with finite mean.

Keywords Random graphs · Gromov–Hausdorff distance · Scaling limits · Continuum random tree · Diameter

Mathematics Subject Classification (2000) 05C80 · 60C05

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Critical random graphs: limiting constructions and distributional properties*†

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Abstract

We consider the Erdős–Rényi random graph $G(n,p)$ inside the critical window, where $p = 1/n + \lambda n^{-4/3}$ for some $\lambda \in \mathbb{R}$. We proved in [1] that considering the connected components of $G(n,p)$ as a sequence of metric spaces with the graph distance rescaled by $n^{-1/3}$ and letting $n \to \infty$ yields a non-trivial sequence of limit metric spaces $\mathscr{C} = (\mathscr{C}_1, \mathscr{C}_2, \dots)$. These limit metric spaces can be constructed from certain random real trees with vertex-identifications. For a single such metric space, we give here two equivalent constructions, both of which are in terms of more standard probabilistic objects. The first is a global construction using Dirichlet random variables and Aldous' Brownian continuum random tree. The second is a recursive construction from an inhomogeneous Poisson point process on $\mathbb{R}_+$. These constructions allow us to characterize the distributions of the masses and lengths in the constituent parts of a limit component when it is decomposed according to its cycle structure. In particular, this strengthens results of Łuczak et al. [29] by providing precise distributional convergence for the lengths of paths between kernel vertices and the length of a shortest cycle, within any fixed limit component.

MSC 2000 subject classifications: primary 05C80; secondary 60C05.

†L.A.B. was supported by an NSERC Discovery Grant throughout the research and writing of this paper. C.G. was funded by EPSRC Postdoctoral Fellowship EP/D065755/1.

Timeline

- Jean-François Le Gall's seminar in October 2007.
- Project crystallized in spring 2008.
- LAB moved back to Montréal in summer 2008.
- Major progress in September 2008 during a visit of NB and CG to LAB at Université de Montréal.
- Finished first paper in March 2009. Less than 2 years! Math is not fast, but this actually feels like it did come together quickly.
- (On the other hand, that paper didn't appear in print in a journal until 2012!)
- Finished follow-up paper in March 2010.

Universality

Our scaling limit has subsequently been shown to be universal, in that a whole host of different critical random graph models have essentially the same limit.

VII. Minimum Spanning Trees

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- Necessarily T is a tree: the *minimum spanning tree* of K_n with these weights.

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- His formulation: Given the *complete graph* K_n together with distinct non-negative *edge weights* $\{w_e, e \in E(K_n)\}$.
- Seek the *unique connected graph* T that minimizes the total length $\sum_{e \in E(T)} w_e$.
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- For *random edge weights*, the structure of the minimum spanning tree of K_n is intimately connected to that of the critical Erdős-Renyi random graph.
- We used this connection to study the metric space limit of the minimum spanning tree.

The metric space limit of the minimum spanning tree

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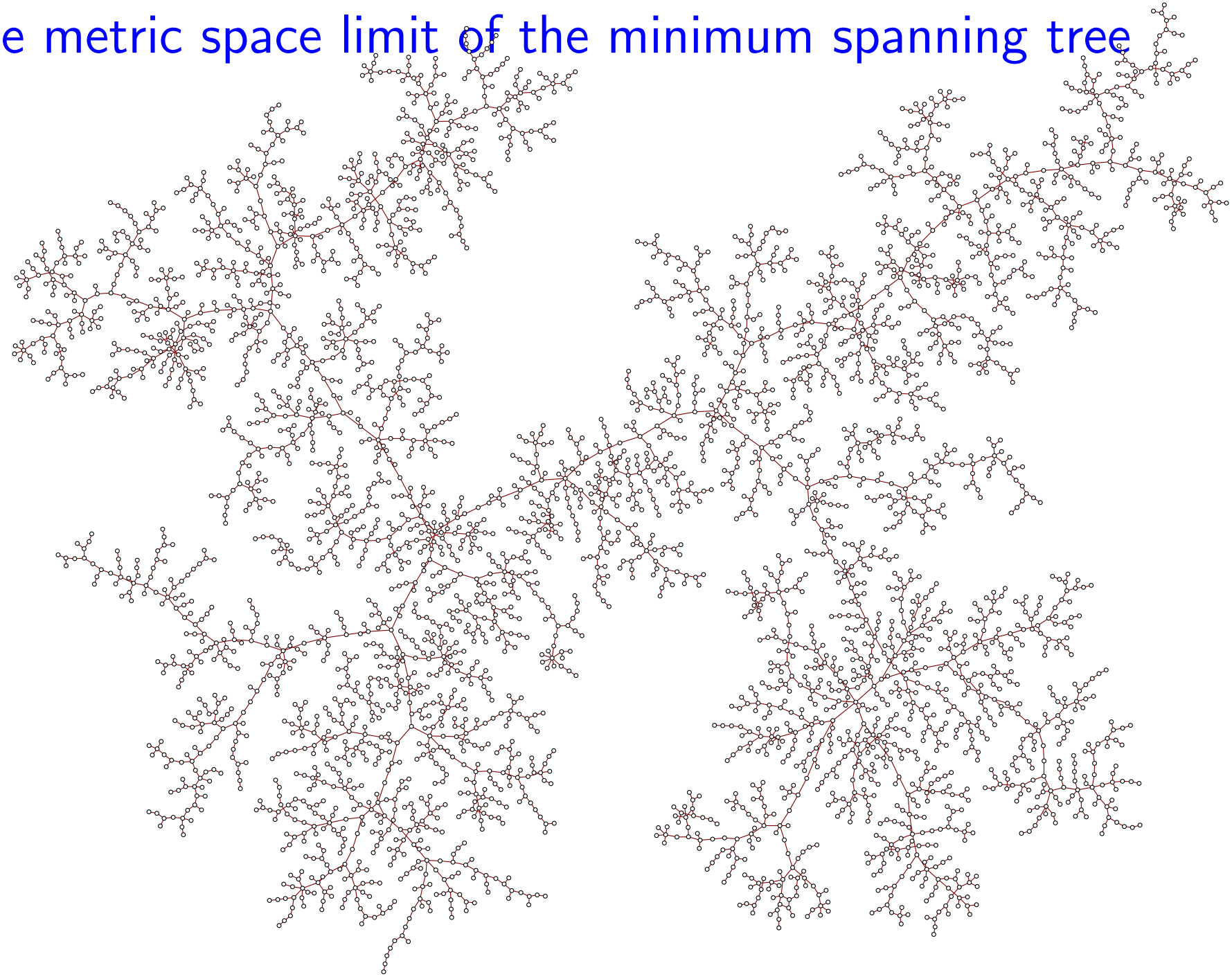

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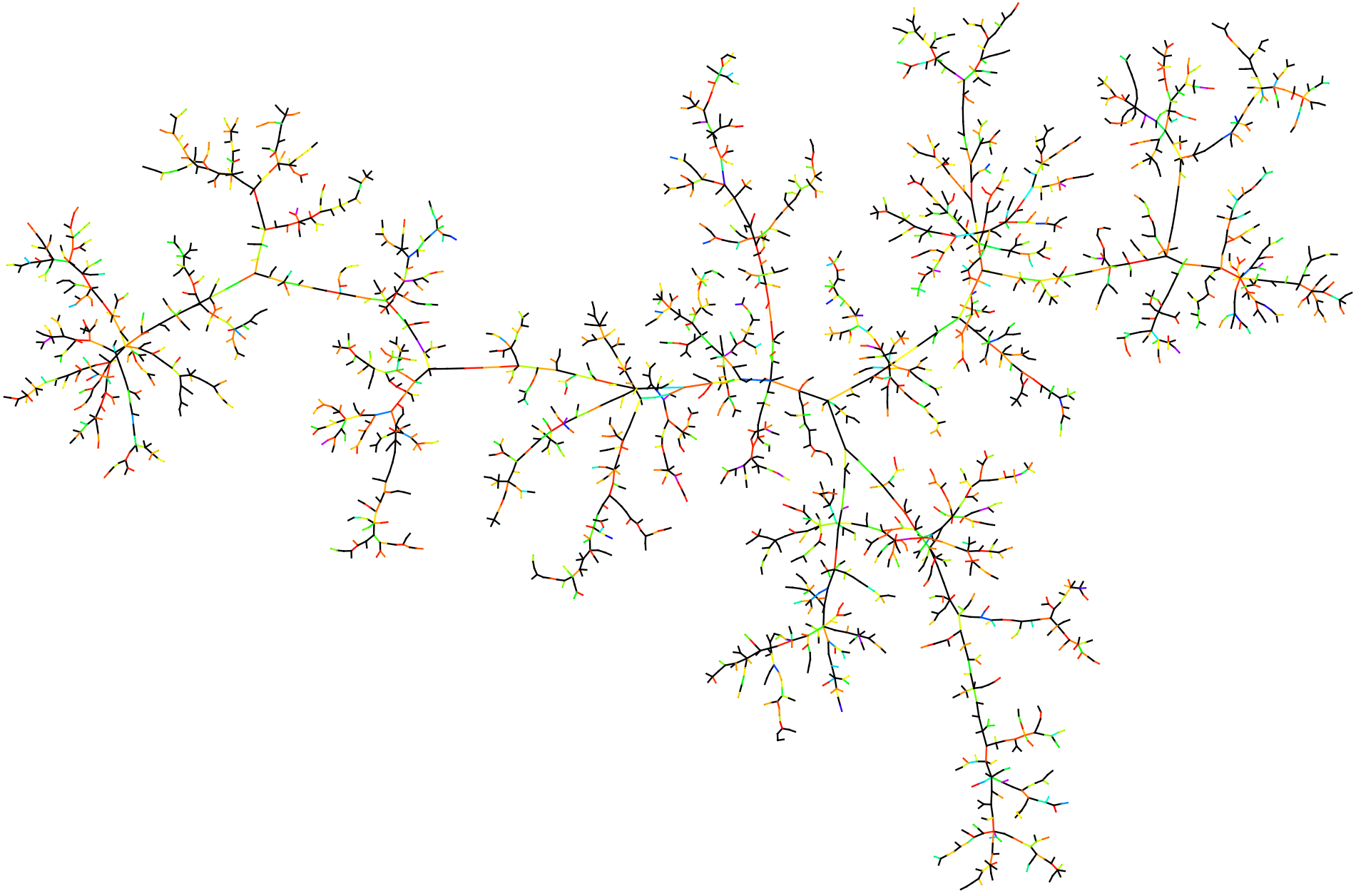

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Grégory Miermont

Kruskal's minimum spanning tree algorithm

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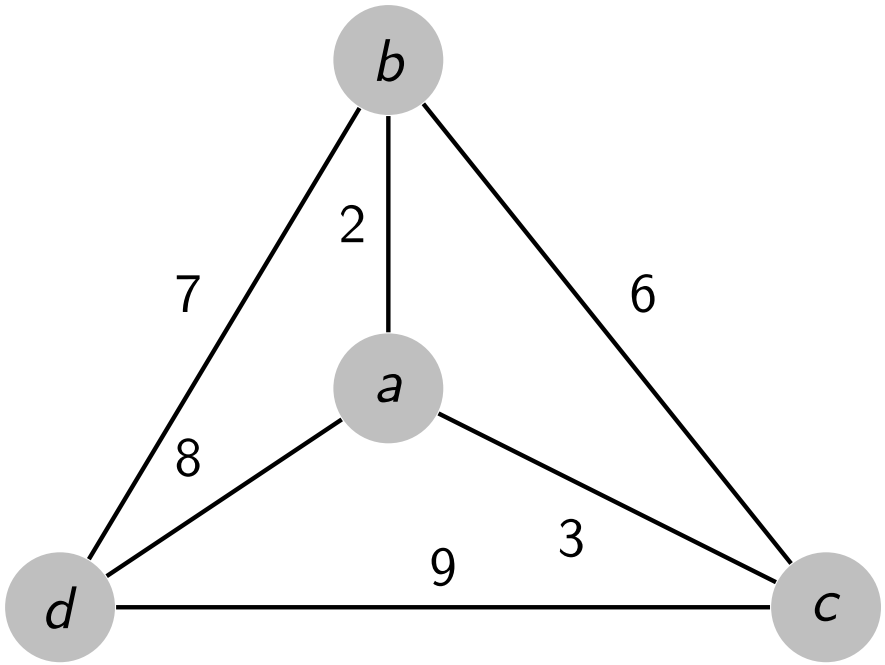

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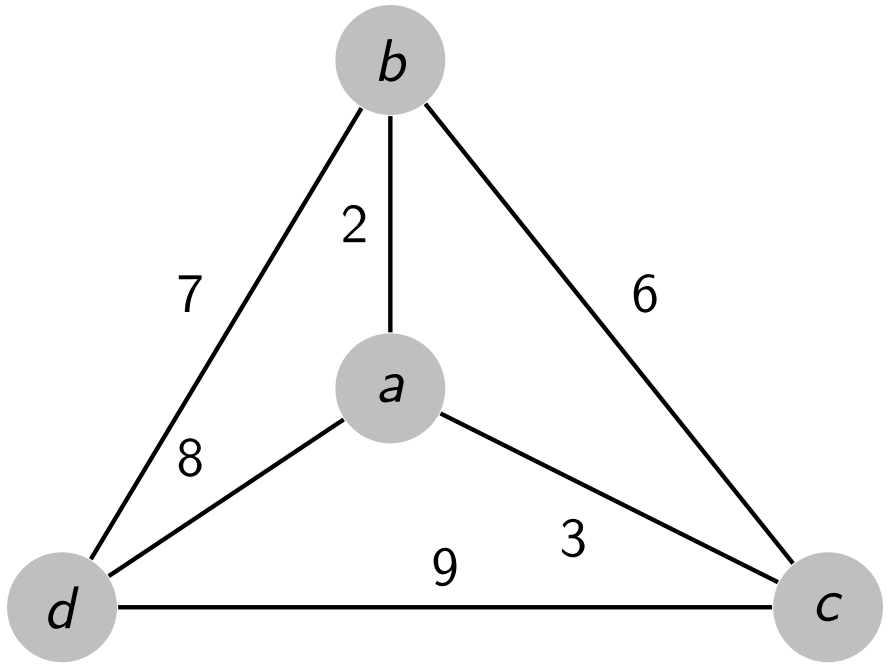

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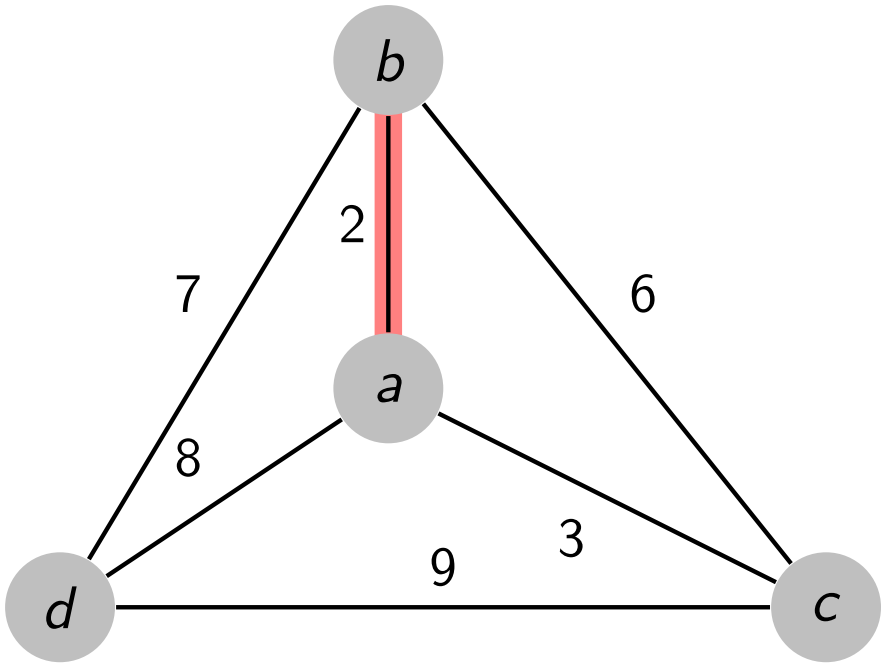

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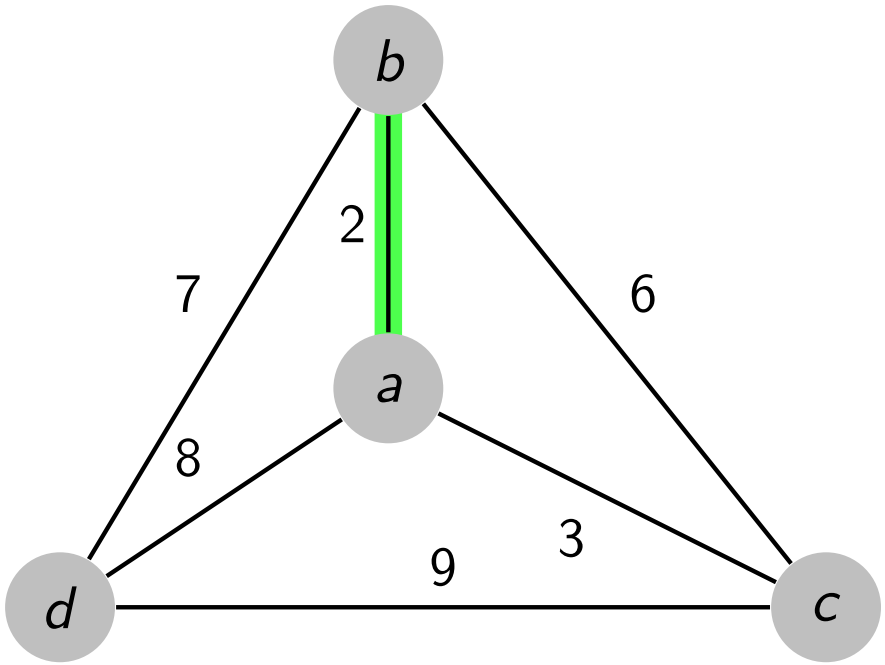

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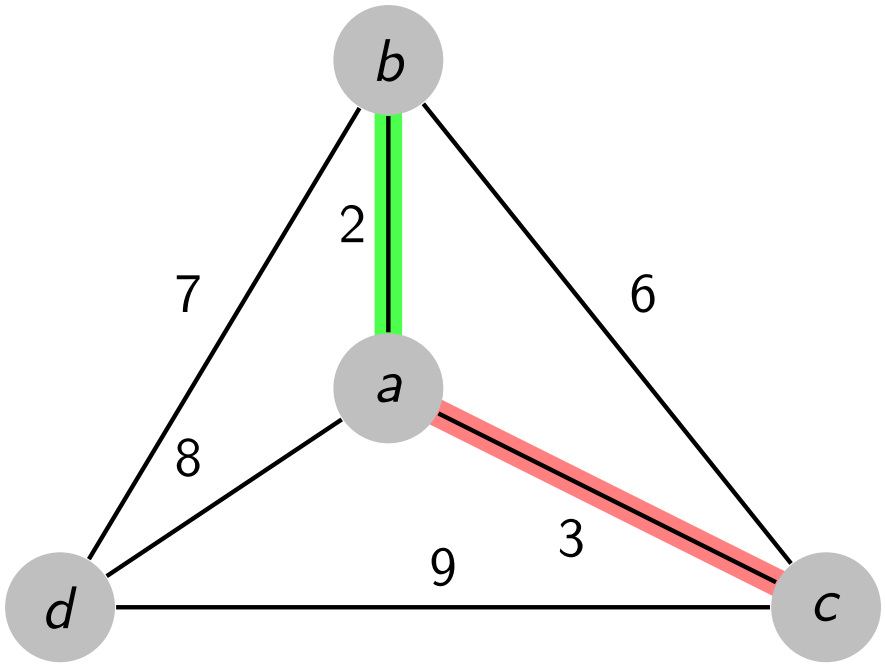

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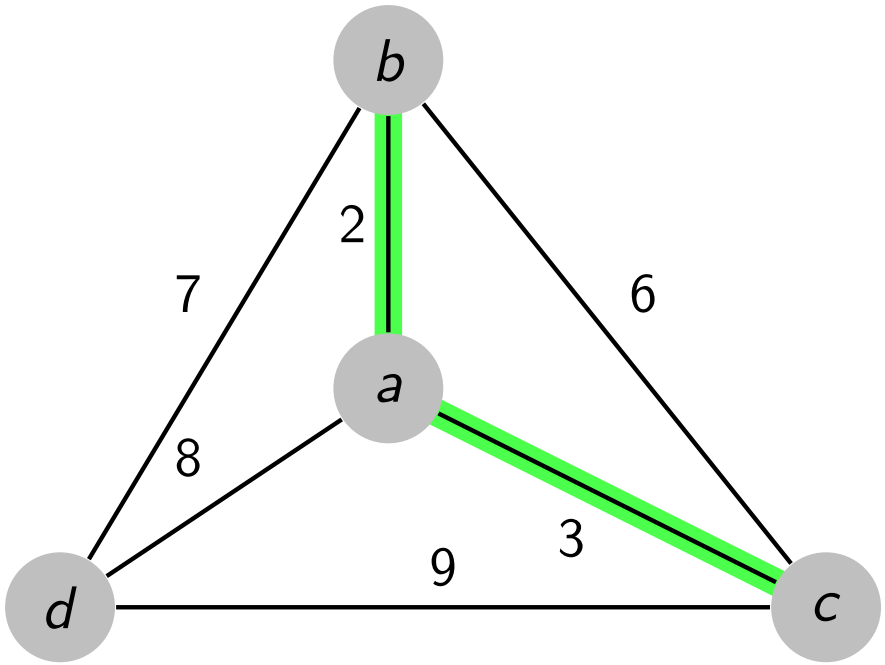

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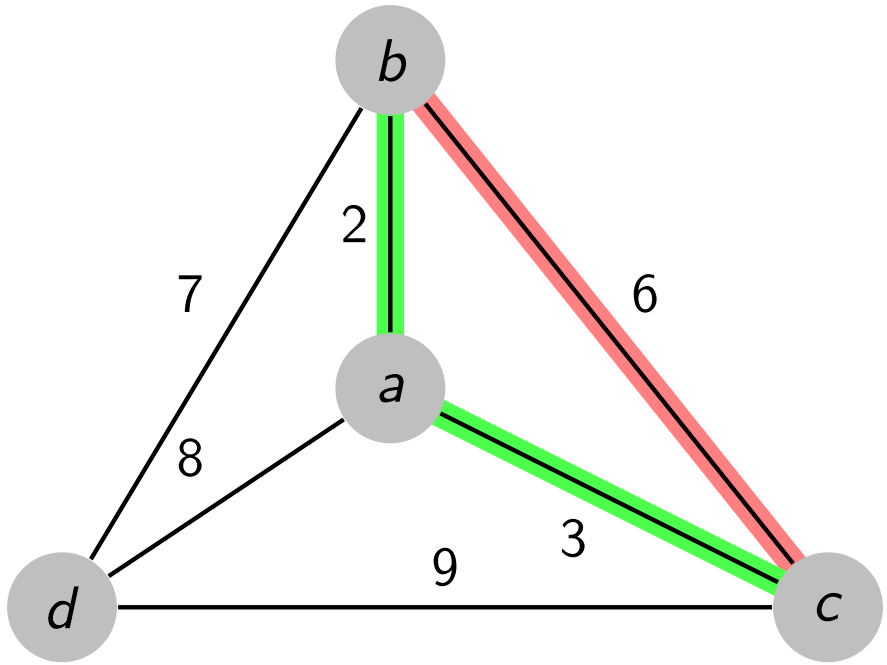

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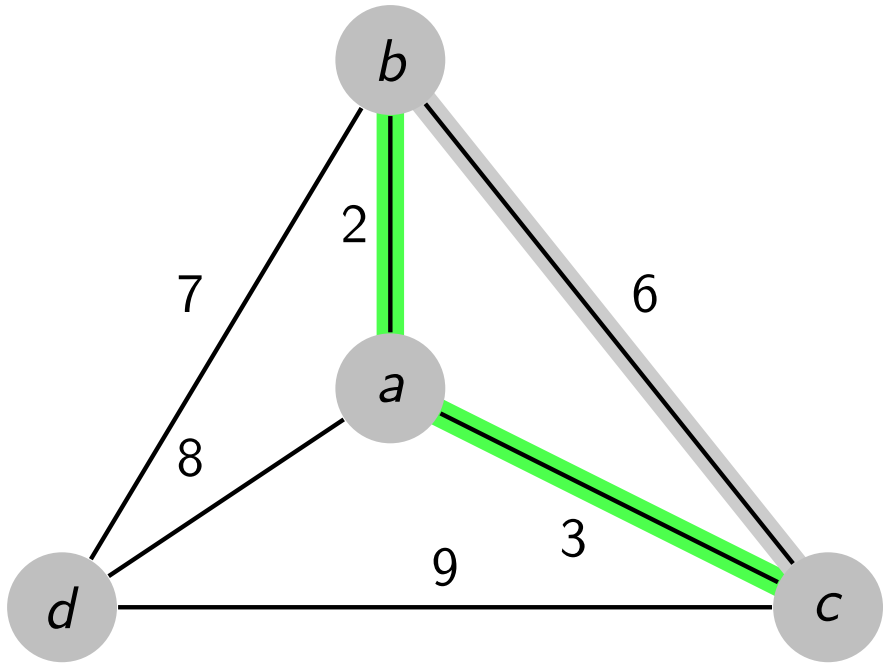

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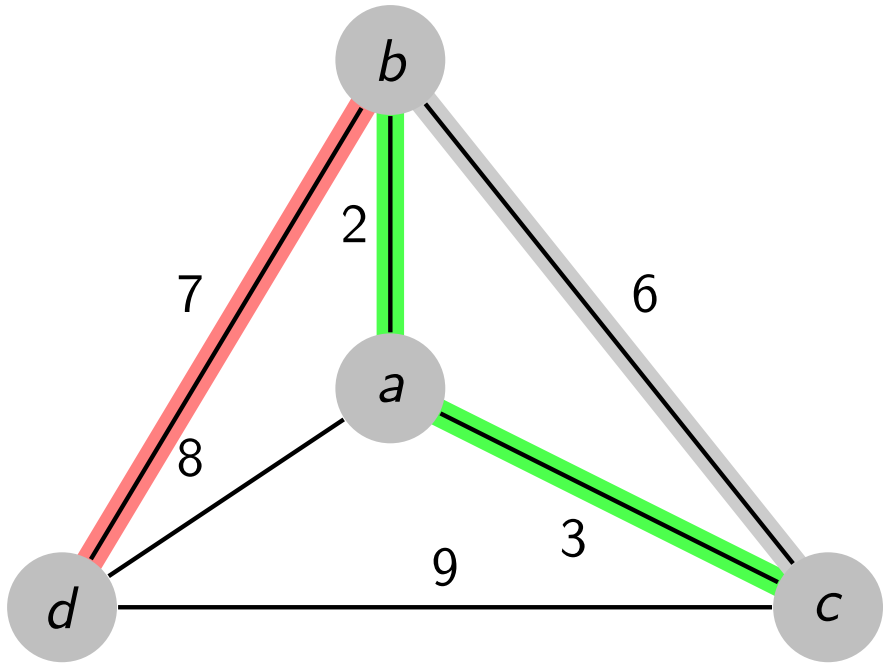

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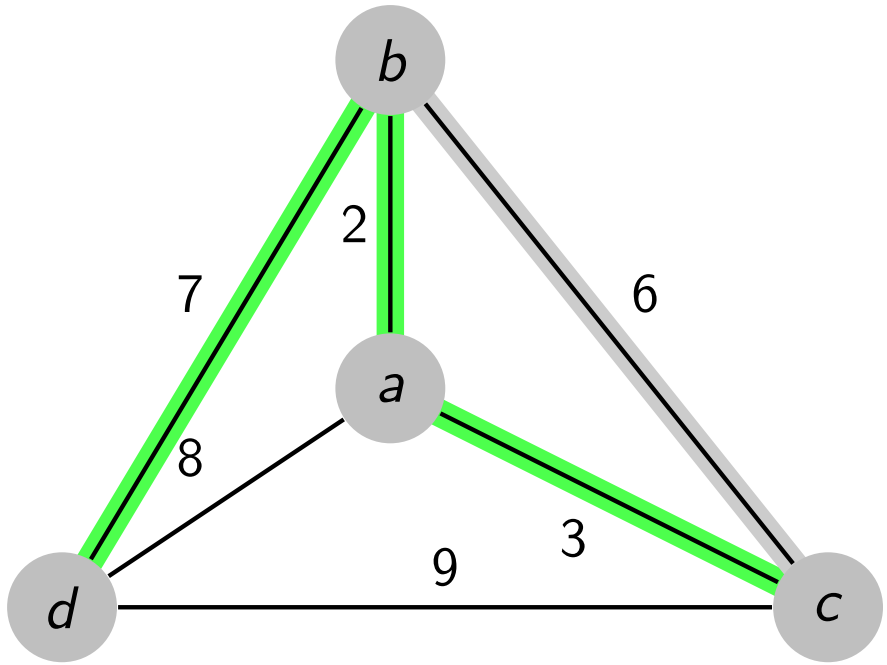

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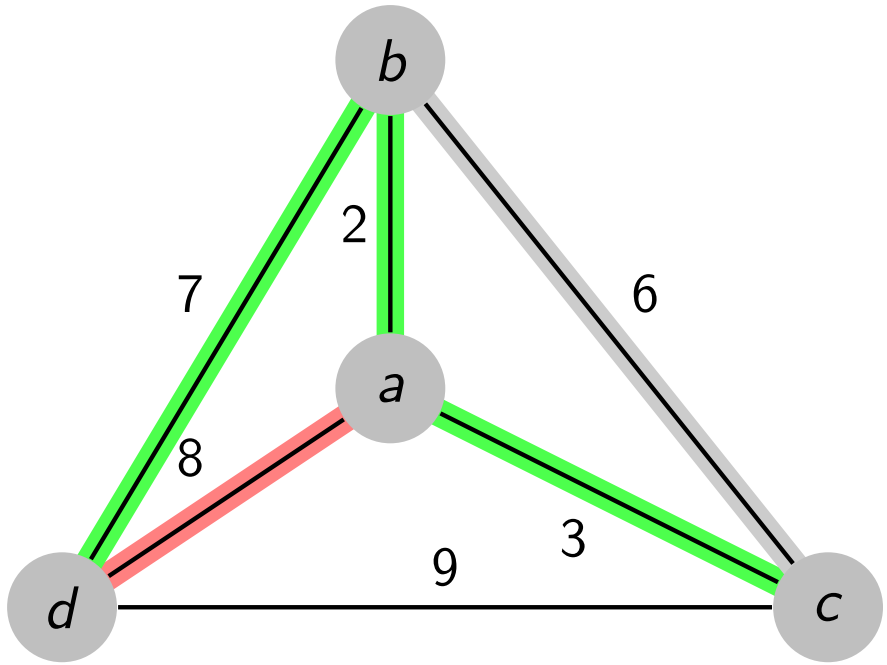

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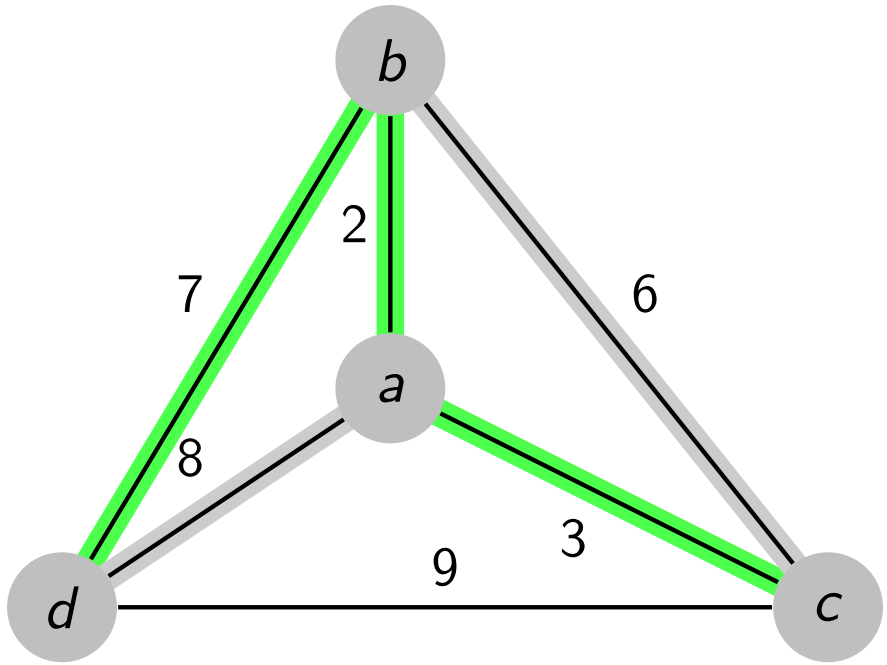

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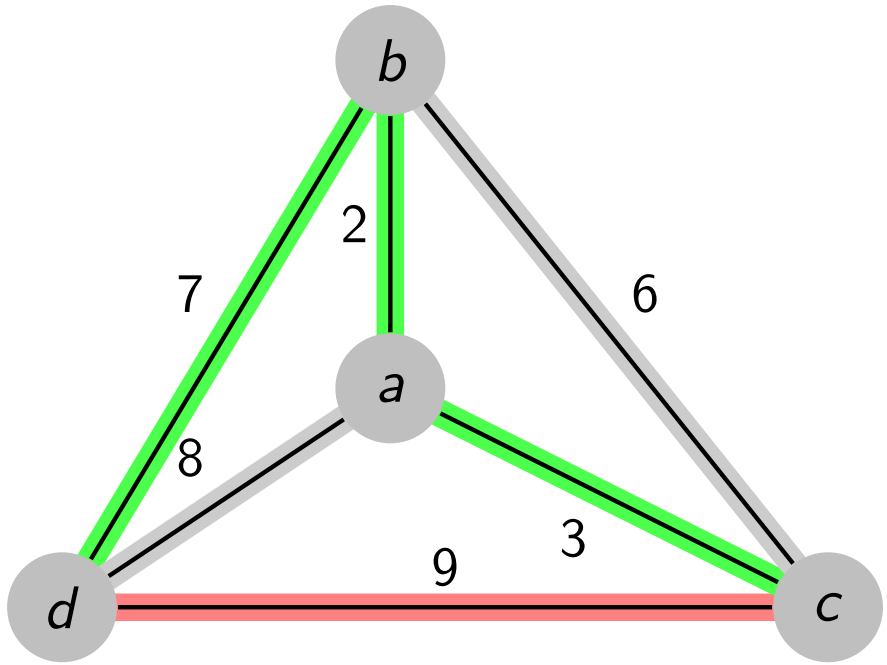

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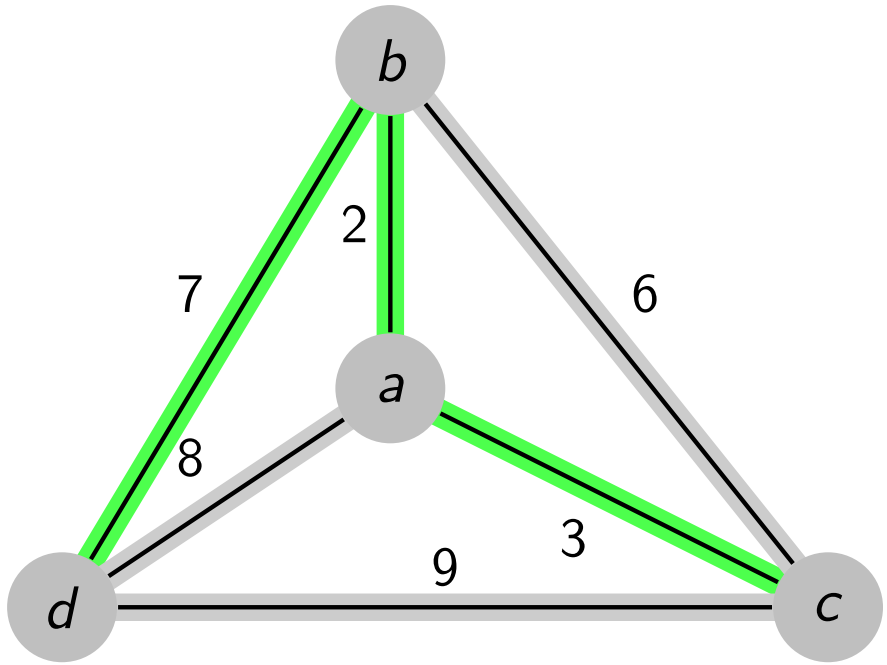

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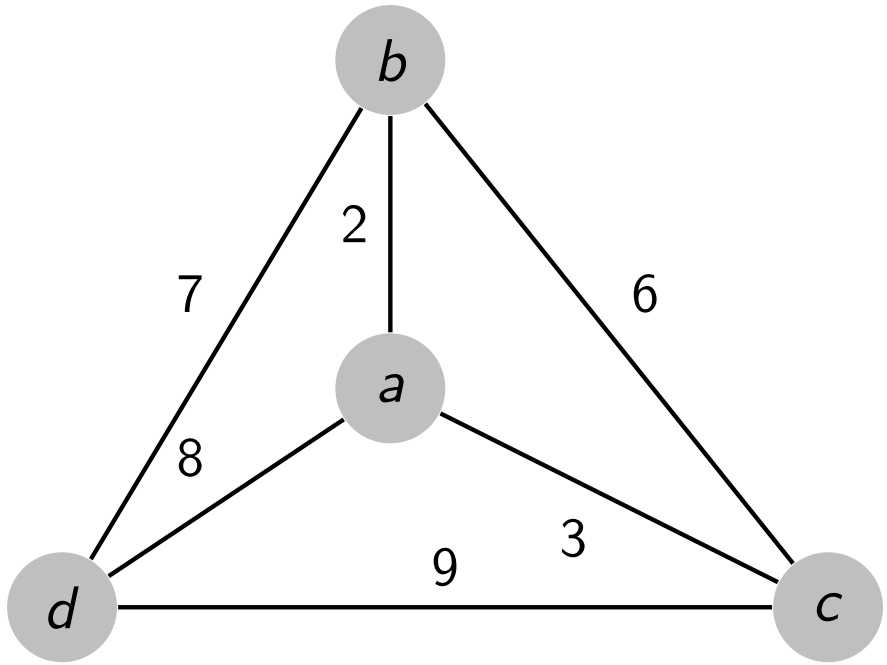

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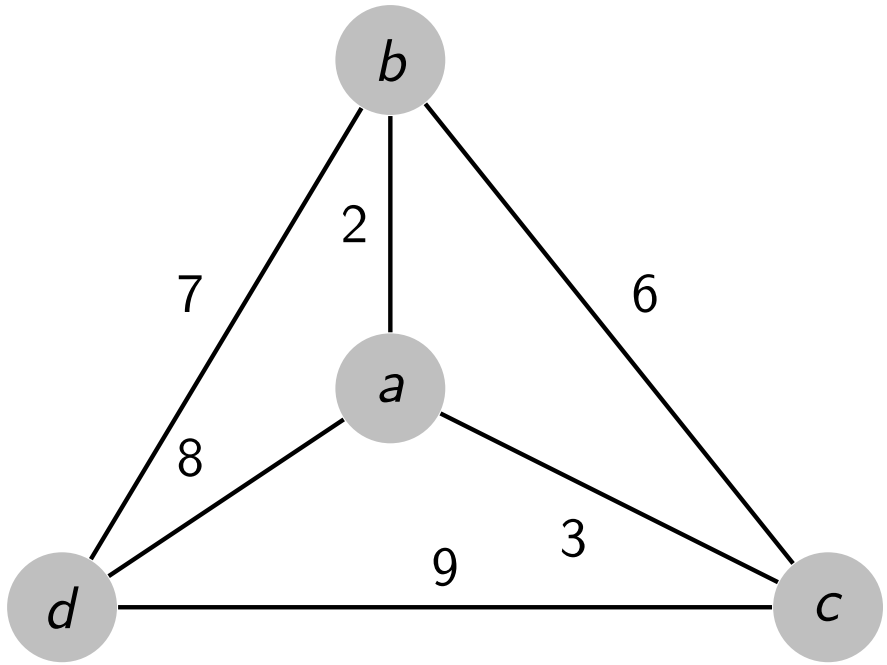

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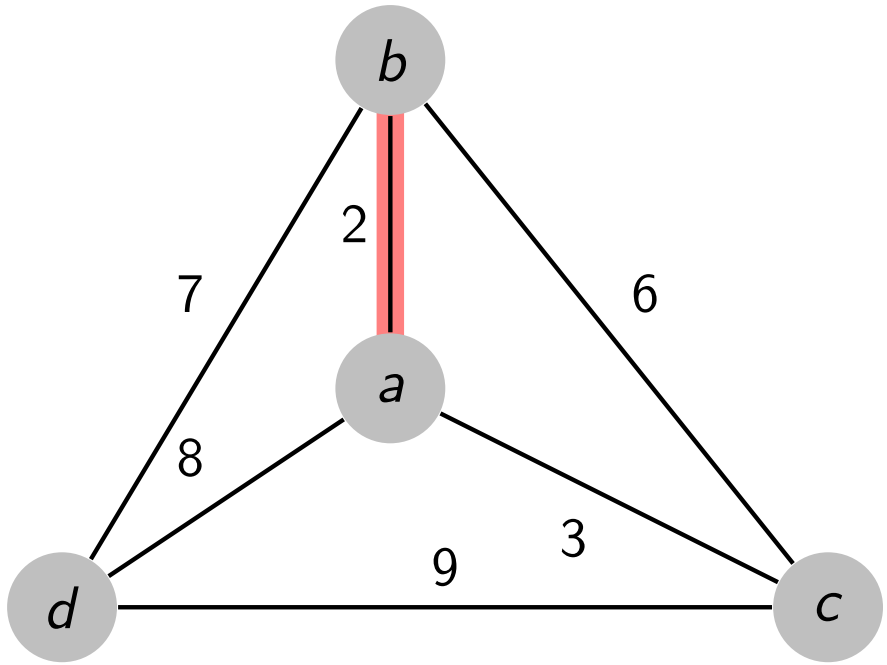

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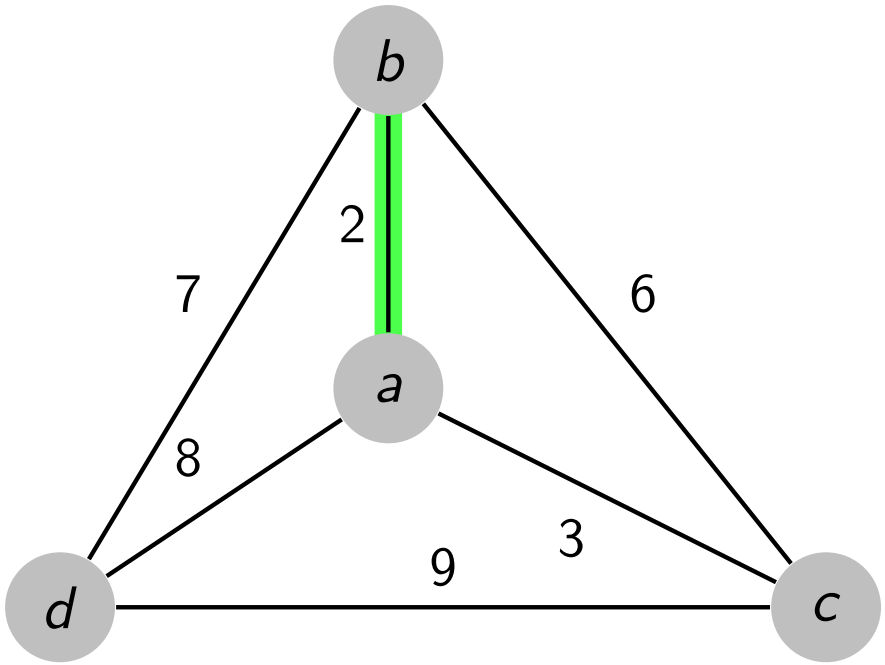

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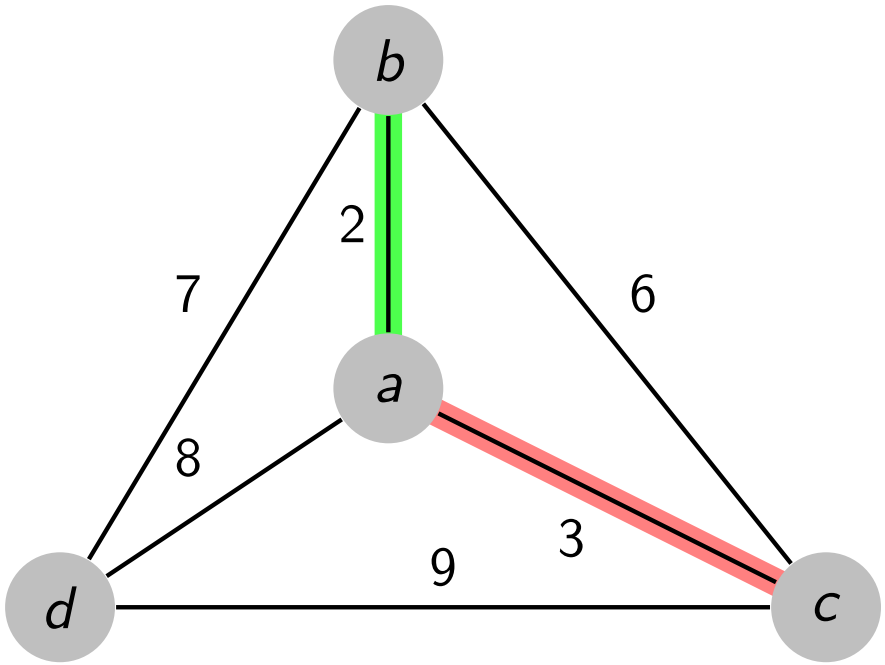

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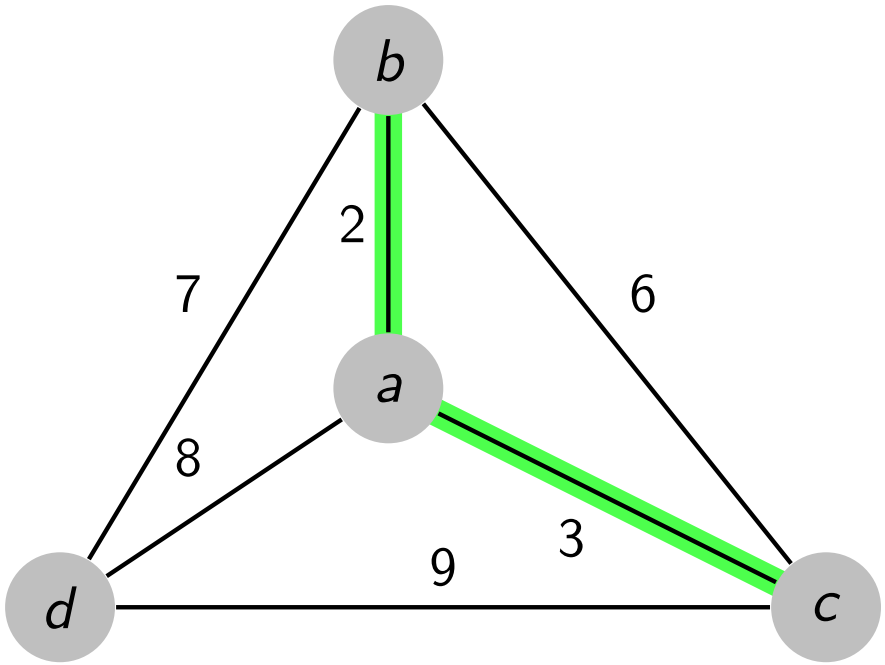

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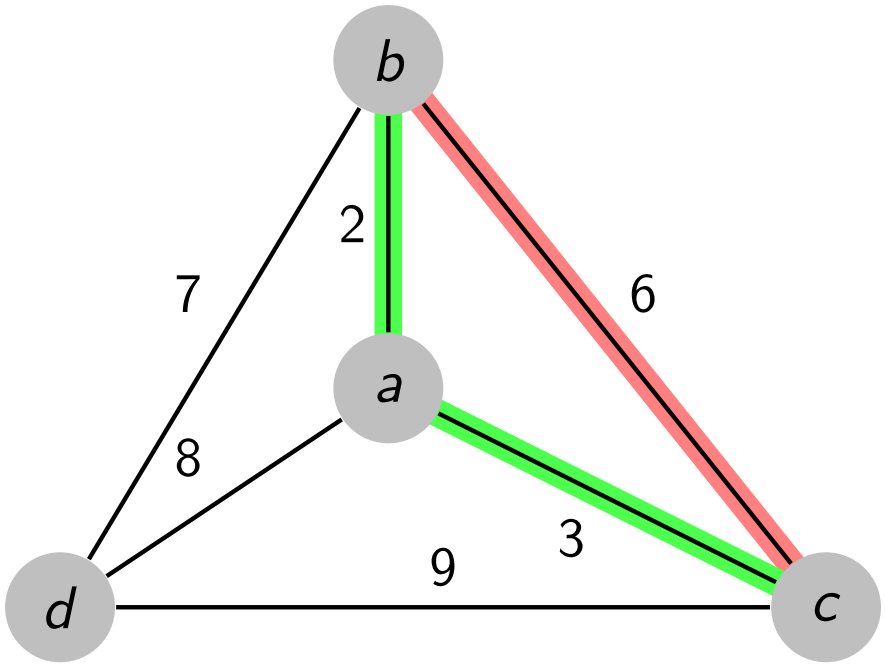

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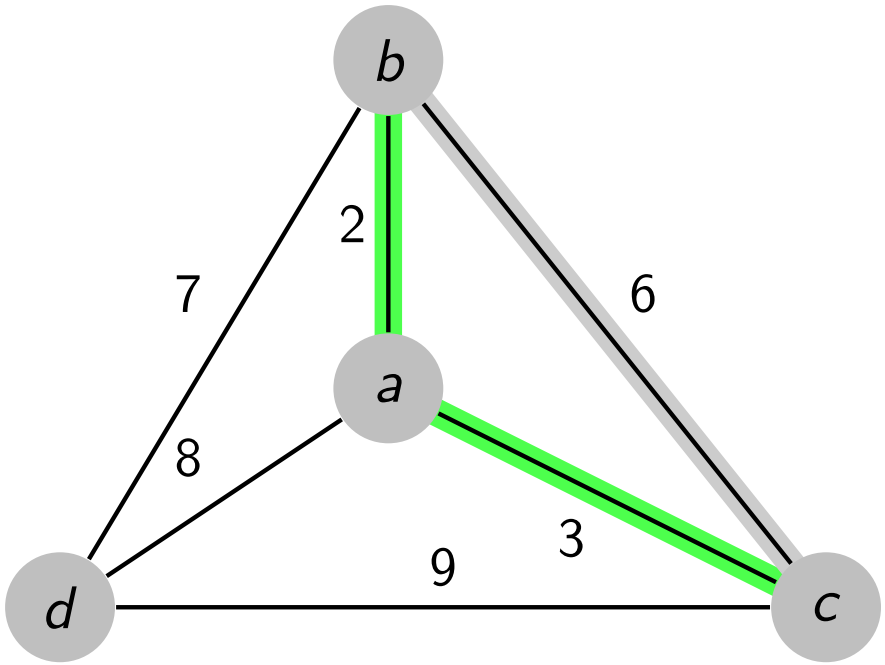

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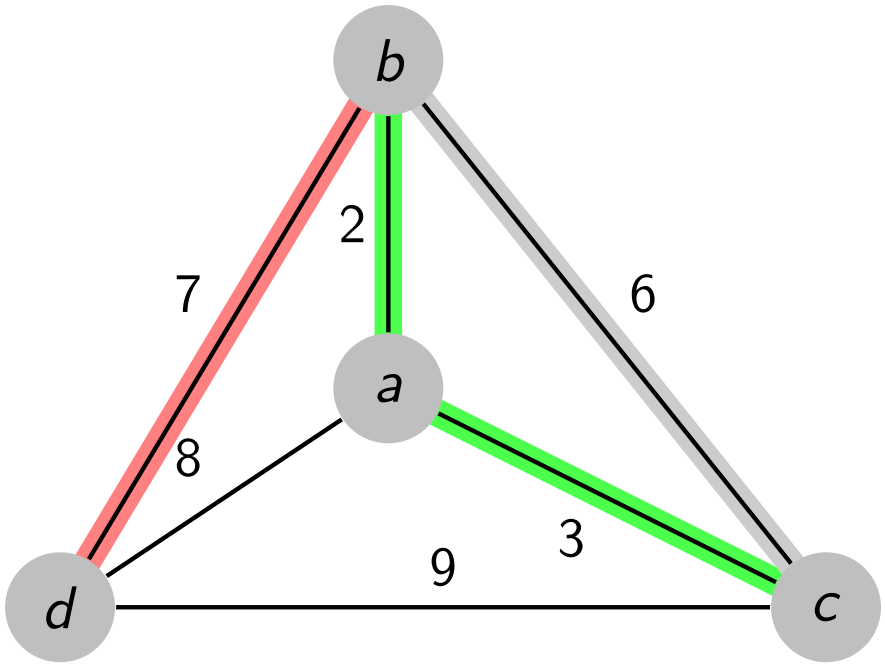

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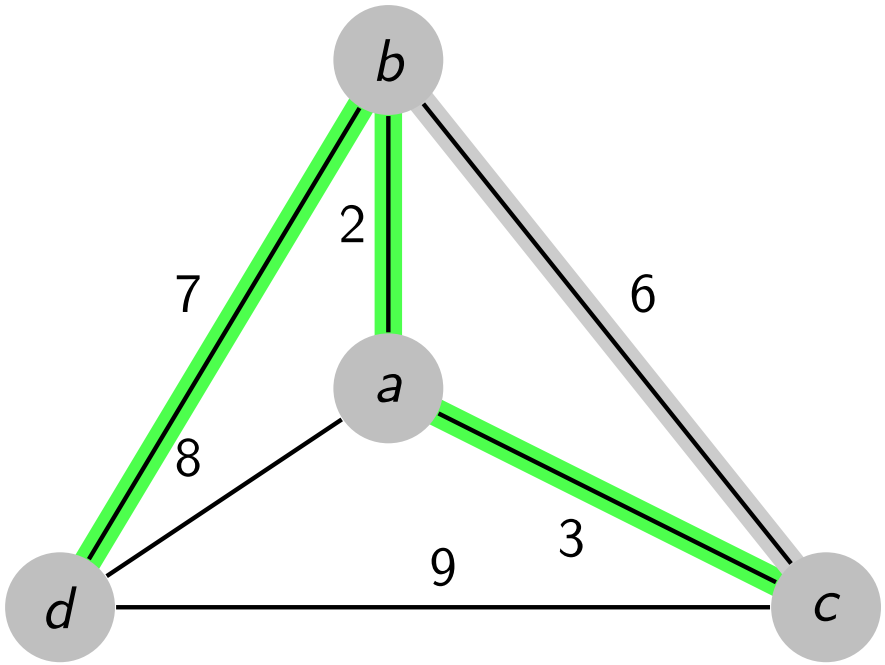

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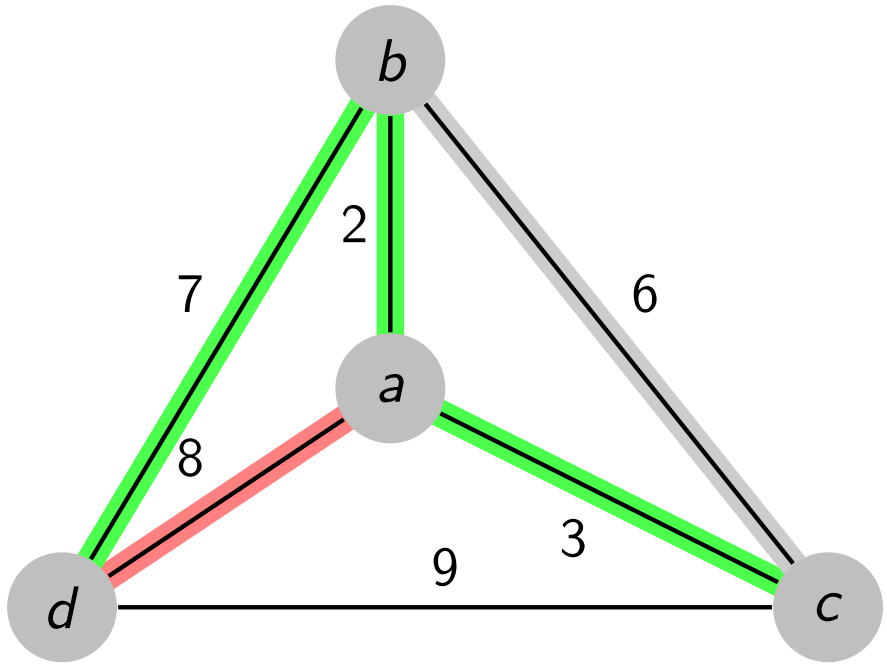

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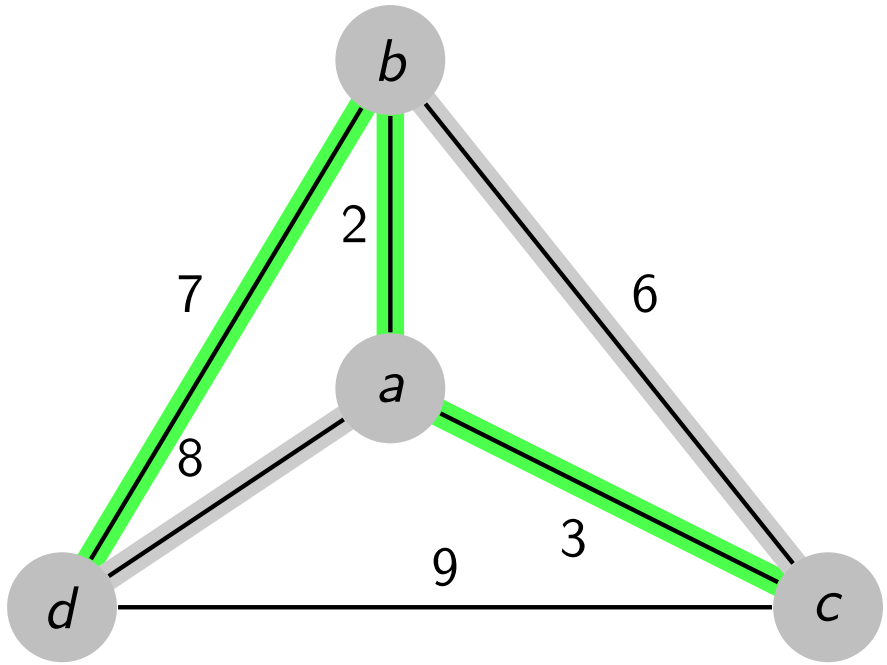

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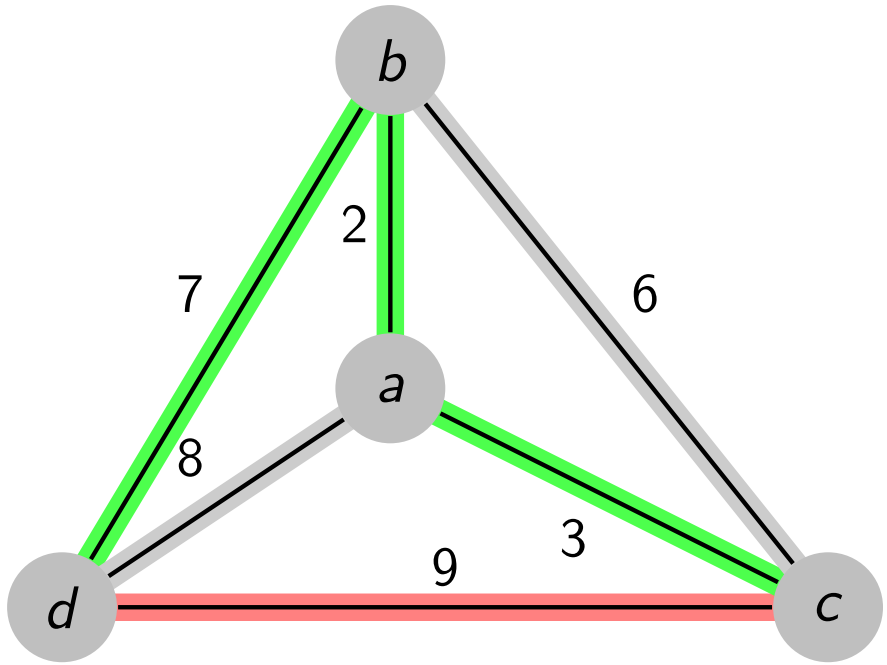

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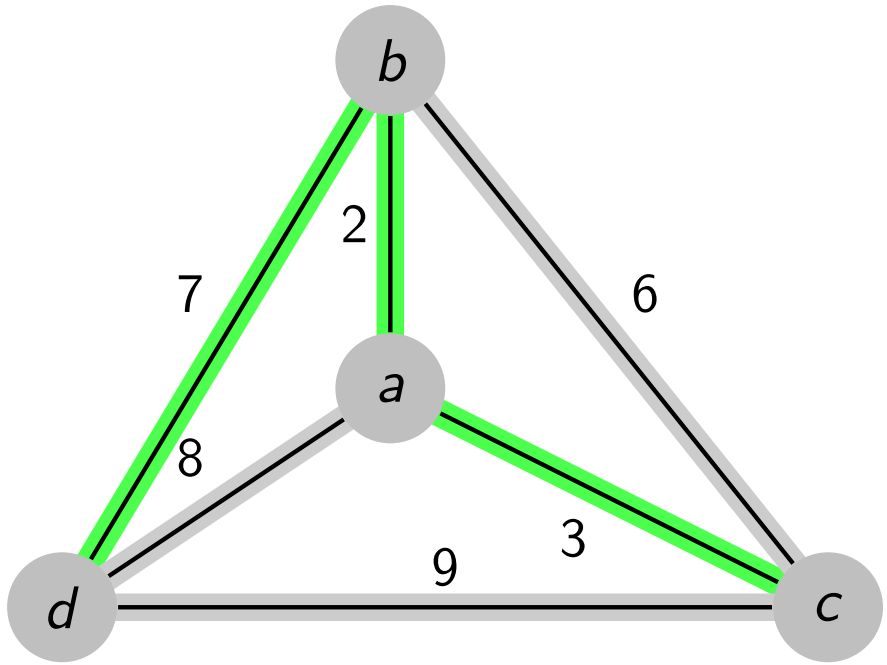

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- $G(n, 1/n)$: the breakpoint. MST structure is built here.

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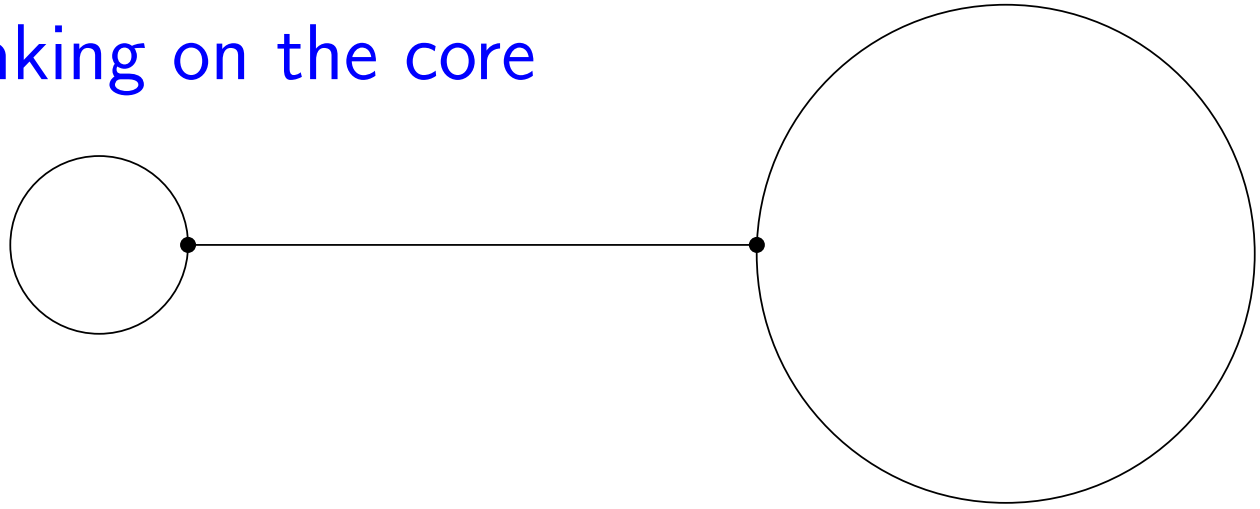
- Cycle breaking is *dual* to Kruskal: instead of adding edges (unless they would create cycles), remove edges (unless they would cause disconnections).

Cycle breaking on the core



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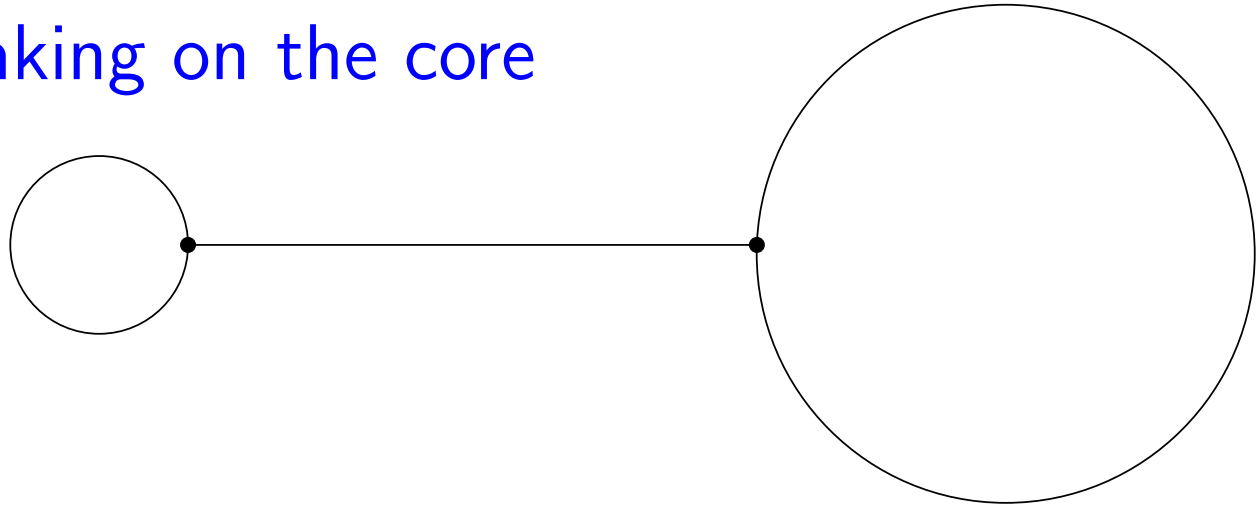

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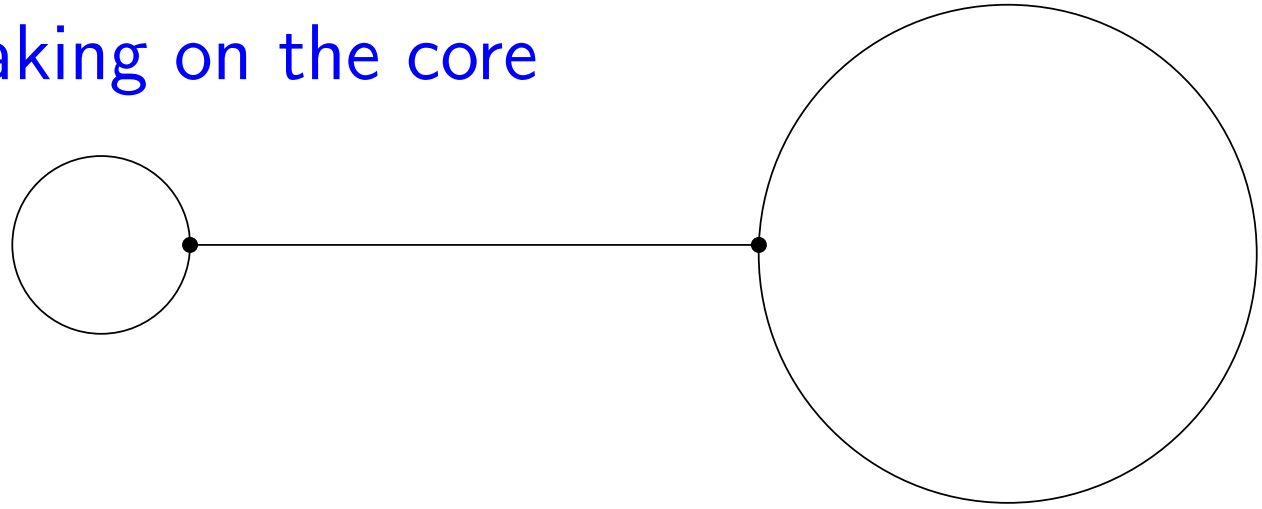

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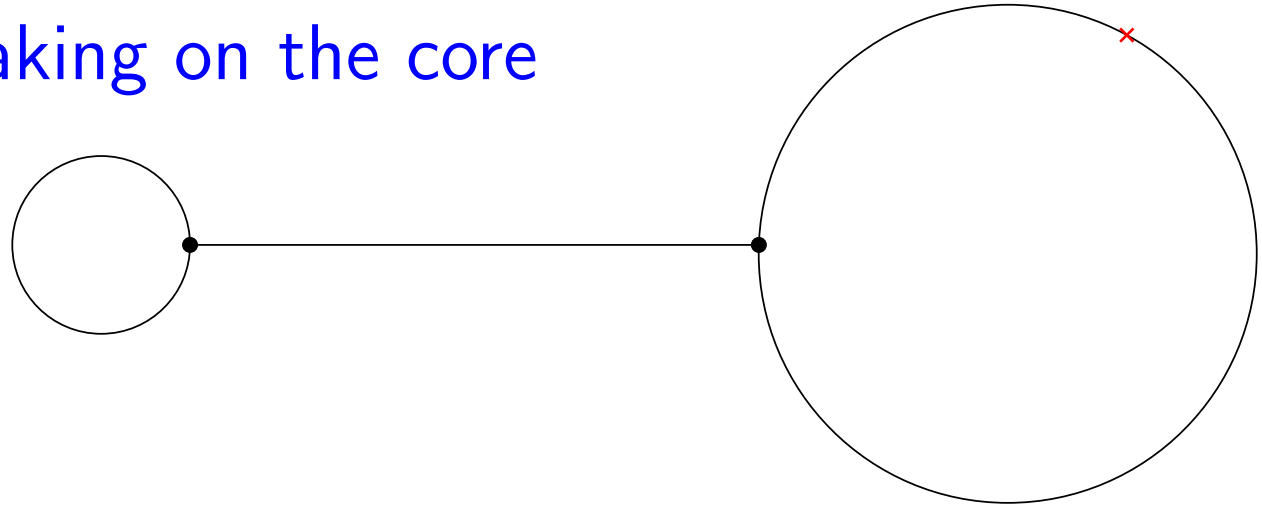
⋆ In the limit, we're choosing a random point on the core.

Cycle breaking on the core

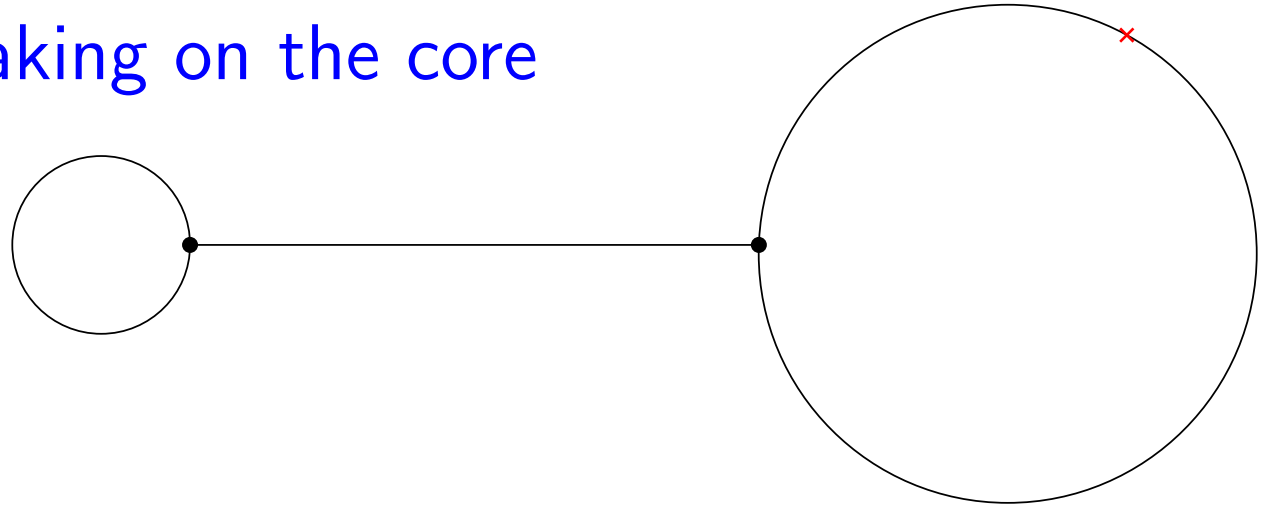


⋆ Choose the largest weight edge in the core, try to remove it.

⋆ It turns out that for components of $G(n, p)$, this edge is equally likely to be any edge of the core.

⋆ In the limit, we're choosing a random point on the core.

Cycle breaking on the core



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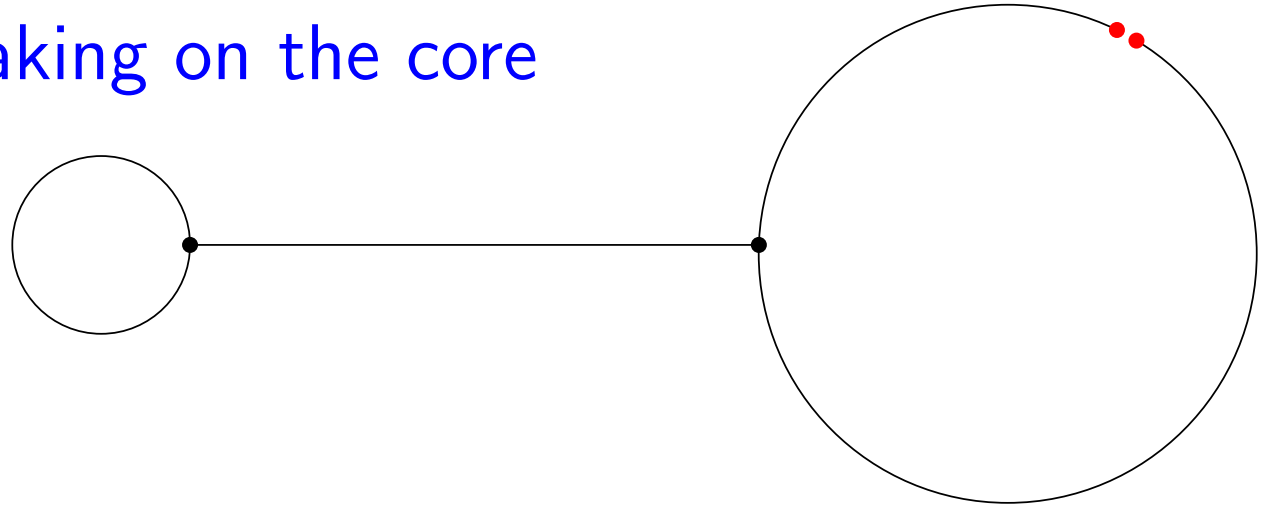

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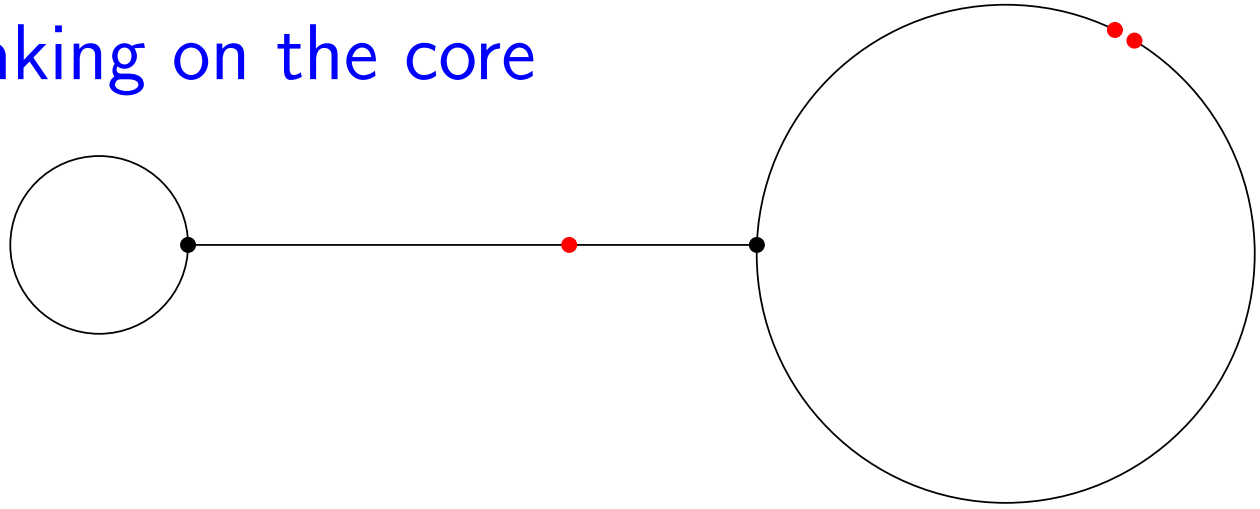

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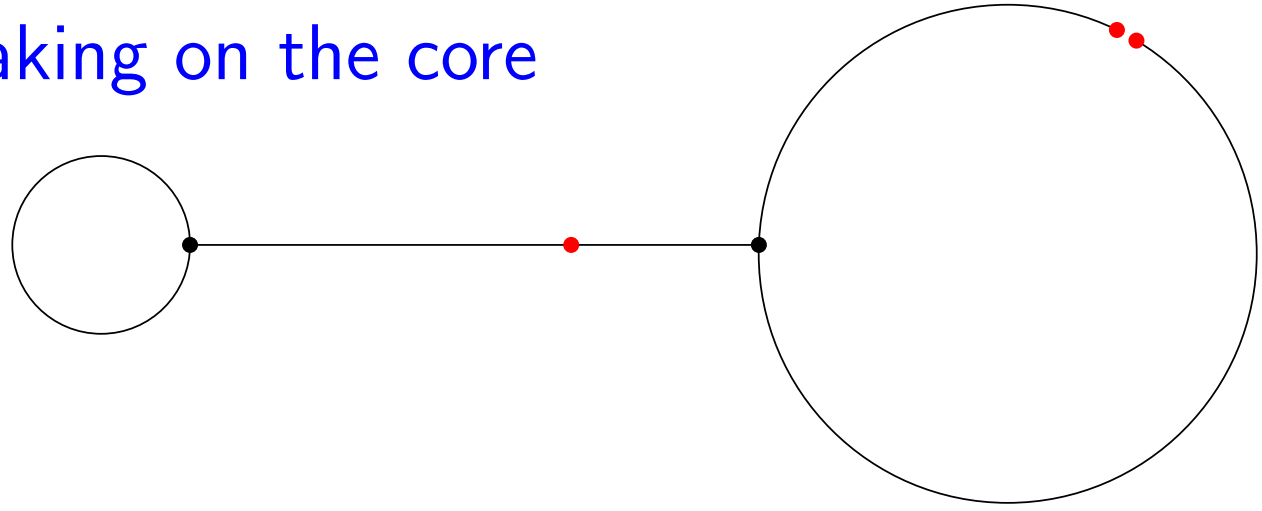

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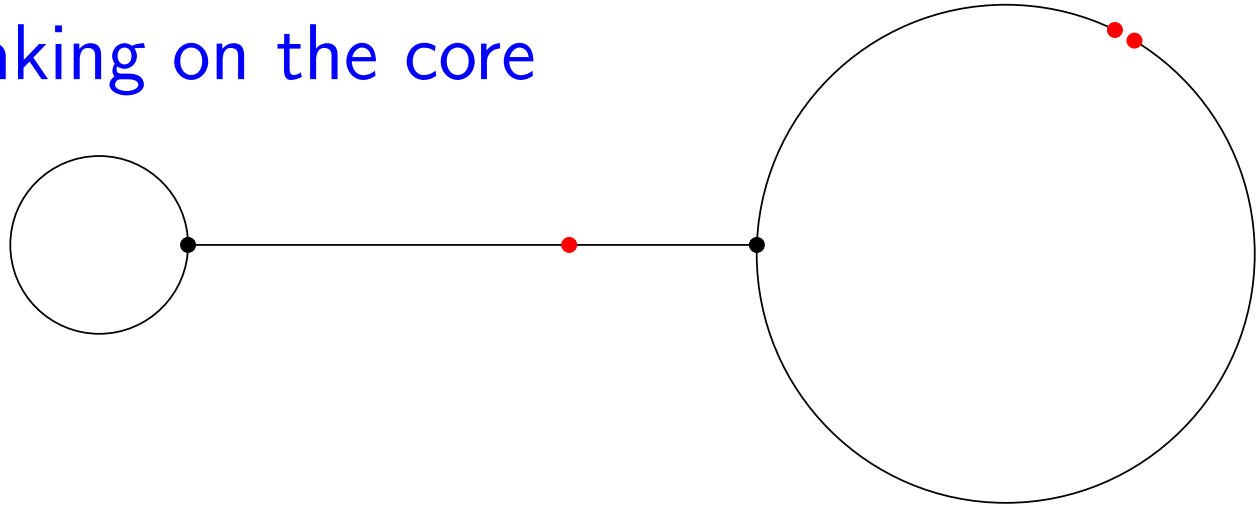

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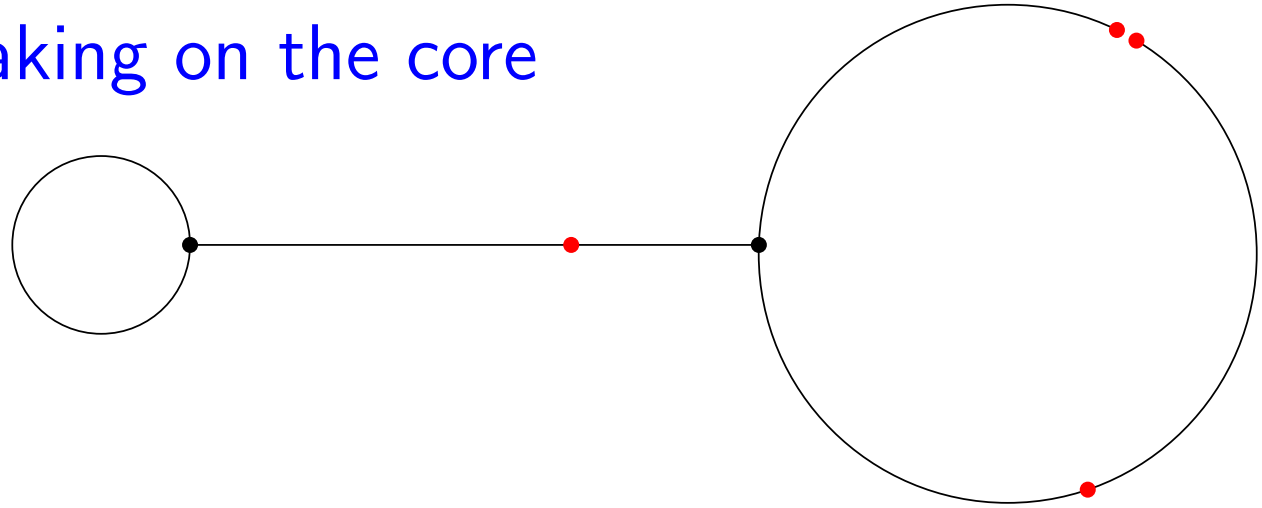
Cycle breaking on the core



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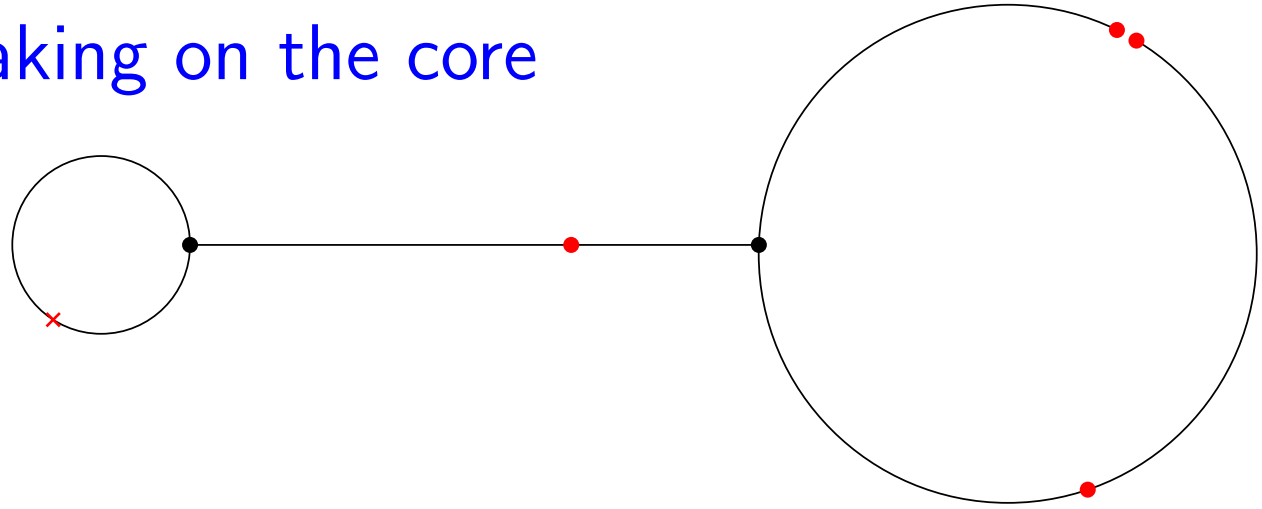

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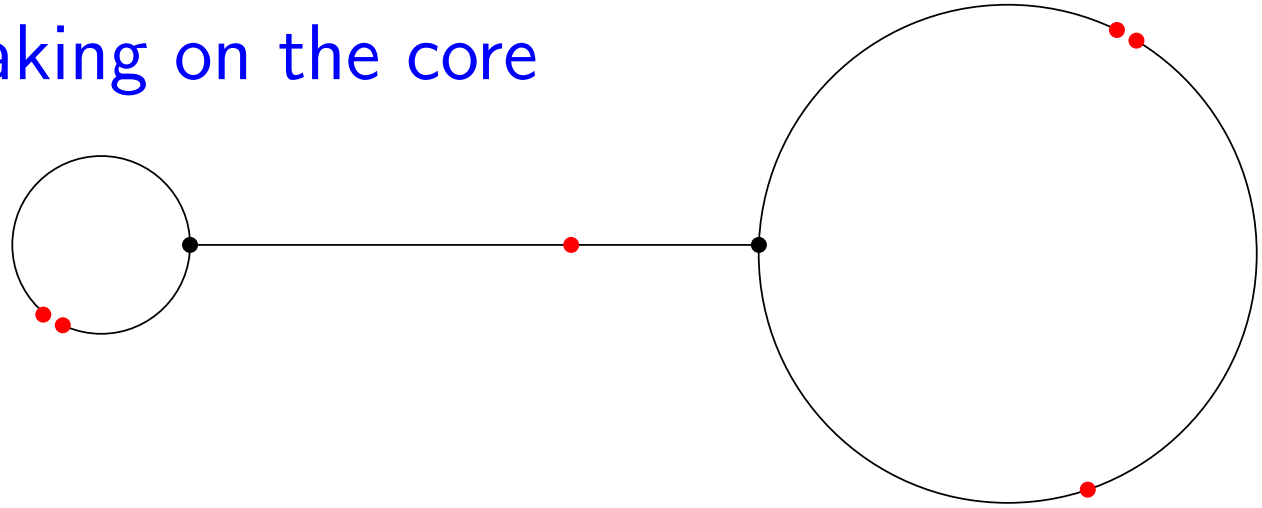

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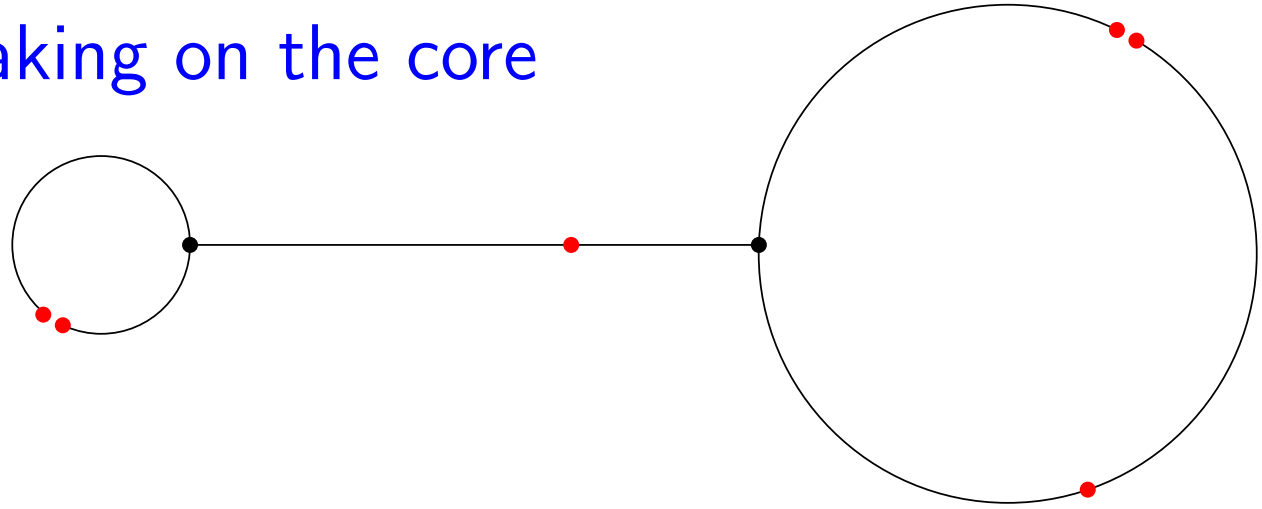

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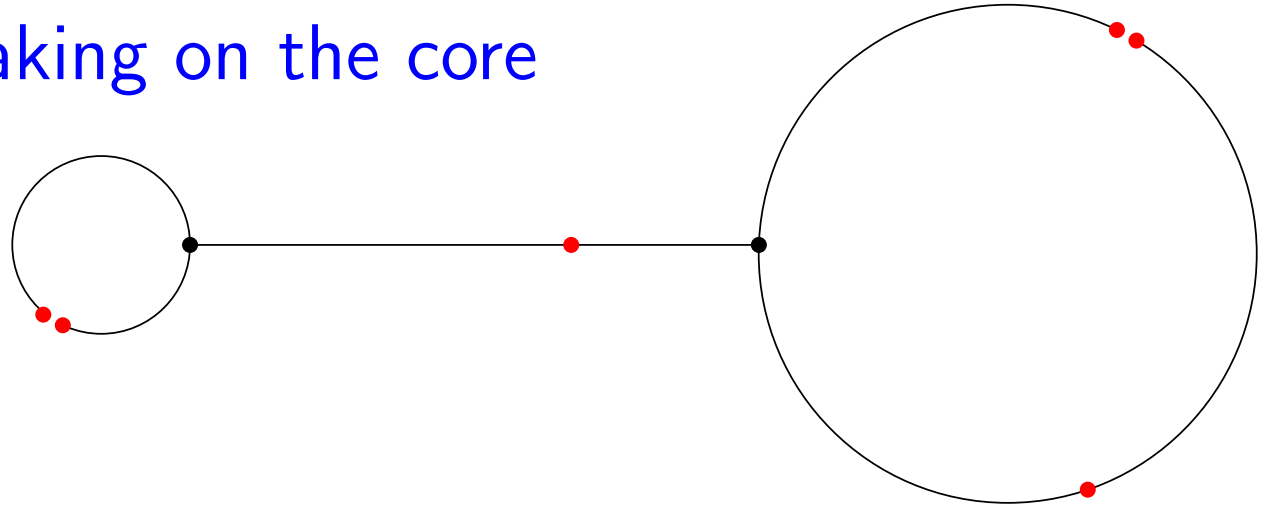
⋆ Repeat until all cycles are broken.

⋆ Now glue all the pendant trees back on.

Cycle breaking on the core



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Result: MST of a component of $G(n,1/n)$ can be built (in the limit) by gluing randomly rescaled Brownian continuum random trees along the edges of a random discrete tree.

Wrapping up

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Because $p = 1/n$ is the critical point for the Erdős-Rényi random graph, and due to the coupling between the random graph (percolation) process and Kruskal's algorithm, proving convergence for the minimum spanning trees of components of $G(n, 1/n)$, as sketched on the previous slide, is the main step in proving convergence of the full minimum spanning tree M_n to a limit $\mathcal{M}$.

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THE SCALING LIMIT OF THE MINIMUM SPANNING TREE OF THE COMPLETE GRAPH

BY LOUIGI ADDARIO-BERRY[1,2,*], NICOLAS BROUTIN[3,†], CHRISTINA GOLDSCHMIDT[2,4,‡] AND GRÉGORY MIERMONT[5,§]

McGill University[*], *Inria Rocquencourt-Paris*[†], *University of Oxford*[‡] *and École Normale Supérieure de Lyon/Institut universitaire de France*[§]

Consider the minimum spanning tree (MST) of the complete graph with n vertices, when edges are assigned independent random weights. Endow this tree with the graph distance renormalized by $n^{1/3}$ and with the uniform measure on its vertices. We show that the resulting space converges in distribution as $n \to \infty$ to a random compact measured metric space in the Gromov–Hausdorff–Prokhorov topology. We additionally show that the limit is a random binary $\mathbb{R}$-tree and has Minkowski dimension 3 almost surely. In particular, its law is mutually singular with that of the Brownian continuum random tree or any rescaled version thereof. Our approach relies on a coupling between the MST problem and the Erdős–Rényi random graph. We exploit the explicit description of the scaling limit of the Erdős–Rényi random graph in the so-called critical window, established in [*Probab. Theory Related Fields* **152** (2012) 367–406], and provide a similar description of the scaling limit for a "critical minimum spanning forest" contained within the MST. In order to accomplish this, we introduce the notion of $\mathbb{R}$-graphs, which generalise $\mathbb{R}$-trees, and are of independent interest.

Universality of the minimum spanning tree limit. . .

. . . still an open problem in general!